Midterm 3

Complete exactly 6 of the 8 problems below. Only 6 will be graded. Indicate which problems you have chosen by circling the problem number.

1. Recall *Newton's method* for finding roots of a function f:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Consider $f(x) = \sin x$, and fix $x_0 \in [5\pi/6, 7\pi/6]$.

(a) Prove that Newton's method will succeed in locating the root π of f: that is, $\lim_{n\to\infty} x_n = \pi$.

(b) The estimate x_n converges exponentially fast to π. What is the exponential rate? Give an explicit bound of the form

$$|x_n - \pi| \leq Ca^n.$$

Solution. (a) Consider the function $\phi(x) = x - \tan x$ defined on $[5\pi/6, 7\pi/6]$. Note that $\phi'(x) = 1 - \sec^2 x \in [-1/3, 0]$. In particular ϕ is decreasing, so

$$\phi([5\pi/6, 7\pi/6]) = [7\pi/6 - 1/\sqrt{3}, 5\pi/6 + 1/\sqrt{3}] \subset [5\pi/6, 7\pi/6].$$

Now ϕ is a contraction mapping, so $x_n = \phi^n(x_0)$ converges to its unique fixed point, π.

(b) The contraction constant is $1/3$. So the estimate x_n satisfies

$$|x_n - x_0| \leq \frac{\sqrt{3}}{2}\left(\frac{1}{3}\right)^n.$$

(The constant $\frac{\sqrt{3}}{2}$ is obtained by maximizing $|x_0 - x_1|$ over $x_0 \in [5\pi/6, 7\pi/6]$.)

2. Let $G(x, y) = x + y + y^4 - 3$.

(a) Show that $G(x, y) = 0$ can be solved for y as a function of x in a neighborhood of $(1, 1)$.

(b) Starting with $f_0(x) = 1$, compute the first two approximations $f_1(x)$, $f_2(x)$ of $y = f(x)$ provided by the implicit function theorem. (Do not simplify.)

Solution. (a) Observe that $G(1, 1) = 0$ and $D_2G(1, 1) = 5 \neq 0$. By the implicit function theorem, $G(x, y) = 0$ can be solved for y as a function of x in a neighborhood of $(1, 1)$.

(b) The first two approximations are

$$
\begin{aligned}
f_1(x) &= 1 - \frac{x-1}{5} \\
f_2(x) &= 1 - \frac{x-1}{5} - \frac{x + 1 - \frac{x-1}{5} + \left(1 - \frac{x-1}{y}\right)^4 - 3}{5}.
\end{aligned}
$$

3. Let $D_r^\infty = \{x \in \mathbb{R}^3 : ||x||_\infty \leq r$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function such that $f(0) = 0$ and

$$||df_x - I||_\infty \leq \epsilon < 1 \qquad \text{for } x \in D_1^\infty. \tag{1}$$

(a) Prove that $f(D_1^\infty) \subset D_{1+\epsilon}^\infty$.

(b) Show that the estimate in (a) is sharp by giving an example of a C^1 function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying (1), such that $f(D_1^\infty) = D_{1+\epsilon}^\infty$.

Solution. Let $h \in D_1^\infty$. Using the function $f - I$ in the mean value theorem we obtain

$$|(f-I)(h) - (f-I)(0)|_\infty \leq |h|_\infty \max_{x \in L} ||d(f-I)_x||,$$

where L is the line segment between 0 and h. Simplifying, we get

$$|f(h) - h|_\infty \leq |h|_\infty \max_{x \in L} ||df_x - I||.$$

Notice that $L \subset D_1^\infty$ since $h \in D_1^\infty$, so

$$|f(h) - h|_\infty \leq \epsilon |h|_\infty.$$

By the triangle inequality,

$$|f(h)|_\infty \leq (1+\epsilon)|h|_\infty \leq 1 + \epsilon.$$

Since $h \in D_1^\infty$ was arbitrary, this shows $f(D_1^\infty) \subset D_{1+\epsilon}^\infty$.

(b) Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x) = (1+\epsilon)x$. Then it is easily verified that $f(D_1^\infty) = D_{1+\epsilon}^\infty$.

4. Compute the (∞-) norm, $||L||$, of the linear map

$$L(x, y, z) = (-2x + 3y - z, 7x + 4z).$$

Solution. The linear map has matrix

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 7 & 0 & 4 \end{pmatrix}.$$

The norm in question is

$$||L|| = ||A|| = \max\{|-2| + |3| + |-1|, |7| + |0| + |4|\} = 11.$$

5. Let A be an $m \times n$ matrix *with all positive entries* and define

$$||A||_{1,\infty} = \max_{x \in \partial D^\infty} |Ax|_1$$

where

$$\begin{aligned} &\partial D^\infty = \{x \in \mathbb{R}^n \,:\, |x|_\infty = 1\} \\ &|y|_1 = |y_1| + \ldots + |y_m| \qquad \text{for } y \in \mathbb{R}^m. \end{aligned}$$

Express $||A||_{1,\infty}$ in terms of the entries of A. Justify your response with proof.

Solution. Let $x \in \partial D^\infty$. Then

$$\begin{aligned} |Ax|_1 &= \left| \sum_{j=1}^n a_{1j} x_j \right| + \ldots + \left| \sum_{j=1}^n a_{nj} x_j \right| \\ &\le \sum_{j=1}^n a_{1j} |x_j| + \ldots + \sum_{j=1}^n a_{nj} |x_j| \\ &\le \sum_{j=1}^n a_{1j} + \ldots + \sum_{j=1}^n a_{nj} \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}. \end{aligned}$$

On the other hand, let $x = (1, 1, \ldots, 1)$. Then $x \in \partial D^\infty$ and

$$\begin{aligned} |Ax|_1 &= \left| \sum_{j=1}^n a_{1j} \right| + \ldots + \left| \sum_{j=1}^n a_{nj} \right| \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}. \end{aligned}$$

It follows that

$$|Ax|_1 = \sum_{i=1}^n \sum_{j=1}^n a_{ij}.$$

6. Show that $\int_0^1 x^2\,dx = 1/3$ by using the definition of volume. You may use the fact that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. By definition, the integral is the volume of the set

$$S = \{(x,y) \,:\, x \in [0,1],\ y \in [0,x^2]\}.$$

See Week 12 Homework solutions for a proof that $v(S) = 1/3$.

7. Let $f : [a, b] \to \mathbb{R}$ be a bounded, nonnegative function and assume

$$G(x) = \int_a^x f$$

exists for all $x \in [a, b]$. Prove that G is continuous.

Solution. Pick M so that $0 \le f \le M$, and observe that

$$|G(y) - G(x)| = \left| \int_a^y f - \int_a^x f \right| = \left| \int_x^y f \right| \le M|y - x|.$$

It follows that G is continuous (in fact Lipschitz continuous, with constant M).

8. Consider the set

$$S = [0, 1/2]^2 \cup ([0, 1]^2 \cap \mathbb{I}^2)$$

where $\mathbb{I}$ is the set of irrational numbers.

(a) Provide intervals $I_1^-, \dots, I_k^-$ and an interval I_1^+ such that:

$$\begin{aligned} &\cup_{j=1}^k I_j^- \subset S, \qquad I_1^+ \supset S \\ &\sum_{j=1}^k v(I_j^-) = 1, \qquad v(I_1^+) = 1. \end{aligned} \tag{2}$$

(b) The set S is not contented. Why doesn't the answer to part (a) contradict the definition of volume?

Solution. (a) Define $I_1^- = \ldots = I_4^- = [0, 1/2]^2$ and $I_1^+ = [0, 1]^2$. These are intervals satisfying (2).

(b) This doesn't contradict the definition because the I_j^-'s in part (a) are *overlapping*.