Week 1 Homework

1. Let V be a vector space of dimension n, and let $v_1, \ldots, v_k$ be linearly independent vectors in V, with $k < n$. Show that there exist vectors $v_{k+1}, \ldots, v_n$ such that $v_1, \ldots, v_n$ is a basis for V.

2. Use the Gram-Schmidt process to find an orthonormal basis for $P^2([-1,1])$, the vector space of polynomial functions on $[-1,1]$ of degree ≤ 2.

3. Let V be a normed vector space[1] such that for each $x, y \in V$,

$$2|x|^2 + 2|y|^2 = |x+y|^2 + |x-y|^2$$

Show that the function $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right)$$

is an inner product on V.

4. Let $T_c : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through the line $y = cx$, and let R_θ be counterclockwise rotation around the origin by the angle θ. Observe that

$$T_c = R_\theta \circ T_0 \circ R_{-\theta}$$

for an appropriately chosen θ, and use this to represent T_c as a matrix.

5. Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$ and define $V^* = \{f : V \to \mathbb{R} \,:\, f \text{ is linear}\}$. Then V^* is also a vector space. Find linear maps $f : V \to V^*$ and $g : V^* \to V$ such that $f \circ g = Id_{V^*}$ and $g \circ f = Id_V$.

Hint: Let $v_1, \ldots, v_n$ be an orthonormal basis for V, and define $f : V \to V^*$ by $f(v) = \phi_v \equiv \langle v, \cdot \rangle$. (So $\phi_v(x) = \langle v, x \rangle$.) Show that $\phi_{v_1}, \ldots, \phi_{v_n}$ is a basis for V^*.

6. Let A be an upper triangular $n \times n$ matrix, i.e. $A = (a_{ij})$ where $a_{ij} = 0$ for $1 \leq j < i \leq n$. Prove that det A $= a_{11}a_{22}\ldots a_{nn}$.

[1] Recall in this class we only consider vector spaces over $\mathbb{R}$.