Week 10 Homework

1. Show that $2 - x - \sin x = 0$ has exactly one solution, x_*, in $[\pi/6, \pi/2]$. Then show that $\phi(x) = 2 - \sin x$ is a contraction mapping on $[\pi/6, \pi/2]$, and calculate x_* to 3 digits of precision.

2. Show that the set of all points (x, y) such that $(x + y)^5 - xy = 1$ is a 1-manifold.

3. Recall Newton's method for finding roots of a differentiable function $f : I \to \mathbb{R}$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_0 \in I. \tag{1}$$

Apply Newton's method, starting with $x_0 = 2$, to estimate $\sqrt{2}$ to three digits of precision *without using the actual (unknown) value of* $\sqrt{2}$ *for comparison.*

Hint: To get an error estimate, prove that the function $\phi(x) = \frac{1}{2}(x + 2/x)$ is a contraction mapping on $[\sqrt{2}, 2]$. Then use our error estimate for contraction mappings.

4. Let $f : [a, b] \to [c, d]$ be a surjective differentiable function such that $0 < m \leq f'(x) \leq M$ for $x \in [a, b]$. Let $y_* \in [c, d]$ and consider the following iteration scheme:

$$x_{n+1} = \phi(x_n), \qquad x_0 = a \tag{2}$$

where

$$\phi(x) = x - \frac{f(x) - y_*}{M}.$$

Prove that ϕ is a contraction map on $[a, b]$. What can one conclude about the sequence $\{x_n\}$?

5. Consider the situation of problem 4, and think of x_n as a function of y_*, say $x_n = f_n(y_*)$. What can be said about the sequence of functions $\{f_n\}$, each defined on $[c, d]$? Do they converge pointwise? Uniformly?

(*) **Error estimate for contraction mappings:** If $\phi : [a, b] \to [a, b]$ is a contraction mapping with contraction constant k and $x_0 \in [a, b]$, then

$$|x_n - x_*| \leq \frac{k^n}{1 - k} |x_1 - x_0|$$

where $x_n \equiv \phi^n(x_0)$ and x_* is the unique fixed point of ϕ.