Week 10 Homework

1. Show that $2 - x - \sin x = 0$ has exactly one solution, x_*, in $[\pi/6, \pi/2]$. Then show that $\phi(x) = 2 - \sin x$ is a contraction mapping on $[\pi/6, \pi/2]$, and calculate x_* to 3 digits of precision.

Solution. Let $f(x) = 2 - x - \sin x$. Observe that $f(\pi/6) = 2 - \pi/6 - 1/2 > 0$ and $f(\pi/2) = 2 - \pi/2 - 1 < 0$, while $f'(x) = -1 - \cos x < 0$ for $x \in [\pi/6, \pi/2]$. So by IVT, $f(x) = 0$ has a solution, and by MVT, $f(x) = 0$ has only one solution.

Now, note that $-\sqrt{3}/2 \le \phi'(x) \le 0$ for $x \in [\pi/6, \pi/2]$. Since ϕ is decreasing on $[\pi/6, \pi/2]$,

$$\phi([\pi/6, \pi/2]) = [\phi(\pi/2), \phi(\pi/6)] = [1, 3/2] \subset [\pi/6, \pi/2].$$

Thus ϕ is a contraction mapping with contraction constant $\sqrt{3}/2$. Let $x_n = \phi^n(\pi/2)$. The contraction mapping theorem then implies

$$|x_n - x_*| \le \frac{\pi/2 - 1}{1 - \sqrt{3}/2} \left(\frac{\sqrt{3}}{2} \right)^n$$

The right hand side is less than 0.001 if $n \ge 59$. The following Matlab code estimates the root:

```
n = 59;
x=pi/2;
for i=1:n
    x = 2-sin(x);
end
disp('root estimate') x
```

The output of the code is $x = 1.106...$. (In fact if one uses $n = 7$ above one finds the root to precision 0.001. So the contraction mapping estimate is not sharp.)

2. Show that the set of all points (x, y) such that $(x + y)^5 - xy = 1$ is a 1-manifold.

Solution. Let $f(x, y) = (x + y)^5 - xy - 1$. Consider the sets

$$S = \{(x, y) \,:\, f(x, y) = 0\}$$
$$S' = \{(x, y) \,:\, f(x, y) = 0,\, \nabla f(x, y) \neq 0\}$$

We have proved that S' is a manifold. We will show here that $S = S'$; since $S' \subset S$ it suffices to show $S \subset S'$. Compute

$$D_1 f(x, y) = 5(x + y)^4 - y$$
$$D_2 f(x, y) = 5(x + y)^4 - x$$

Assume $(x,y) \in S \setminus S'$. Then $D_1 f(x,y) = D_2 f(x,y) = 0$ so $y = x$. In particular from $D_2 f(x,y) = 0$ we get $5(2x)^4 = x$. Now from $f(x,y) = 0$ we obtain:

$$\begin{aligned}
&(2x)^5 - x^2 - 1 = 0 \\
&5(2x)^5 - 5x^2 = 5 \\
&2x^2 - 5x^2 = 5
\end{aligned}$$

and so $-3x^2 = 5 > 0$, a contradiction. We conclude that $S \setminus S' = \emptyset$ so $S = S'$ as desired.

3. Recall Newton's method for finding roots of a differentiable function $f : I \to \mathbb{R}$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_0 \in I. \tag{1}$$

Apply Newton's method, starting with $x_0 = 2$, to estimate $\sqrt{2}$ to three digits of precision *without using the actual (unknown) value of $\sqrt{2}$ for comparison.*

Hint: To get an error estimate, prove that the function $\phi(x) = \frac{1}{2}(x + 2/x)$ is a contraction mapping on $[\sqrt{2}, 2]$. Then use our error estimate for contraction mappings.

Solution. We want to find roots of $f(x) = x^2 - 2$. Define

$$\begin{aligned}
\phi(x) &= x - f(x)/f'(x) \\
&= \frac{1}{2}\left(x + \frac{2}{x}\right).
\end{aligned}$$

If x_* is a fixed point of ϕ then $f(x_*) = 0$, meaning $x_* = \sqrt{2}$. We show ϕ is a contraction mapping on $[\sqrt{2}, 2]$. Compute

$$|\phi'(x)| = \frac{1}{2} - \frac{1}{x^2} \in \left[0, \frac{1}{4}\right] \qquad \text{for } x \in [\sqrt{2}, 2].$$

In particular ϕ is increasing on $[\sqrt{2}, 2]$ so

$$\phi(x) \in [\phi(\sqrt{2}), \phi(2)] = [\sqrt{2}, 3/2] \subset [\sqrt{2}, 2].$$

Thus, ϕ is a contraction mapping with contraction constant $1/4$. So with $x_0 = 2$ and $x_n = \phi^n(x_0)$ we have

$$|x_n - \sqrt{2}| \leq \frac{2}{3}\left(\frac{1}{4}\right)^n.$$

The right hand side is less that 0.001 if $n \geq 5$. The following Matlab code can be used:

```
n=5;
x=2;
for i=1:n
    x = 0.5*(x+2/x);
end
```

disp('root estimate') x

We get the output $x = 1.414...$, while $\sqrt{2} = 1.414....$ (In fact with $n = 5$ we obtain an estimate which is precise up to at least 15 digits after the decimal point!)

4. Let $f : [a, b] \to [c, d]$ be a surjective differentiable function such that $0 < m \le f'(x) \le M$ for $x \in [a, b]$. Let $y_* \in [c, d]$ and consider the following iteration scheme:

$$x_{n+1} = \phi(x_n), \qquad x_0 = a \tag{2}$$

where

$$\phi(x) = x - \frac{f(x) - y_*}{M}.$$

Prove that ϕ is a contraction map on $[a, b]$. What can one conclude about the sequence $\{x_n\}$?

Solution. Note that

$$\phi'(x) \in \left[0, 1 - \frac{m}{M}\right).$$

Observe that since f is increasing and surjective we must have $f(a) = c$ and $f(b) = d$. So since ϕ is also increasing,

$$a \le a - \frac{f(a) - y_*}{M} \le \phi(x) \le b - \frac{f(b) - y_*}{M} \le b.$$

Thus ϕ is a contraction mapping of $[a, b]$. One concludes that $\{x_n\}$ converges to the unique fixed point x_* of ϕ, which is the unique point $x_* \in [a, b]$ such that $f(x_*) = y_*$.

5. Consider the situation of problem 4, and think of x_n as a function of y_*, say $x_n = f_n(y_*)$. What can be said about the sequence of functions $\{f_n\}$, each defined on $[c, d]$? Do they converge pointwise? Uniformly?

Solution. In Problem 4 think of x_n and x_* as being functions of $y_* \in [c, d]$:

$$\begin{aligned} x_n &= f_n(y_*) \\ x_* &= f(y_*). \end{aligned}$$

By the contraction mapping theorem,

$$|f_n(y_*) - y_*| \le \frac{k^n}{1-k}(b - a)$$

where $k \equiv 1 - m/M$. This inequality holds uniformly in $y_* \in [c, d]$, showing $\{f_n\}$ converges uniformly to f.

One application of this is to prove that f is continuous: one first shows (by induction) that each f_n is continuous, then uses the fact that if a sequence of continuous functions converges uniformly on an interval, then the limit function is continuous.

(*) **Error estimate for contraction mappings:** If $\phi : [a, b] \to [a, b]$ is a contraction mapping with contraction constant k and $x_0 \in [a, b]$, then

$$|x_n - x_*| \leq \frac{k^n}{1-k}|x_1 - x_0|$$

where $x_n \equiv \phi^n(x_0)$ and x_* is the unique fixed point of ϕ.