Week 11 Homework

1. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that $G(a,b) = 0$ and $D_2G(a,b) \neq 0$. Then the implicit function theorem yields a function f such that the graph of $y = f(x)$ agrees with the zero set of G in a neighborhood of (a,b). Show that

$$f'(x) = -\frac{D_1G(x, f(x))}{D_2G(x, f(x))}$$

for x in a neighborhood of a.

2. The p-norm on $\mathbb{R}^n$ is defined by

$$|x|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}.$$

Show that

$$\lim_{p \to \infty} |x|_p = |x|_\infty$$

where

$$|x|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

3. Let $a \in \mathbb{R}^n$ and consider the linear map $L_a : \mathbb{R}^n \to \mathbb{R}$ defined by

$$L_a(x) = a \cdot x.$$

For a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ define

$$||L||_p = \max_{x \in \partial D^p} |L(x)|_p$$

where $D^p = \{x \in \mathbb{R}^n \, : \, |x|_p \leq 1\}$. Express $||L_a||_1$ and $||L_a||_\infty$ in terms of the appropriate norms of a.

4. For a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ define

$$||L||_{p,q} = \max_{x \in \partial D^p} |L(x)|_q.$$

Show that if A is the matrix of L, then

$$||L||_{1,\infty} = \max\{|a_{ij}| \, : \, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

5. Show that the linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if

$$M \equiv \min_{x \in \partial D^\infty} |L(x)|_\infty$$

is positive. Use this to show that L is one-to-one if and only if there exists $M > 0$ such that $|L(x)|_\infty \geq M|x|_\infty$ for all $x \in \mathbb{R}^n$.