Week 11 Homework

1. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function such that $G(a,b) = 0$ and $D_2G(a,b) \neq 0$. Then the implicit function theorem yields a C^1 function f such that the graph of $y = f(x)$ agrees with the zero set of G in a neighborhood of (a,b). Show that

$$f'(x) = -\frac{D_1G(x,f(x))}{D_2G(x,f(x))}$$

for x in a neighborhood of a.

Solution. Let $\phi(x) = (x, f(x))$. Then $G \circ \phi \equiv 0$ on a neighborhood U of a, and we may assume U is small enough so that $D_2G(x,f(x)) \neq 0$ for $x \in U$. Thus

$$0 = (G \circ \phi)'(x) = \nabla G(\phi(x)) \cdot \phi'(x) = D_1G(x,f(x)) + f'(x)D_2G(x,f(x)).$$

The result follows.

2. The p-norm on $\mathbb{R}^n$ is defined by

$$|x|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}.$$

Show that

$$\lim_{p\to\infty} |x|_p = |x|_\infty$$

where

$$|x|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

Solution. Let $|x_k|$ be the greatest of $|x_1|, \ldots, |x_n|$. Then

$$|x|_p \leq (n|x_k|^p)^{1/p} \to |x_k| = |x|_\infty \text{ as } p \to \infty,$$

and

$$|x|_p \geq (|x_k|^p)^{1/p} = |x_k| = |x|_\infty$$

The result follows by the squeeze theorem.

3. Let $a \in \mathbb{R}^n$ and consider the linear map $L_a : \mathbb{R}^n \to \mathbb{R}$ defined by

$$L_a(x) = a \cdot x.$$

For a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ define

$$||L||_p = \max_{x\in\partial D^p} |L(x)|_p$$

where $D^p = \{x \in \mathbb{R}^n : |x|_p \leq 1\}$. Express $||L_a||_1$ and $||L_a||_\infty$ in terms of the appropriate norms of a.

Solution. First consider $||L_a||_1$. Let $|a_k|$ be the largest of $|a_1|, \ldots, |a_n|$. Then

$$\begin{aligned} ||L_a||_1 &= \max_{x \in \partial D^1} |a \cdot x|_1 \\ &= \max_{x \in \partial D^1} |a_1 x_1 + \ldots + a_n x_n| \\ &\leq \max_{x \in \partial D^1} (|a_1||x_1| + \ldots + |a_n||x_n|) \\ &\leq |a_k| = |a|_\infty. \end{aligned}$$

On the other hand, let $x = e_k$. Then $x \in \partial D^1$ and

$$|L(x)|_1 = |a \cdot e_k|_1 = |a_k| = |a|_\infty$$

which shows that $||L_a||_1 \geq |a|_\infty$. We conclude that $||L_a||_1 = |a|_\infty$.

Now consider $||L_a||_\infty$. Compute

$$\begin{aligned} ||L_a||_\infty &= \max_{x \in \partial D^\infty} |a \cdot x|_\infty \\ &= \max_{x \in \partial D^\infty} |a_1 x_1 + \ldots + a_n x_n| \\ &\leq \max_{x \in \partial D^\infty} (|a_1||x_1| + \ldots + |a_n||x_n|) \\ &\leq |a_1| + \ldots + |a_n| = |a|_1. \end{aligned}$$

On the other hand, for $j = 1, \ldots, n$, let $x_j = 1$ if $a_j > 0$ and $x_j = -1$ if $a_j < 0$. Then $x \in \partial D^\infty$ and

$$|L(x)|_\infty = |a \cdot x|_\infty = |a_1| + \ldots + |a_n| = |a|_1.$$

which shows $||L_a||_\infty \geq |a|_1$. We conclude that $||L_a||_\infty = |a|_1$.

4. For a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ define

$$||L||_{p,q} = \max_{x \in \partial D^p} |L(x)|_q.$$

Show that if A is the matrix of L, then

$$||L||_{1,\infty} = \max\{|a_{ij}| \,:\, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Solution. Let $x \in \mathbb{R}^n$, let $y = L(x)$ and let $|y_k|$ be the largest of $|y_1|, \ldots, |y_m|$. Then

$$\begin{aligned} ||L||_{1,\infty} &= \max_{x \in \partial D^1} |L(x)|_\infty \\ &= \max_{x \in \partial D^1} |y_k| \\ &= \max_{x \in \partial D^1} |\sum_{j=1}^n a_{kj} x_j| \\ &\leq \max_{x \in \partial D^1} \sum_{j=1}^n |a_{kj}||x_j| \\ &\leq \max\{|a_{ij}| \,:\, 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

On the other hand assume $|a_{kl}|$ is the largest of $|a_{ij}|$, $1 \leq i \leq m$, $1 \leq j \leq n$, and let $x = e_l$. Then $x \in \partial D^1$ and

$$\begin{aligned}
|L(x)|_\infty &= |Ax|_\infty \\
&= \left| \begin{pmatrix} a_{1l} \\ \vdots \\ a_{ml} \end{pmatrix} \right|_\infty \\
&= |a_{kl}| = \max\{|a_{ij}| \,:\, 1 \leq i \leq m, 1 \leq j \leq n\},
\end{aligned}$$

which shows

$$||L||_{1,\infty} \geq \max\{|a_{ij}| \,:\, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The result follows.

5. Show that the linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if

$$M \equiv \min_{x \in \partial D^\infty} |L(x)|_\infty$$

is positive. Use this to show that L is one-to-one if and only if there exists $M > 0$ such that $|L(x)|_\infty \geq M|x|_\infty$ for all $x \in \mathbb{R}^n$.

Solution. Assume $\min_{x \in \partial D^\infty} |L(x)|_\infty = 0$. Then $|L(x)|_\infty = 0$ for some x, which means $L(x) = 0$. Thus Ker $L \neq 0$ and L is not one-to-one. Conversely assume L is not one-to-one. Then Ker $L = 0$, so there is $x \neq 0$ such that $L(x) = 0$. Thus

$$0 = L(x) = |x|_\infty L\left(\frac{x}{|x|_\infty}\right)$$

and since $x/|x|_\infty \in \partial D^\infty$ this shows $\min_{x \in \partial D^\infty} |L(x)|_\infty = 0$.

Now assume there is $K > 0$ such that $|L(x)|_\infty \geq K|x|_\infty$ for all $x \in \mathbb{R}^n$. Let $x \in \partial D^\infty$ be arbitrary. Then $|L(x)|_\infty \geq K|x|_\infty = K$ which shows $\min_{x \in \partial D^\infty} |L(x)|_\infty \geq K > 0$. Conversely, assume $M \equiv \min_{x \in \partial D^\infty} |L(x)|_\infty > 0$, and let $x \in \mathbb{R}^n$ be arbitrary but not zero. Then

$$|L(x)|_\infty = |x|_\infty \left| L\left(\frac{x}{|x|_\infty}\right) \right| \geq M|x|_\infty.$$

If $x = 0$ the same equation is trivially true. The result follows.