Week 12 Homework

1. Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function with a C^1 inverse locally near a. That is, there is a neighborhood U of a and a C^1 function $g : f(U) \to U$ such that $g(f(x)) = x$ for $x \in U$. Show that then $f'(a)$ is nonsingular.

2. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (x, y^3, z^5)$.

(i) f has a *global* inverse, g. Give a formula for g.

(ii) Compute $f'(0)$ and observe that it is singular. Why doesn't this contradict Problem 1?

3. Using the definition of area (see below), calculate the area of the set

$$\{(x, y) \,:\, x \in [0, 1],\ y \in [0, x^2]\}.$$

4. Prove from the definition of area that if $S, T \subset \mathbb{R}^2$ both have area and $S \subset T$, then $v(S) \leq v(T)$.

5. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and let $a = x_0 < x_1 < \ldots < x_n = b$. Use the mean value theorem to show that there exist rectangles $R_1, \ldots, R_n$ of the form

$$R_i = [x_{i-1}, x_i] \times [0, f'(x_i^*)] \quad \text{where } x_i^* \in [x_{i-1}, x_i]$$

such that

$$\sum_{i=1}^{n} v(R_i) = f(b) - f(a).$$

6. Prove that if f and g are admissible, so are $f + g$ and cf $(c \in \mathbb{R})$.

Definition of area: A set $S \subset \mathbb{R}^2$ has *area* $v(S) = \alpha$ if for each $\epsilon > 0$, there exist rectangles $R_1^+, \ldots, R_k^+$ and nonoverlapping rectangles $R_1^-, \ldots, R_l^-$ such that

$$R_1^+ \cup \ldots \cup R_k^+ \supset S \qquad \text{and} \qquad R_1^- \cup \ldots \cup R_l^- \subset S$$
$$\sum_{i=1}^{k} v(R_i^+) < \alpha + \epsilon \qquad \text{and} \qquad \sum_{i=1}^{l} v(R_i^-) > \alpha - \epsilon$$

where for a rectangle $R = [c, d] \times [e, f]$ we define $v(R) \equiv (d - c)(f - e)$.