Week 12 Homework

1. Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function with a C^1 inverse locally near a. That is, there is a neighborhood U of a and a C^1 function $g : f(U) \to U$ such that $g(f(x)) = x$ for $x \in U$. Show that then $f'(a)$ is nonsingular.

Solution. Observe that $g \circ f = I$, (where I is the identity transformation), so

$$dg_{f(a)} \circ df_a = d(g \circ f)_a = dI_a = I. \tag{1}$$

To see that $f'(a)$ is nonsingular, note that if $f'(a)x = 0$ then

$$x = I(x) = dg_{f(a)}(df_a(x)) = dg_{f(a)}(f'(a)x) = dg_{f(a)}(0) = 0.$$

2. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (x, y^3, z^5)$.

(i) f has a *global* inverse, g. Give a formula for g.

(ii) Compute $f'(0)$ and observe that it is singular. Why doesn't this contradict Problem 1?

Solution. (i) The inverse of f is, by inspection, $g(x, y, z) = (x, y^{1/3}, z^{1/5})$.
(ii) Compute

$$f'(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and observe that for the nonzero vector $x = (0, 1, 0)^t$ we have $f'(0)x = 0$. There is no contradiction to (i) since g is not C^1: it is not differentiable at $(0, 0, 0)$.

3. Using the definition of area (see below), calculate the area of the set

$$S = \{(x, y) \,:\, x \in [0, 1],\, y \in [0, x^2]\}.$$

Solution. Let $\epsilon > 0$, $n \in \mathbb{N}$, and define for $j = 1, \ldots, n$:

$$R_j^- = \left[\frac{j-1}{n}, \frac{j}{n}\right] \times \left[0, \frac{(j-1)^2}{n^2}\right], \qquad R_j^+ = \left[\frac{j-1}{n}, \frac{j}{n}\right] \times \left[0, \frac{j^2}{n^2}\right]$$

Then the R_j^-'s and R_j^+'s are rectangles (with the R_j^-'s nonoverlapping) such that

$$\cup_{j=1}^n R_j^- \subset S, \qquad \cup_{j=1}^n R_j^+ \supset S.$$

Furthermore

$$\sum_{j=1}^n v(R_j^-) = \frac{1}{n^3} \sum_{j=1}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6n^3}$$

$$\sum_{j=1}^n v(R_j^+) = \frac{1}{n^3} \sum_{j=1}^{n-1} j^2 = \frac{n(n+1)(2n+1)}{6n^3}$$

To see that $v(S) = 1/3$, choose n large enough so that

$$\frac{(n-1)n(2n-1)}{6n^3} > 1/3 - \epsilon$$
$$\frac{n(n+1)(2n+1)}{6n^3} < 1/3 + \epsilon.$$

4. Assume $S, T \subset \mathbb{R}^2$ both have area and $S \subset T$. Prove that then $v(S) \leq v(T)$.

Solution. Assume for contradiction that $v(S) > v(T)$, and let $\epsilon = v(S) - v(T)$. Choose nonoverlapping intervals $I_1, \ldots, I_p$ and intervals $J_1, \ldots, J_q$ such that

$$\cup_{i=1}^{p} I_j \subset S$$
$$\cup_{i=1}^{q} J_j \supset S$$

and

$$\sum_{i=1}^{p} v(I_j) > v(S) - \epsilon/2$$
$$\sum_{i=1}^{q} v(J_j) < v(T) + \epsilon/2.$$

Then

$$v(T) - v(S) > \left[\sum_{i=1}^{q} v(J_j) - \sum_{i=1}^{p} v(I_j)\right] - \epsilon.$$

To obtain a contradiction, it suffices to show that the quantity in square brackets above is nonnegative. To do this, subdivide the I_j's and J_j's wherever they intersect, so that we obtain nonoverlapping subintervals $\tilde{I}_1, \ldots, \tilde{I}_{\tilde{p}}$ whose union is the same as the union of the I_j's, and nonoverlapping intervals $\tilde{J}_1, \ldots, \tilde{J}_{\tilde{q}}$ whose union is the same as the union of the J_j's, such that the $\tilde{I}_j$'s are a subcollection of the $\tilde{J}_j$'s. Then

$$\sum_{i=1}^{q} v(J_j) - \sum_{i=1}^{p} v(I_j) = \sum_{j=1}^{\tilde{q}} v(\tilde{J}_j) - \sum_{j=1}^{\tilde{p}} v(\tilde{I}_j) \geq 0.$$

5. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and let $a = x_0 < x_1 < \ldots < x_n = b$. Use the mean value theorem to show that there exist rectangles $R_1, \ldots, R_n$ of the form

$$R_i = [x_{i-1}, x_i] \times [0, f'(x_i^*)] \quad \text{where } x_i^* \in [x_{i-1}, x_i]$$

such that

$$\sum_{i=1}^{n} v(R_i) = f(b) - f(a).$$

Solution. Choose x_i^* in $[x_{i-1}, x_i]$ such that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

Then

$$\sum_{i=1}^{n} v(R_i) = \sum_{i=1}^{n} (x_i - x_{i-1}) f'(x_i^*) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] = f(b) - f(a).$$

6. Prove that if f and g are admissible, so are $f + g$ and cf ($c \in \mathbb{R}$).

Solution. Assume f and g are admissible. Then f is zero outside a bounded set D, and g is zero outside a bounded set E, so $f + g$ is zero outside the bounded set $D \cup E$. Also f is continuous outside a negligible set S, and g is continuous outside a negligible set T, so $f + g$ is continuous outside the negligible set $S \cup T$. Finally f and g are bounded, say $|f| \leq M$ and $|g| \leq L$, so $|f + g| \leq M + L$, showing $f + g$ is bounded.

To see that cf is admissible, note that cf is zero outside D, $|cf| \leq cM$ and cf is continuous outside the negligible set S.

Definition of area: A set $S \subset \mathbb{R}^2$ has *area* $v(S) = \alpha$ if for each $\epsilon > 0$, there exist rectangles $R_1^+, \ldots, R_k^+$ and nonoverlapping rectangles $R_1^-, \ldots, R_l^-$ such that

$$R_1^+ \cup \ldots \cup R_k^+ \supset S \qquad \text{and} \qquad R_1^- \cup \ldots \cup R_l^- \subset S$$
$$\sum_{i=1}^{k} v(R_i^+) < \alpha + \epsilon \qquad \text{and} \qquad \sum_{i=1}^{l} v(R_i^-) > \alpha - \epsilon$$

where for a rectangle $R = [c, d] \times [e, f]$ we define $v(R) \equiv (d - c)(f - e)$.