Week 14 Homework

1. If $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are both contented sets, then $A \times B$ is contented and $v(A \times B) = v(A)v(B)$. Prove this in the following two ways:

(a) Using the definition of volume;

(b) Using Fubini's theorem for the function $\phi_{A\times B} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$.

Recall that if I and J are intervals, then *by definition* $v(I \times J) = v(I)v(J)$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be class C^2. Prove that $D_1D_2f = D_2D_1f$. (Assume for contradiction that $D_1D_2f \neq D_2D_1f$. Then WLOG $D_1D_2f - D_2D_1f > 0$ in a rectangle R around 0. Compute $\int_R [D_1D_2f - D_2D_1f]$ using Fubini's theorem.)

3. Define $f : [0,1]^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q} \\ 1/q, & \text{if } x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}, \text{ with } y = p/q, \quad p, q \text{ relatively prime} \end{cases}$$

and let $f_y : [0,1] \to \mathbb{R}$ be defined by $f_y(x) = f(x,y)$. Show that f is integrable and $\int_{[0,1]^2} f = 0$, but that f_y is not integrable for $y \in \mathbb{Q}$.

4. Use Cavalieri's principle to compute the volume of the intersection of the cylinders $x^2+z^2 \leq 1$ and $y^2 + z^2 \leq 1$.