Week 14 Homework

1. If $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are both contented sets, then $A \times B$ is contented and $v(A \times B) = v(A)v(B)$. Prove this in the following two ways:

(a) Using the definition of volume;

(b) Using Fubini's theorem for the function $\phi_{A\times B} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$.

Recall that if I and J are intervals, then *by definition* $v(I \times J) = v(I)v(J)$.

Solution. (a) Let $\epsilon > 0$ and choose $\delta < 1$ such that

$$\delta < \epsilon \left(1 + \min\{v(A), v(B)\}\right)^{-1}.$$

Choose nonoverlapping intervals $I_1^A, \ldots, I_p^A$ and intervals $J_1^A, \ldots, J_q^A$ such that

$$\cup_{j=1}^p I_j^A \subset A, \quad \cup_{j=1}^q J_j^A \supset A$$
$$\sum_{j=1}^p v(I_j^A) > v(A) - \delta, \quad \sum_{j=1}^q v(J_j^A) < v(A) + \delta.$$

Choose nonoverlapping intervals $I_1^B, \ldots, I_r^B$ and intervals $J_1^B, \ldots, J_s^B$ such that

$$\cup_{j=1}^r I_j^B \subset B, \quad \cup_{j=1}^s J_j^B \supset B$$
$$\sum_{j=1}^r v(I_j^B) > v(B) - \delta, \quad \sum_{j=1}^s v(J_j^B) < v(B) + \delta.$$

Now define intervals:

$$I_{j,k} = I_j^A \times I_k^B, \quad 1 \le j \le p,\, 1 \le k \le r$$
$$J_{j,k} = J_j^A \times J_k^B, \quad 1 \le j \le q,\, 1 \le k \le s.$$

Notice that the intervals $\{I_{j,k}\}_{1\le j\le p,\, 1\le k\le r}$ are nonoverlapping,

$$\cup_{1\le j\le p,\, 1\le k\le r}\, I_{j,k} \subset A \times B$$
$$\cup_{1\le j\le q,\, 1\le k\le s}\, I_{j,k} \supset A \times B,$$

and

$$\begin{aligned}\sum_{1\le j\le p,\, 1\le k\le r} v(I_{j,k}) &= \sum_{1\le j\le p,\, 1\le k\le r} v(I_j^A)v(I_k^B) \\ &= \sum_{j=1}^p v(I_j^A) \sum_{k=1}^r v(I_k^B) \\ &> (v(A) - \delta)(v(B) - \delta) \\ &> v(A)v(B) - \epsilon,\end{aligned}$$

while

$$
\begin{aligned}
\sum_{1\le j\le q,\,1\le k\le s} v(J_{j,k}) &= \sum_{1\le j\le q,\,1\le k\le s} v(J_j^A)v(J_k^B)\\
&= \sum_{j=1}^{q} v(J_j^A)\sum_{k=1}^{s} v(J_k^B)\\
&< (v(A)+\delta)(v(B)+\delta)\\
&< v(A)v(B)+\epsilon.
\end{aligned}
$$

(b) To use Fubini's theorem we must first show that $\phi_{A\times B}$ is integrable and for fixed x, $\phi_{A\times B}(x,y)$ is integrable. To see the latter, note that for fixed x, $\phi_{A\times B}(x,y)$ is either the zero function or is equal to $\phi_B(y)$. The zero function is integrable and $\phi_B(y)$ is integrable since B is contented. To see that $\phi_{A\times B}$ is integrable we need to show $A\times B$ is contented. For this one could show for instance that $\partial(A\times B)=\partial A\times\partial B$ is negligible, using an argument like in part (a). Then from Fubini's theorem we obtain

$$
\begin{aligned}
v(A\times B) &= \int_{A\times B}\phi_{A\times B}\\
&= \int_A\left(\int_B \phi_{A\times B}(x,y)\,dy\right)\,dx\\
&= \int_A\left(\int_B \phi_A(x)\phi_B(y)\,dy\right)\,dx\\
&= \int_A (\phi_A(x)v(B))\,dx\\
&= v(B)\int_A \phi_A(x)\,dx\\
&= v(B)v(A).
\end{aligned}
$$

2. Let $f:\mathbb{R}^2\to\mathbb{R}$ be class C^2. Prove that $D_1D_2f=D_2D_1f$. (Assume for contradiction that $D_1D_2f\neq D_2D_1f$. Then WLOG $D_1D_2f-D_2D_1f>0$ in a rectangle R around 0. Compute $\int_R[D_1D_2f-D_2D_1f]$ using Fubini's theorem.)

Solution. We adopt the contradiction assumption above. By Fubini's theorem,

$$
\begin{aligned}
\int_{\mathbb{R}^2} D_1D_2f &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} D_1D_2f(x,y)\,dy\right)\,dx\\
&= \int_{\mathbb{R}} D_1f(x,y)\,dx\\
&= f(x,y)
\end{aligned}
$$

while

$$\begin{aligned}\int_{\mathbb{R}^2} D_2 D_1 f &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} D_2 D_1 f(x,y)\, dx \right) dy \\ &= \int_{\mathbb{R}} D_2 f(x,y)\, dy \\ &= f(x,y)\end{aligned}$$

So

$$\int_{\mathbb{R}^2} (D_1 D_2 f - D_2 D_1 f) = \int_{\mathbb{R}^2} D_1 D_2 f - \int_{\mathbb{R}^2} D_2 D_1 f = 0,$$

contradiction.

3. Define $f : [0,1]^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q} \\ 1/q, & \text{if } x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}, \text{ with } y = p/q, \quad p, q \text{ relatively prime} \end{cases}$$

and let $f(x,0) \equiv 0$. Define $f_y : [0,1] \to \mathbb{R}$ by $f_y(x) = f(x,y)$. Show that f is integrable and $\int_{[0,1]^2} f = 0$, but that f_y is not integrable for $y \in \mathbb{Q}$.

Solution. First we prove f_y is not integrable for $y \in \mathbb{Q}$. Fix $y \in \mathbb{Q} \cap [0,1]$ and write $y = p_0/q_0$ in lowest terms. Then

$$f_y(x) = \begin{cases} 1/q_0, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

So

$$O(f_y) = \{(x,y) \in \mathbb{R}^2 \,:\, x \in \mathbb{Q},\, y \in (0, 1/q_0]\}.$$

Only trivial intervals are subsets of $O(f_y)$, and any collection of intervals whose union contains $O(f_y)$ must have total volume at least $1/q_0$. This shows $O(f_y)$ is not contented, or equivalently, f_y is not integrable.

Now we show that f is integrable and $\int_{[0,1]^2} f = 0$. It suffices to show

$$O(f) = \{(x,y,z) \in [0,1]^3 \,:\, x \in \mathbb{Q},\, y = p_y/q_y \in \mathbb{Q} \text{ (lowest terms)},\, z \in (0, 1/q_y]\}$$

is negligible. Let $\epsilon > 0$. Define

$$S_n = \{y \in \mathbb{Q} \cap (0,1] \,:\, y = p/q \text{ in lowest terms and } 1/q \geq 1/n\}.$$

Note that

$$S_n = \{i/j \,:\, 1 \leq j \leq n,\, 1 \leq i \leq j\}$$

and so $|S_n| \leq n^2$. In particular S_n is finite for each n. Choose n such that $1/n < \epsilon/2$, and let $x_1, \ldots, x_k$ be the all the elements of S_n. Define $I_0^+ = [0,1] \times [0,1] \times [0, \epsilon/2]$ and

$$I_j^+ = [0,1] \times [x_j - \epsilon/(4k), x_j + \epsilon/(4k)] \times [0,1], \quad 1 \leq j \leq k$$

Then

$$\cup_{j=0}^{k} I_j^+ \supset O(f)$$

and

$$\sum_{j=0}^{k} v(I_j^+) = \epsilon/2 + k\epsilon/(2k) = \epsilon.$$

4. Use Cavalieri's principle to compute the volume of the intersection A of the cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$.

Solution. Use "slices" with $z = t$ to obtain

$$v(A) = \int_{-1}^{1} v(A_t)\, dt$$

where A_t is a square:

$$\begin{aligned} A_t &= \{(x, y) \,:\, x^2 \leq 1 - t^2\} \cap \{(x, y) \,:\, y^2 \leq 1 - t^2\} \\ &= \{(x, y) \,:\, -\sqrt{1 - t^2} \leq x, y \leq \sqrt{1 - t^2}\} \end{aligned}$$

so that

$$v(A_t) = \left(2\sqrt{1 - t^2}\right)^2 = 4(1 - t^2)$$

and

$$v(A) = \int_{-1}^{1} v(A_t)\, dt = \int_{-1}^{1} 4(1 - t^2)\, dt = 16/3.$$