Week 1 Homework

1. Let V be a vector space of dimension n, and let $v_1, \ldots, v_k$ be linearly independent vectors in V, with $k < n$. Show that there exist vectors $v_{k+1}, \ldots, v_n$ such that $v_1, \ldots, v_n$ is a basis for V.

Solution. Every basis for V has exactly n vectors, so $v_1, \ldots, v_k$ is not a basis for V; since $v_1, \ldots, v_k$ are linearly independent this means they do not span V. Thus, there is a vector $v_{k+1} \in V$ such that $v_{k+1} \notin \operatorname{span}(v_1, \ldots, v_k)$. We claim that $v_1, \ldots, v_{k+1}$ are linearly independent. To see this, suppose $c_1 v_1 + \ldots c_{k+1} v_{k+1} = 0$. If $c_{k+1} = 0$ then $c_1 = \ldots = c_k = 0$ since $v_1, \ldots, v_k$ are linearly independent. If $c_{k+1} \neq 0$ then v_{k+1} can be written as a linear combination of $v_1, \ldots, v_k$, contrary to $v_{k+1} \notin \operatorname{span}(v_1, \ldots, v_k)$. By induction, we can find linearly independent vectors $v_1, \ldots, v_n$ in V, and since $\dim V = n$ this must be a basis for V.

2. Use the Gram-Schmidt process to find an orthonormal basis for $P^2([-1,1])$, the vector space of polynomial functions on $[-1,1]$ of degree ≤ 2.

Solution. Observe that $1, x, x^2$ is a basis for $P^2([-1,1])$, and in fact 1 and x are orthogonal. (Recall the inner product is $\langle f, g\rangle = \int_{-1}^1 fg\,dx$.) To find a vector which is orthogonal to both 1 and x we use Gram-Schmidt:

$$x^2 - \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 x^2\,dx}x - \frac{\int_{-1}^1 x^2\,dx}{\int_{-1}^1 dx}1 = x^2 - \frac{1}{3}.$$

The last step is to normalize 1, x, and $x^2 - 1/3$; we obtain

$$\frac{\sqrt{2}}{2},\ \sqrt{\frac{3}{2}}x,\ \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right).$$

3. Let V be a normed vector space[1] such that for each $x, y \in V$,

$$2|x|^2 + 2|y|^2 = |x+y|^2 + |x-y|^2. \tag{1}$$

Show that the function $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y\rangle = \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right) \tag{2}$$

is an inner product on V.

Solution. Observe that $\langle u, u\rangle = |u|^2 > 0$ if $x \neq 0$ by positivity of norms. Also $\langle u, v\rangle = \langle u, v\rangle$ follows from $|-x| = |x|$. We will show $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$. Observe that by putting $x = u + w$ and $y = v$ in (1),

$$|u+v+w|^2 = 2|u+w|^2 + 2|v|^2 - |u-v+w|^2$$

[1] Recall in this class we only consider vector spaces over $\mathbb{R}$.

and swapping u and v we get

$$|u+v+w|^2 = 2|v+w|^2 + 2|u|^2 - |v-u+w|^2 = 2|v+w|^2 + 2|u|^2 - |u-v-w|^2.$$

By averaging the right hand sides of the last two equations we obtain

$$|u+v+w|^2 = |u+w|^2 + |v+w|^2 + |u|^2 + |v|^2 - \frac{1}{2}|u-v+w|^2 - \frac{1}{2}|u-v-w|^2.$$

Now,

$$\begin{aligned}
\langle u+v, w\rangle &= \frac{1}{4}\left(|u+v+w|^2 - |u+v-w|^2\right) \\
&= \frac{1}{4}\left(|u+w|^2 + |v+w|^2 - |u-w|^2 - |v-w|^2\right) \\
&= \langle u, w\rangle + \langle v, w\rangle.
\end{aligned}$$

It follows immediately that $\langle nu, v\rangle = n\langle u, v\rangle$ holds for any nonnegative integer n, and from (2) we easily see that $\langle -u, v\rangle = -\langle u, v\rangle$, so it holds for every integer n. From this it follows that $\langle (1/n)u, v\rangle = (1/n)\langle u, v\rangle$ for any nonzero integer n and so $\langle ru, v\rangle = r\langle u, v\rangle$ for any rational number r. To finish the proof, observe that the map $t \to |tx+y|$ is continuous (this follows from the triangle inequality), so from (2) the map $t \to \langle tx, y\rangle$ is continuous. Thus, $t \to (1/t)\langle tx, y\rangle$ is continuous when $t \neq 0$, and this map gives the constant value $\langle x, y\rangle$ whenever t is rational. Thus, $\langle tx, y\rangle = t\langle x, y\rangle$ for every real number t. This completes the proof.

4. Let $T_c : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through the line $y = cx$, and let R_θ be counterclockwise rotation around the origin by the angle θ. Observe that

$$T_c = R_\theta \circ T_0 \circ R_{-\theta}$$

for an appropriately chosen θ, and use this to represent T_c as a matrix.

Solution. Observe that

$$\begin{aligned}
M_{T_0} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
M_{R_\theta} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
M_{R_{-\theta}} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
\end{aligned}$$

and compute

$$M_{R_\theta} M_{T_0} M_{R_{-\theta}} = \begin{pmatrix} 1 - 2\sin^2\theta & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta - 1 \end{pmatrix}.$$

To finish the problem let $\theta = \tan^{-1} c$ and note that

$$T_c(x) = M_{R_\theta} M_{T_0} M_{R_{-\theta}} x.$$

5. Let V be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$ and define $V^* = \{f : V \to \mathbb{R} : f \text{ is linear}\}$. Then V^* is also a vector space. Find linear maps $f : V \to V^*$ and $g : V^* \to V$ such that $f \circ g = Id_{V^*}$ and $g \circ f = Id_V$.

Hint: Let $v_1, \ldots, v_n$ be an orthonormal basis for V, and define $f : V \to V^*$ by $f(v) = \phi_v \equiv \langle v, \cdot \rangle$. (So $\phi_v(x) = \langle v, x \rangle$.) Show that $\phi_{v_1}, \ldots, \phi_{v_n}$ is a basis for V^*.

Solution. Let $v_1, \ldots, v_n$ be an orthonormal basis of V and define $f : V \to V^*$ by $f(v) = \phi_v \equiv \langle v, \cdot \rangle$. Note that f and ϕ_v are linear because of bilinearity of inner products. Write $\phi_j \equiv \phi_{v_j}$ for $j = 1, \ldots, n$, and observe that

$$\phi_j(c_1 v_1 + \ldots + c_n v_n) = c_j. \tag{3}$$

We claim that $\phi_1, \ldots, \phi_n$ is a basis for V^*. To see that they are linearly independent, suppose

$$\phi \equiv c_1\phi_1 + \ldots + c_n\phi_n = 0,$$

that is, ϕ is the zero function on V. Then for $j = 1, \ldots, n$,

$$0 = \phi(v_j) = c_j.$$

To see that $\phi_1, \ldots, \phi_n$ span V, observe that for any $\phi \in V^*$ we have

$$\phi = \phi(v_1)\phi_1 + \ldots + \phi(v_n)\phi_n.$$

Now define $g : V^* \to V$ by

$$g(\phi) = \phi(v_1)v_1 + \ldots + \phi(v_n)v_n.$$

Then g is linear and $g(\phi_j) = v_j$. Now $f(g(\phi_j)) = \phi_j$ and $g(f(v_j)) = v_j$, so linearity of f and g imply the result.

6. Let A be an upper triangular $n \times n$ matrix, i.e. $A = (a_{ij})$ where $a_{ij} = 0$ for $1 \le j < i \le n$. Prove that det A $= a_{11}a_{22}\ldots a_{nn}$.

Solution. The result is trivially true for 1×1 matrices. Assume it is true for $(n-1) \times (n-1)$ matrices. Let A be an upper triangular $n \times n$ matrix, a_{ij} its (i, j)th entry and A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column from A. We compute the determinant of A by expanding along the bottom row:

$$\det A = \sum_{j=1}^{n} (-1)^{n+j} a_{nj} \det A_{nj} = (-1)^{2n} a_{nn} \det A_{nn} = a_{11} \ldots a_{nn},$$

with the last equality coming from the induction assumption. By induction we are done.