Week 3 Homework

1. For $C, D \subset \mathbb{R}^n$ define $d(C,D) = \inf_{x \in C,\, y \in D} |x - y|$. If C is compact and D is closed, prove there exist $x \in C$ and $y \in D$ such that $|x - y| = d(C,D)$.

Hint: First show that the statement is true when $C = \{a\}$ is a single point.

2. Let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable, $F = -\nabla V$, and suppose $\phi : \mathbb{R} \to \mathbb{R}^n$ satisfies

$$F(\phi(t)) = m\phi''(t)$$

where $m > 0$ is constant. Let

$$K(t) = \frac{1}{2}m|\phi'(t)|^2, \qquad P(t) = V(\phi(t)),$$

and prove $K + P$ is a constant function.

Hint: Show that $(K + P)' = 0$. You may use without justification the fact that $P'(t) = \nabla V(\phi(t)) \cdot \phi'(t)$ (a consequence of the *chain rule*, to be proved next week).

3. Give an example to show that MVT does not hold for differentiable functions $\mathbb{R} \to \mathbb{R}^n$. That is, find a differentiable function $f : [a,b] \to \mathbb{R}^n$ such that there is no $c \in (a,b)$ satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4. Define $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(x,y) = xy$. Prove that p is differentiable everywhere with

$$dp_{(a,b)}(x,y) = bx + ay.$$

5. If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, show that F is continuous at a.

Hint: Let $R(h) = [F(a+h) - F(a) - dF_a(h)]/|h|$ for $h \neq 0$, so that
$F(a+h) = F(a) + dF_a(h) + |h|R(h)$.

6. Let

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Prove that f has directional derivatives $D_v f(0,0)$ for every $v \in \mathbb{R}^2$ (so in particular its partial derivatives exist) but that f is not differentiable at $(0,0)$.