Week 3 Homework

1. For $C, D \subset \mathbb{R}^n$ define $d(C,D) = \inf_{x\in C,\, y\in D} |x-y|$. If C is compact and D is closed, prove there exist $c \in C$ and $d \in D$ such that $|c-d| = d(C,D)$.

Hint: First show that the statement is true when $C = \{a\}$ is a single point.

Remark: Below we write $d(x,D)$ instead of $d(\{x\},D)$. Observe that $d(x,D) = \inf_{y\in D} |x-y|$.

Solution. We first prove the hint. Let $D \subset \mathbb{R}^n$ be closed and let $a \in \mathbb{R}^n$. Choose r large enough so that the closed ball $B = \{x \in \mathbb{R}^n : |x-a| \le r\}$ intersects D. Let $K = B \cap D$, observe that K is compact, and consider the continuous function $f : K \to \mathbb{R}$ defined by $f(x) = |a-x|$. As f attains an absolute minimum on K, there is $b \in K$ such that $|a-b| = d(a,K)$. Since $|a-x| \ge |a-b|$ for any $x \in D \setminus K = D \setminus B$, it follows that $|a-b| = d(a,D)$.

Now let $C \subset \mathbb{R}^n$ be compact and define a function $g : C \to \mathbb{R}$ by setting $g(x) = d(x,D)$. We show below that g is continuous. So since C is compact, g attains an absolute minimum at some $c \in C$. Thus $g(c) = d(c,D) = d(C,D)$. And by the hint there is $d \in D$ such that $|c-d| = d(c,D)$, which completes the proof.

To see that g is continuous, let $x_0 \in C$ and $\epsilon > 0$. From the hint there is $y_0 \in D$ such that $|x_0 - y_0| = g(x_0)$. Let x be such that $|x - x_0| < \delta \equiv \epsilon$. Then by the triangle inequality

$$|x - y_0| \le |x_0 - y_0| + |x - x_0| < g(x_0) + \epsilon.$$

and so $g(x) = d(x,D) < g(x_0) + \epsilon$. Now assume (for contradiction) that

$$g(x) \le g(x_0) - \epsilon.$$

By the hint there is $y \in D$ such that $|x-y| = g(x)$, so

$$|x_0 - y| \le |x_0 - x| + |x - y| < \epsilon + g(x) \le g(x_0)$$

which contradicts the definition of g. So

$$g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$$

and g is continuous.

An alternate proof uses sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ in C and D such that $|c_n - d_n| \to d(C,D)$. One shows that $\{d_n\}_{n=1}^\infty$ is bounded, so since D is closed there is a subsequence $\{d_{n_k}\}_{k=1}^\infty$ converging to $d \in D$. Then since C is compact there is a subsequence $\{c_{n_{k_l}}\}_{l=1}^\infty$ of $\{c_{n_k}\}_{k=1}^\infty$ converging to $c \in D$. So by continuity of the norm $|\cdot|$,

$$\lim_{l\to\infty} \left| c_{n_{k_l}} - d_{n_{k_l}} \right| = |c-d| = d(C,D).$$

2. Let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable, $F = -\nabla V$, and suppose $\phi : \mathbb{R} \to \mathbb{R}^n$ satisfies

$$F(\phi(t)) = m\phi''(t)$$

where $m > 0$ is constant. Let

$$K(t) = \frac{1}{2}m|\phi'(t)|^2, \qquad P(t) = V(\phi(t)),$$

and prove $K + P$ is a constant function.

Hint: Show that $(K + P)' = 0$. You may use without justification the fact that $P'(t) = \nabla V(\phi(t)) \cdot \phi'(t)$ (a consequence of the *chain rule*, to be proved next week).

Solution. Differentiating, we obtain

$$K'(t) = m\phi'(t) \cdot \phi''(t) = F(\phi(t)) \cdot \phi'(t)$$
$$P'(t) = \nabla V(\phi(t)) \cdot \phi'(t) = -F(\phi(t)) \cdot \phi'(t)$$

and so $(K + P)' = 0$.

3. Give an example to show that MVT does not hold for differentiable functions $\mathbb{R} \to \mathbb{R}^n$. That is, find a differentiable function $f : [a, b] \to \mathbb{R}^n$ such that there is no $c \in (a, b)$ satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

Solution. Define $f : \mathbb{R} \to \mathbb{R}^3$ by $f(x) = (\cos x, \sin x, x)$. Note that $f(0) = (1, 0, 0)$ and $f(2\pi) = (1, 0, 2\pi)$, so

$$\frac{f(2\pi) - f(0)}{2\pi - 0} = (0, 0, 1).$$

On the other hand the first two components of $f'(x) = (-\sin x, \cos x, 1)$ are never both zero, so there is no c satisfying (1) on $[0, 2\pi]$.

4. Define $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(x) = x_1x_2$. Prove that p is differentiable everywhere with

$$dp_a(h) = a_2h_1 + a_1h_2.$$

Solution. Let $a \in \mathbb{R}^2$, define $L_a : \mathbb{R}^2 \to \mathbb{R}$ by $L_a(h) = a_2h_1 + a_1h_2$. Now

$$\frac{|p(a + h) - p(a) - L_a(h)|}{|h|} = \frac{|(a_1 + h_1)(a_2 + h_2) - a_1a_2 - (a_2h_1 + a_1h_2)|}{\sqrt{h_1^2 + h_2^2}}$$
$$= \frac{|h_1h_2|}{\sqrt{h_1^2 + h_2^2}} \leq \frac{|h_1h_2|}{\min\{|h_1|, |h_2|\}} \leq \max\{|h_1|, |h_2|\}$$

which shows $\lim_{h\to 0}[p(a + h) - p(a) - L_a(h)]/|h| = 0$. Thus, p is differentiable with $dp_a = L_a$.

5. If $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, show that F is continuous at a.

Hint: Let $R(h) = [F(a+h) - F(a) - dF_a(h)]/|h|$ for $h \neq 0$, so that
$F(a+h) - F(a) = dF_a(h) + |h|R(h)$.

Solution. It suffices to show

$$\lim_{h \to 0} |F(a+h) - F(a)| = 0.$$

By the triangle inequality,

$$|F(a+h) - F(a)| \leq |dF_a(h)| + |h||R(h)|.$$

Since F is differentiable, $\lim_{h\to 0} ||R(h)| = 0$. Furthermore, dF_a is a linear map, hence (Lipschitz) continuous – as we show below – so $\lim_{h\to 0} |dF_a(h)| = 0$. Thus,

$$\lim_{h \to 0} |F(a+h) - F(a)| = 0.$$

To see that linear maps are Lipschitz[1] continuous, let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

be its matrix, with rows $A_1, \ldots, A_m$. Write $C = \max_j |A_j|$. Then

$$\begin{aligned} |L(h)| &= |Ah| \\ &= \left| \begin{pmatrix} A_1 \cdot h \\ \vdots \\ A_m \cdot h \end{pmatrix} \right| \\ &= \sqrt{(A_1 \cdot h)^2 + \ldots + (A_m \cdot h)} \\ &\leq \sqrt{|A_1|^2|h|^2 + \ldots + |A_m|^2|h|^2} \\ &\leq \sqrt{m}\, C\, |h| \end{aligned}$$

where the second-to-last step uses Cauchy-Schwartz. Linearity of L now implies Lipschitz continuity.

6. Let

$$f(x) = \begin{cases} \frac{x_2^3}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases}.$$

Prove that f has directional derivatives $D_v f(0)$ for every $v \in \mathbb{R}^2$ (so in particular its partial derivatives exist) but that f is not differentiable at 0.

[1] A map $F : \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz continuous if there is $C > 0$ such that $|F(x) - F(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}^n$.

Solution. Let $v \neq 0 \in \mathbb{R}^2$ and compute

$$\begin{aligned} D_v f(0) &= \lim_{t\to 0} \frac{f(tv) - f(0)}{t} \\ &= \lim_{t\to 0} \frac{(tv_2)^3}{t[(tv_1)^2 + (tv_2)^2]} \\ &= \frac{v_2^3}{v_1^2 + v_2^2} \end{aligned} \tag{2}$$

so that in particular

$$D_1 f(0) = 0, \qquad D_2 f(0) = 1$$

(Recall $D_j \equiv D_{e_j}$.) Assume for contradiction that f is differentiable at 0. Then the directional derivatives at 0 can be written in terms of partial derivatives:

$$D_v f(0) = v_1 D_1 f(0) + v_2 D_2 f(0) = v_2,$$

which contradicts (2).