Week 4 Homework

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and assume

$$(*) \qquad f(tx) = tf(x) \qquad \text{for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}$$

(i) Show that[1] $f(x) = \nabla f(0) \cdot x$, so that f is linear.

(ii) Assume $g : \mathbb{R}^n \to \mathbb{R}$ satisfies (*) but is *not* linear (i.e. not additive). Show that g has directional derivatives at 0 but is not differentiable. (Problem 6 of HW3 is a special case of this.)

2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}^n$ be differentiable, and let $h = g \circ f \circ \phi$. Show that

$$h'(t) = \nabla g(f(\phi(t))) \cdot D_{\phi'(t)} f(\phi(t)).$$

3. Show that if $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable, then $\nabla(fg) = g\nabla f + f\nabla g$.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} x_1 x_2 (x_1^2 - x_2^2)/(x_1^2 + x_2^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that $D_1 f(0, x_2) = -x_2$ and $D_2 f(x_1, 0) = x_1$ for all x_1, x_2. Conclude that $D_1 D_2 f(0,0)$ and $D_2 D_1 f(0,0)$ exist but are unequal.

5(a) For $a, x \in \mathbb{R}^2$ with $|x| = 1$, show that $|a \cdot x| \leq |a|$ by finding the maximum and minimum values of $f(x) = a \cdot x$ on the unit circle.

(b) Use (a) to show $|a \cdot b| \leq |a||b|$ for $a, b \in \mathbb{R}^2$.

[1] Recall $\nabla f = (D_1 f, \ldots, D_n f)$.