Week 4 Homework

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and assume

$$(*) \qquad f(tx) = tf(x) \qquad \text{for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}$$

(i) Show that[1] $f(x) = \nabla f(0) \cdot x$, so that f is linear.

(ii) Assume $g : \mathbb{R}^n \to \mathbb{R}$ satisfies (*) but is *not* linear (i.e. not additive). Show that g has directional derivatives at 0 but is not differentiable. (Problem 6 of HW3 is a special case of this.)

Solution. Note that (*) implies $f(0) = 0$. Let $x \in \mathbb{R}^n$ and compute

$$\nabla f(0) \cdot x = D_x f(0) = \frac{f(tx) - f(0)}{t} = f(x)$$

This shows f is linear with matrix $\nabla f(0)$. So if g satisfies (*) but is not linear, then g cannot be differentiable though it has directional derivatives $D_x f(0) = f(x)$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}^n$ be differentiable, and let $h = g \circ f \circ \phi$. Show that

$$h'(t) = \nabla g(f(\phi(t))) \cdot D_{\phi'(t)} f(\phi(t)).$$

Solution. By the chain rule,

$$\begin{aligned} h'(t) &= g'((f \circ \phi)(t)) \cdot (f \circ \phi)'(t) \\ &= \nabla g(f(\phi(t))) \cdot (f'(\phi(t))\phi'(t)) \end{aligned}$$

Now observe that

$$f'(\phi(t))\phi'(t) = df_{\phi(t)}(\phi'(t)) = D_{\phi'(t)} f(\phi(t))$$

which gives the desired result.

3. Show that if $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable, then $\nabla(fg) = g\nabla f + f\nabla g$.

Solution. We must show that

$$\lim_{h \to 0} \frac{1}{|h|} \left(f(a+h)g(a+h) - f(a)g(a) - g(a)\nabla f(a) \cdot h - f(a)\nabla g(a) \cdot h \right) = 0.$$

Observe that

$$f(a+h)g(a+h) = f(a+h)\left(g(a+h) - g(a)\right) + g(a)\left(f(a+h) - f(a)\right)$$

[1] Recall $\nabla f = (D_1 f, \ldots, D_n f)$.

Substituting the second display into the first, we get

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{|h|}\left(f(a+h)g(a+h)-f(a)g(a)-g(a)\nabla f(a)\cdot h-f(a)\nabla g(a)\cdot h\right)\\
&=\lim_{h\to 0}\frac{1}{|h|}\left[g(a)\left(f(a+h)-f(a)-\nabla f(a)\cdot h\right)+f(a+h)\left(g(a+h)-g(a)\right)-f(a)\nabla g(a)\cdot h\right]\\
&=\lim_{h\to 0}\frac{1}{|h|}\left[f(a+h)\left(g(a+h)-g(a)\right)-f(a)\nabla g(a)\cdot h\right]\\
&=\lim_{h\to 0}\frac{1}{|h|}\left[f(a)\left(g(a+h)-g(a)-\nabla g(a)\cdot h\right)+\left(f(a+h)-f(a)\right)\left(g(a+h)-g(a)\right)\right]\\
&=\lim_{h\to 0}\frac{1}{|h|}(f(a+h)-f(a))(g(a+h)-g(a))
\end{aligned}\tag{1}$$

We claim that $(f(a+h)-f(a))/|h|$ is bounded as $h\to 0$. To see this, let

$$\phi(h)=\frac{1}{|h|}\left(f(a+h)-f(a)-\nabla f(a)\cdot h\right)$$

Then

$$\begin{aligned}
\frac{|f(a+h)-f(a)|}{|h|}&=\left|\phi(h)+\frac{f(a)\cdot h}{|h|}\right|\\
&\le|\phi(h)|+\frac{|f(a)\cdot h|}{|h|}\\
&\le|\phi(h)|+|f(a)|
\end{aligned}$$

and since $\phi(h)\to 0$ as $h\to 0$ the claim follows. Now from continuity of g at a the limit in (1) must be zero.

4. Define $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x)=\begin{cases}xy(x^2-y^2)/(x^2+y^2), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Show that $D_1f(0,y)=-y$ and $D_2f(x,0)=x$ for all x,y. Conclude that $D_1D_2f(0,0)$ and $D_2D_1f(0,0)$ exist but are unequal.

Solution. Compute

$$\begin{aligned}
D_1f(0,y)&=\lim_{t\to 0}\frac{1}{t}\frac{ty(t^2-y^2)}{t^2+y^2}=-y\\
D_2f(x,0)&=\lim_{t\to 0}\frac{1}{t}\frac{xt(x^2-t^2)}{x^2+t^2}=x
\end{aligned}$$

Thus

$$\begin{aligned}
D_2D_1f(0,0)&=-1\\
D_1D_2f(0,0)&=1.
\end{aligned}$$

5(a) For $a, x \in \mathbb{R}^2$ with $|x| = 1$, show that $|a \cdot x| \leq |a|$ by finding the maximum and minimum values of $f(x) = a \cdot x$ on the unit circle.

(b) Use (a) to show $|a \cdot b| \leq |a||b|$ for $a, b \in \mathbb{R}^2$.

Solution. Let $f(x) = a \cdot x$ and $g(x) = |x|^2 - 1$. We will maximize f on the unit circle $S = \{x : g(x) = 0\}$. To do this we use Lagrange multipliers:

$$\nabla f(x) = \lambda \nabla g(x)$$

which gives

$$\begin{aligned} a_1 &= 2\lambda x_1 \\ a_2 &= 2\lambda x_2 \end{aligned}$$

and so

$$\frac{a_1}{a_2} = \frac{x_1}{x_2}$$

Using $|x|^2 = 1$ and solving for x_1, x_2 gives

$$x_1 = \frac{\pm a_1}{|a|}, \quad x_2 = \frac{\pm a_2}{|a|}$$

which shows the max/min values of f on S are ± 1. Now if $b \in \mathbb{R}^2$, then $b/|b| \in S$ and

$$\left| a \cdot \frac{b}{|b|} \right| \leq 1,$$

so $|a \cdot b| \leq |b|$.