Week 5 Homework

1. Find the shortest distance from the point $(1, 0)$ to the parabola $y^2 = 4x$.

2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable, and suppose $g(a, b) = 0$, $D_2 g(a, b) \neq 0$. Then by the implicit function theorem, there is a rectangular neighborhood[1] $Q = U \times V$ of (a, b) and a continuously differentiable function $h : U \to \mathbb{R}$ such that

$$Q \cap \{(x, y) \,:\, y = h(x)\} = Q \cap \{(x, y) \,:\, g(x, y) = 0\},$$

that is, the graph of h agrees with the zero set of g inside Q. Use this to show there is an open set $W \subset \mathbb{R}$ containing 0 and a differentiable curve $\phi : W \to \mathbb{R}^2$ such that

$$\begin{aligned} &\phi(0) = (a, b) \\ &\phi'(0) \neq (0, 0) \\ &\phi(W) = Q \cap \{(x, y) \,:\, g(x, y) = 0\}, \end{aligned}$$

that is, ϕ traces out the zero set of g in a neighborhood of (a, b).

3. Let C be the curve $g(x, y) = 0$ (with $g : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable), and suppose $\nabla g \neq 0$ at each point of C. Take a point p not on C, and let q be the point of C closest[2] to p. Show that the line through p and q is orthogonal to C at q.

4. Using Lagrange multipliers λ to maximize/minimize a quadratic form

$$q(x, y) = ax^2 + 2bxy + cy^2$$

on the unit circle $x^2 + y^2 = 1$, we obtain the equations

$$ax + by = \lambda x \tag{1}$$
$$bx + cy = \lambda y \tag{2}$$

and solutions (x_i, y_i, λ_i), $i = 1, 2$. If $\lambda_1 \neq \lambda_2$, show that (x_1, y_1) and (x_2, y_2) are orthogonal.

(Hint: First, substitute (x_1, y_1, λ_1) into equations (1) and (2), multiply the equations by x_2 and y_2, respectively, then add. Next, substitute (x_2, y_2, λ_2) into (1) and (2), multiply by x_1 and y_1, respectively, then add. Finally subtract the two results.)

5. Provide charts to prove that the unit sphere $S^1 = \{(x, y) \,:\, x^2 + y^2 = 1\}$ is a 1-dimensional manifold.

[1] I.e., U and V are open intervals containing a and b, respectively.
[2] This point is guaranteed to exist by HW3 #1.