Week 5 Homework

1. Find the shortest distance from the point $(1, 0)$ to the parabola $y^2 = 4x$.

Solution. The graph of $y^2 = 4x$ is a closed set so there is a minimum distance. Let $f(x, y) = (x-1)^2 + y^2$ and $g(x, y) = y^2 - 4x$; we must minimize f on the zero set of g. If (x, y) is such a minimum then $\nabla f(x, y) = \lambda \nabla g(x, y)$ and so

$$\begin{aligned} 2(x-1) &= -4\lambda x \\ 2y &= 2\lambda y \end{aligned}$$

The second equation implies $\lambda = 1$ unless $y = 0$. But if $\lambda = 1$, then from the first equation $x = -1$, which is impossible because $y^2 - 4x = 0$. So $y = 0$ which implies $x = 0$. Thus $(0, 0)$ is the only critical point of f on the zero set of g, so f must attain an absolute minimum at $(0, 0)$. The shortest distance from $(1, 0)$ to $y^2 = 4x$ is then $f(0, 0) = 1$.

2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable, and suppose $g(a, b) = 0$, $D_2 g(a, b) \neq 0$. Then by the implicit function theorem, there is a rectangular neighborhood[1] $Q = U \times V$ of (a, b) and a continuously differentiable function $h : U \to \mathbb{R}$ such that

$$Q \cap \{(x, y) \,:\, y = h(x)\} = Q \cap \{(x, y) \,:\, g(x, y) = 0\},$$

that is, the graph of h agrees with the zero set of g inside Q. Use this to show there is an open set $W \subset \mathbb{R}$ containing 0 and a differentiable curve $\phi : W \to \mathbb{R}^2$ such that

$$\begin{aligned} &\phi(0) = (a, b) \\ &\phi'(0) \neq (0, 0) \\ &\phi(W) = Q \cap \{(x, y) \,:\, g(x, y) = 0\}, \end{aligned} \tag{1}$$

that is, ϕ traces out the zero set of g in a neighborhood of (a, b).

Solution. Let

$$\phi(t) = (t + a, h(t + a))$$

Then $\phi(0) = (a, h(a)) = (a, b)$ and $\phi'(0) = (1, 0) \neq (0, 0)$. To satisfy the last condition of (1), let

$$W = \{t \in \mathbb{R} \,:\, t + a \in U \cap h^{-1}(V)\}.$$

3. Let C be the curve $g(x, y) = 0$ (with $g : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable) and suppose $\nabla g \neq 0$ at each point of C. Take a point p not on C, and let q be the point of C closest to p. Show that the line through p and q is orthogonal to C at q.

[1] I.e., U and V are open intervals containing a and b, respectively.

Solution. By Problem 2, there is[2] a differentiable function $\phi : U \to C$, with $U \subset \mathbb{R}$ open, such that $\phi(0) = q$ and $\phi'(0) \neq 0$. (That is, ϕ traces out C in a neighborhood of q, with nonzero velocity as it passes through q.)

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = |x - p|^2$. By assumption, $f|_C$ has a minimum at q. Thus $\psi \equiv f \circ \phi$ has a minimum at $t = 0$, so

$$0 = \psi'(0) = (f \circ \phi)'(0) = \nabla f(q) \cdot \phi'(0),$$

which shows $\nabla f(q)$ is orthogonal to $\phi'(0)$, or equivalently $\nabla f(q)$ is orthogonal to C at q. But observe that

$$\nabla f(q) = 2(q - p)$$

so $q - p$ is orthogonal to C at q, as desired.

4. Using Lagrange multipliers λ to maximize/minimize a quadratic form

$$q(x, y) = ax^2 + 2bxy + cy^2$$

on the unit circle $x^2 + y^2 = 1$, we obtain the equations

$$ax + by = \lambda x \tag{2}$$
$$bx + cy = \lambda y \tag{3}$$

and solutions (x_i, y_i, λ_i), $i = 1, 2$. If $\lambda_1 \neq \lambda_2$, show that (x_1, y_1) and (x_2, y_2) are orthogonal.

(Hint: First, substitute (x_1, y_1, λ_1) into equations (2) and (3), multiply the equations by x_2 and y_2, respectively, then add. Next, substitute (x_2, y_2, λ_2) into (2) and (3), multiply by x_1 and y_1, respectively, then add. Finally subtract the two results.)

Solution. Following the hint we have

$$ax_1x_2 + by_1x_2 = \lambda_1 x_1 x_2$$
$$bx_1y_2 + cy_1y_2 = \lambda_1 y_1 y_2$$

and

$$ax_2x_1 + by_2x_1 = \lambda_2 x_2 x_1$$
$$bx_2y_1 + cy_2y_1 = \lambda_2 y_2 y_1$$

Adding the first two equations and subtracting the second two, we get

$$(\lambda_1 - \lambda_2)(x_1x_2 + y_1y_2) = 0,$$

so since $\lambda_1 \neq \lambda_2$,

$$(x_1, y_1) \cdot (x_2, y_2) = 0.$$

[2]See also Theorem 4.1.

5. Provide charts to prove that the unit sphere $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ is a 1-dimensional manifold.

Solution. To prove $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ is a 1 dimensional manifold we must show that for each $a \in S^1$, there is an open set $U \subset R^2$ such that $U \cap S^1$ is a patch. Define charts $h_+ : (-1, 1) \to \mathbb{R}$ by $h_+(x) = \sqrt{1 - x^2}$ and $h_- : (-1, 1) \to \mathbb{R}$ by $h_-(x) = -\sqrt{1 - x^2}$. Define the following patches:

$$
\begin{aligned}
P_1 &= \{(x, y) : x \in (-1, 1),\ y = h_+(x)\} \\
P_2 &= \{(x, y) : x \in (-1, 1),\ y = h_-(x)\} \\
P_3 &= \{(x, y) : y \in (-1, 1),\ x = h_+(x)\} \\
P_4 &= \{(x, y) : y \in (-1, 1),\ x = h_-(x)\}.
\end{aligned}
$$

Geometrically, P_1, P_2, P_3 and P_4 are the upper, lower, right, and left (open) half-circles. Consider the open half-planes

$$
\begin{aligned}
U_1 &= \{(x, y) : y > 0\} \\
U_2 &= \{(x, y) : y < 0\} \\
U_3 &= \{(x, y) : x > 0\} \\
U_4 &= \{(x, y) : x < 0\}
\end{aligned}
$$

If $(a, b) \in S^1$, then either $a \neq 0$ or $b \neq 0$. If $b > 0$ then $(a, b) \in U_1 \cap S^1 = P_1$. If $b < 0$ then $(a, b) \in U_2 \cap S^1 = P_2$. If $a > 0$ then $(a, b) \in U_3 \cap S^1 = P_3$. If $a < 0$ then $(a, b) \in U_4 \cap S^1 = P_4$. This completes the proof.