Week 7 Homework

1. Show that the maximum value of $f(x) = x_1^2 x_2^2 \dots x_n^2$ on the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $(1/n)^n$. Use this to prove the arithmetic/geometric mean inequality:

$$\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \dots + a_n}{n}$$

for positive real numbers $a_1, a_2, \dots, a_n$.

2. The planes $x + 2y + z = 4$ and $3x + y + 2z = 3$ intersect in a straight line L. Find the point of L which is closest to the origin.

3. Suppose $G : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable, and let

$$M = \{x \in \mathbb{R}^n : G(x) = 0,\ \nabla G_1(x), \dots, \nabla G_m(x) \text{ are linearly independent}\}$$

and

$$T_a = \{\phi'(0) : \phi : \mathbb{R} \to M \text{ is differentiable and } \phi(0) = a\}.$$

Then M is a manifold and $a + T_a$ is the tangent plane to M at a. Show that $\nabla G_1(a), \dots, \nabla G_m(a)$ are orthogonal to T_a.

4. Find the maximal volume of a closed rectangular box whose total surface area is 54.

5. Suppose $f^{(k+1)}$, $g^{(k+1)}$ exist and are continuous in a neighborhood of a. Assume also that $f^{(m)}(a) = g^{(m)}(a) = 0$ for $m = 0, 1, \dots, k-1$ and $g^{(k)}(a) \neq 0$. Use Taylor's theorem to show that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}.$$

6. Example 4 of the textbook shows that if $f : \mathbb{R} \to \mathbb{R}$ is such that $f''(x) - f(x) = 0$ for all x and $f(0) = f'(0) = 0$, then f is identically zero. Use this to show that if $h : \mathbb{R} \to \mathbb{R}$ satisfies $h''(x) = h(x)$ for all x, then $h(x) = ae^x + be^{-x}$ for some $a, b \in \mathbb{R}$.

Hint: Let $f(x) = h(x) - ae^x - be^{-x}$, and choose a, b such that $f(0) = f'(0) = 0$.