Week 7 Homework

1. Show that the maximum value of $f(x) = x_1^2 x_2^2 \ldots x_n^2$ on the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $(1/n)^n$. Use this to prove the arithmetic/geometric mean inequality:

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}$$

for positive real numbers $a_1, a_2, \ldots, a_n$.

Solution. S^{n-1} is the zero set of $g(x) = |x| - 1 = x_1^2 + \ldots + x_n^2 - 1$. Note that $\nabla f(x) = \lambda \nabla g(x)$ implies either $x_i = 0$ for some i, or $x_i \neq 0$ for all i and

$$x_1^2 \ldots \hat{x}_i^2 \ldots x_n^2 = \lambda \tag{1}$$

for $i = 1, \ldots, n$, where the notation $\hat{x}_i^2$ means that term is missing from the product. In the former case we have $f(x) = 0$ which is not a maximum. In the latter case, dividing the ith equation of the form (1) by the jth equation gives

$$x_j^2 / x_i^2 = 1$$

so that $x_i = \pm x_j$. Thus $x_1 = \pm x_2 = \ldots = \pm x_n$ which shows the maximum value of f is attained when, for example, $x_1 = \ldots = x_n = 1/\sqrt{n}$, in which case $f(x) = (1/n)^n$. Now given positive numbers $a_1, \ldots, a_n$, let $x_i = \sqrt{a_i/(a_1 + \ldots + a_n)}$ for $i = 1, \ldots, n$. Then $|x| = 1$, so by the above

$$f(x) = \frac{a_1 \ldots a_n}{(a_1 + \ldots + a_n)^n} \leq \frac{1}{n^n}$$

Taking nth roots and multiplying by $a_1 + \ldots + a_n$ gives the result.

2. The planes $x + 2y + z = 4$ and $3x + y + 2z = 3$ intersect in a straight line L. Find the point of L which is closest to the origin.

Solution. We want to minimize $f(x, y, z) = x^2 + y^2 + z^2$ on L. Using Lagrange multipliers we get

$$\begin{aligned} 2x &= \lambda_1 + 3\lambda_2 \\ 2y &= 2\lambda_1 + \lambda_2 \\ 2z &= \lambda_1 + 2\lambda_2 \end{aligned}$$

Eliminating λ_1 and λ_2, and using the two constraint equations, we obtain

$$\begin{aligned} 3x + y - 5z &= 0 \\ x + 2y + z &= 4 \\ 3x + y + 2z &= 3 \end{aligned}$$

Solving, we get $x = 1/7$, $y = 12/7$, $z = 3/7$. This is the only stationary point, and from the geometric picture it is clear this gives the minimum distance.

3. Suppose $G : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable, and let

$$M = \{x \in \mathbb{R}^n \,:\, G(x) = 0,\ \nabla G_1(x), \dots, \nabla G_m(x) \text{ are linearly independent}\}$$

and

$$T_a = \{\phi'(0) \,:\, \phi : \mathbb{R} \to M \text{ is differentiable and } \phi(0) = a\}.$$

Then M is a manifold and $a+T_a$ is the tangent plane to M at a. Show that $\nabla G_1(a), \dots, \nabla G_m(a)$ are orthogonal to T_a.

Solution. Let $\phi : \mathbb{R} \to M$ be differentiable with $\phi(0) = a$. It suffices to show that $\nabla G_1(a), \dots, \nabla G_m(a)$ are all orthogonal to $\phi'(0)$. Note that $G \circ \phi \equiv 0$ by definition of M, so

$$0 = (G \circ \phi)'(0) = G'(a)\phi'(0) = \begin{pmatrix} \nabla G_1(a) \\ \vdots \\ \nabla G_m(a) \end{pmatrix} \phi'(0)$$

This shows that $\nabla G_1(a) \cdot \phi'(0) = 0, \dots, \nabla G_m(a) \cdot \phi'(0) = 0$, as required.

4. Find the maximal volume of a closed rectangular box whose total surface area is 54.

Solution. We want to maximize $f(x, y, z) = xyz$ on S, where S is the zero set of $g(x, y, z) = 2(xy + yz + xz) - 54$. Using Lagrange multipliers we get

$$\begin{aligned} yz &= \lambda(y + z) \\ xz &= \lambda(x + z) \\ xy &= \lambda(x + y) \end{aligned} \tag{2}$$

Subtracting the second equation in (2) from the first, we get $z(y - x) = \lambda(y - x)$, so either $\lambda = z$ or $y = x$. In the former case, putting $\lambda = z$ into the second equation in (2) gives $\lambda = 0$ which means $x = y = z = 0$ and this is clearly not our maximum. So we must have $y = x$. Arguing similarly on the last two equations in (2) gives $y = z$, so we have $x = y = z$ as the only nontrivial stationary point. It is clear this maximizes f; for example we can imagine doing the maximization in the intersection of S with the first octant, which is a compact set, on the boundary of which f is identically zero.

5. Suppose $f^{(k+1)}$, $g^{(k+1)}$ exist and are continuous in a neighborhood of a. Assume also that $f^{(m)}(a) = g^{(m)}(a) = 0$ for $m = 0, 1, \dots, k - 1$ and $g^{(k)}(a) \neq 0$. Use Taylor's theorem to show that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}.$$

Solution. By Taylor's theorem

$$f(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(\tau)}{(k+1)!}(x-a)^{k+1}$$
$$g(x) = \frac{g^{(k)}(a)}{k!}(x-a)^k + \frac{g^{(k+1)}(\gamma)}{(k+1)!}(x-a)^{k+1}$$

where τ and γ are between a and x. Dividing both equations by $(x-a)^k$ gives

$$\frac{f(x)}{(x-a)^k} = \frac{f^{(k)}(a)}{k!} + \frac{f^{(k+1)}(\tau)}{(k+1)!}(x-a)$$
$$\frac{g(x)}{(x-a)^k} = \frac{g^{(k)}(a)}{k!} + \frac{g^{(k+1)}(\gamma)}{(k+1)!}(x-a)$$

Taking limits as $x \to a$, and using continuity of $f^{(k+1)}$, $g^{(k+1)}$ at a gives

$$\lim_{x\to a} \frac{f(x)}{(x-a)^k} = \frac{f^{(k)}(a)}{k!}$$
$$\lim_{x\to a} \frac{g(x)}{(x-a)^k} = \frac{g^{(k)}(a)}{k!}$$

It follows that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}.$$

6. Example 4 of the textbook shows that if $f : \mathbb{R} \to \mathbb{R}$ is such that $f''(x) - f(x) = 0$ for all x and $f(0) = f'(0) = 0$, then f is identically zero. Use this to show that if $h : \mathbb{R} \to \mathbb{R}$ satisfies $h''(x) = h(x)$ for all x, then $h(x) = ae^x + be^{-x}$ for some $a, b \in \mathbb{R}$.

Hint: Let $f(x) = h(x) - ae^x - be^{-x}$, and choose a, b such that $f(0) = f'(0) = 0$.

Solution. Assume $h''(x) = h(x)$ for all x, and let $f(x) = h(x) - ae^x - be^{-x}$ where

$$a = \frac{h(0) + h'(0)}{2}$$
$$b = \frac{h(0) - h'(0)}{2}.$$

Then $f''(x) = f(x)$ and $f(0) = f'(0) = 0$, so $f \equiv 0$. Thus, $h(x) = ae^x + be^{-x}$ as desired.