Week 8 Homework

1. Consider the function

$$f(x) = \left(e^{-x} - 1\right)\left(\tan^{-1}(x) - x\right).$$

Show that the 4th order Taylor expansion of f around 0 is

$$f(x) = \frac{1}{3}x^4 + R_4(x).$$

(Hint: First compute the Taylor expansions of e^{-x} and $\tan^{-1}$ around zero.) Observe that 0 is a critical point of f, and show also that $\lim_{x\to 0} R_4(x)/x^4 = 0$. What kind of critical point is 0?

2. Let $f(x) = (x_1 + \ldots + x_n)^k$.

(a) Show that $D_1^{j_1} \ldots D_n^{j_n} f(x) = k!$ if $j_1 + \ldots + j_n = k$.
(b) Show that if $i_1 + \ldots + i_n = j_1 + \ldots + j_n = k$, then

$$D_1^{j_1} \ldots D_n^{j_n} x_1^{i_1} \ldots x_n^{i_n} = \begin{cases} j_1! \ldots j_n!, & \text{if } i_1 = j_1, \ldots, i_n = j_n \\ 0, & \text{else} \end{cases}$$

(c) Conclude that

$$\binom{k}{j_1 \ldots j_n} = \frac{k!}{j_1! \ldots j_n!}.$$

Recall that $\binom{k}{j_1 \ldots j_n}$ is defined by the condition

$$(x_1 + \ldots + x_n)^k = \sum_{j_1+\ldots+j_n=k} \binom{k}{j_1 \ldots j_n} x_1^{j_1} \ldots x_n^{j_n}.$$

3. Find the third order Taylor polynomial of the following functions at the points given:

(a) $f(x, y) = (x + y)^3$ at $(1, 1)$.
(b) $f(x, y, z) = xy^2z^3$ at $(1, 0, -1)$.

4. Classify the critical point $(-1, \pi/2, 0)$ of $f(x, y, z) = x \sin z + z \sin y$.