Week 8 Homework

1. Consider the function

$$f(x) = \left(e^{-x} - 1\right)\left(\tan^{-1}(x) - x\right).$$

Show that the 4th order Taylor expansion of f around 0 is

$$f(x) = \frac{1}{3}x^4 + R_4(x).$$

(Hint: First compute the Taylor expansions of e^{-x} and $\tan^{-1}$ around zero.) Observe that 0 is a critical point of f, and show also that $\lim_{x\to 0} R_4(x)/x^4 = 0$. What kind of critical point is 0?

Solution. Observe that

$$\begin{aligned} e^{-x} &= 1 - x + \bar{R}_1(x) \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \tilde{R}_3(x) \end{aligned}$$

where

$$\begin{aligned} \lim_{x\to 0} \frac{\bar{R}_1(x)}{x} &= 0 \\ \lim_{x\to 0} \frac{\tilde{R}_3(x)}{x^3} &= 0 \end{aligned}$$

Thus,

$$f(x) = \left(e^{-x} - 1\right)\left(\tan^{-1}(x) - x\right) = \frac{1}{3}x^4 + R(x)$$

where

$$R(x) = -\frac{1}{3}x^3\bar{R}_1(x) - x\tilde{R}_3(x) + \bar{R}_1(x)\tilde{R}_3(x).$$

Observe that $\lim_{x\to 0} R(x)/x^4 = 0$, which shows $x^4/3$ is the 4th order Taylor expansion of f at 0. It follows that $f'(0) = 0$, and since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x^4} = \lim_{x\to 0} \frac{f(x)}{x^4} = \frac{1}{3} > 0$$

we see that $f(x) > f(0)$ for $x \neq 0$ in a neighborhood of 0. Thus f has a local minimum at 0.

2. Let $f(x) = (x_1 + \ldots + x_n)^k$.

(a) Show that $D_1^{j_1} \ldots D_n^{j_n} f(x) = k!$ if $j_1 + \ldots + j_n = k$.

(b) Show that if $i_1 + \ldots + i_n = j_1 + \ldots + j_n = k$, then

$$D_1^{j_1} \ldots D_n^{j_n} x_1^{i_1} \ldots x_n^{i_n} = \begin{cases} j_1! \ldots j_n!, & \text{if } i_1 = j_1, \ldots, i_n = j_n \\ 0, & \text{else} \end{cases}$$

(c) Conclude that

$$\binom{k}{j_1 \ldots j_n} = \frac{k!}{j_1! \ldots j_n!}.$$

Recall that $\binom{k}{j_1 \ldots j_n}$ is defined by the condition

$$(x_1 + \ldots + x_n)^k = \sum_{j_1+\ldots+j_n=k} \binom{k}{j_1 \ldots j_n} x_1^{j_1} \ldots x_n^{j_n}. \tag{1}$$

Solution. Note that, for any $i \in \{1, \ldots, n\}$ and any $j \in \mathbb{N}$,

$$D_i(x_1 + \ldots + x_n)^j = j(x_1 + \ldots + x_n)^{j-1}.$$

To see (a), note that if $j_1 + \ldots + j_n = k$ then

$$D_1^{j_1} \ldots D_n^{j_n} f(x) = D_1^{j_1+\ldots+j_n} f(x) = D_1^k f(x) = k!$$

Now assume $i_1 + \ldots + i_n = j_1 + \ldots + j_n = k$. If $i_1 = j_1, \ldots, i_n = j_n$ then

$$D_1^{j_1} \ldots D_n^{j_n} x_1^{i_1} \ldots x_n^{i_n} = (D_1^{j_1} x_1^{i_1}) \ldots (D_n^{j_n} x_n^{i_n}) = j_1! \ldots j_n!$$

Otherwise $j_k > i_k$ for some k, in which case

$$D_1^{j_1} \ldots D_n^{j_n} x_1^{i_1} \ldots x_n^{i_n} = (D_1^{j_1} x_1^{i_1}) \ldots (D_k^{j_k} x_k^{i_k}) \ldots (D_n^{j_n} x_n^{i_n}) = 0.$$

This proves (b). Now consider (c). Assume $j_1 + \ldots + j_n = k$, and apply $D_1^{j_1} \ldots D_n^{j_n}$ to both sides of equation (1). We obtain

$$\begin{aligned} k! &= D_1^{j_1} \ldots D_n^{j_n} \sum_{i_1+\ldots+i_n=k} \binom{k}{i_1 \ldots i_n} x_1^{i_1} \ldots x_n^{i_n} \\ &= \sum_{i_1+\ldots+i_n=k} \binom{k}{i_1 \ldots i_n} D_1^{j_1} \ldots D_n^{j_n} x_1^{i_1} \ldots x_n^{i_n} \\ &= \binom{k}{j_1 \ldots j_n} j_1! \ldots j_n! \end{aligned}$$

This proves (c).

3. Find the third order Taylor polynomial of the following functions at the points given:

(a) $f(x, y) = (x + y)^3$ at $(1, 1)$.
(b) $f(x, y, z) = xy^2z^3$ at $(1, 0, -1)$

Solution. (a) The desired expansion is

$$f(a + h) = f(a) + D_h f(a) + \frac{1}{2} D_h^2 f(a) + \frac{1}{6} D_h^3 f(a) + R_3(h).$$

with $a = (1,1)$ and $h = (h_1, h_2) = (x-1, y-1)$. Computing

$$\begin{aligned}
D_h f(x,y) &= h_1 D_1 f(x,y) + h_2 D_2 f(x,y) \\
D_h^2 f(x,y) &= h_1^2 D_1^2 f(x,y) + 2h_1 h_2 D_1 D_2 f(x,y) + h_2^2 D_2^2 f(x,y) \\
D_h^3 f(x,y) &= h_1^3 D_1^3 f(x,y) + 3h_1^2 h_2 D_1^2 D_2 f(x,y) + 3h_1 h_2^2 D_1 D_2^2 f(x,y) + h_2^3 D_2^3 f(x,y)
\end{aligned}$$

we get

$$\begin{aligned}
D_h f(x,y) &= 3(h_1 + h_2)(x+y)^2 \\
D_h^2 f(x,y) &= 6(h_1^2 + 2h_1 h_2 + h_2^2)(x+y) \\
D_h^3 f(x,y) &= 6(h_1^3 + 3h_1^2 h_2 + 3h_1 h_2^2 + h_2^3).
\end{aligned}$$

Higher order derivatives are all zero so $R_3(h) \equiv 0$. Using $(x,y) = a = (1,1)$,

$$\begin{aligned}
D_h f(a) &= 12(h_1 + h_2) \\
D_h^2 f(a) &= 12(h_1^2 + 2h_1 h_2 + h_2^2) \\
D_h^3 f(a) &= 6(h_1^3 + 3h_1^2 h_2 + 3h_1 h_2^2 + h_2^3).
\end{aligned}$$

Thus, the third order Taylor polynomial of f at a is

$$\begin{aligned}
&12(x-1) + 12(y-1) + 6(x-1)^2 + 12(x-1)(y-1) + 6(y-1)^2 \\
&+ (x-1)^3 + 3(x-1)^2(y-1) + 3(x-1)(y-1)^2 + (y-1)^3.
\end{aligned}$$

(b) We could use the same procedure as in (a), but instead we note that since f is polynomial we may simply write it in powers of $(x-1)$, y, and $(z+1)$ to obtain the appropriate Taylor polynomial. Thus,

$$xy^2z^3 = y^2z^3 + (x-1)y^2z^3$$

and since

$$\begin{aligned}
z^3 &= (z+1)^3 - 3z^2 - 3z - 1 \\
&= (z+1)^3 - 3(z+1)^2 + 3z + 2 \\
&= -1 + 3(z-1) - 3(z-1)^2 + (z-1)^3
\end{aligned}$$

we obtain

$$\begin{aligned}
xy^2z^3 &= (x-1)y^2z^3 + y^2z^3 \\
&= -y^2 + 3y^2(z-1) - 3y^2(z-1)^2 + y^2(z-1)^3 \\
&- (x-1)y^2 + 3(x-1)y^2(z-1) - 3(x-1)y^2(z-1)^2 + (x-1)y^2(z-1)^3.
\end{aligned}$$

The last expression is the 4th order Taylor polynomial for f at $a = (1,0,1)$.

4. Classify the critical point $(-1, \pi/2, 0)$ of $f(x,y,z) = x\sin z + z\sin y$.

Solution. The Hessian matrix is

$$H(x,y,z) = \begin{pmatrix} 0 & 0 & \cos z \\ 0 & -z\sin y & \cos y \\ \cos z & \cos y & -x\sin z \end{pmatrix}$$

At $(-1,\pi/2,0)$ this is

$$H\left(-1,\frac{\pi}{2},0\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which is degenerate, thus giving no information about the character of the critical point. We must understand the character of the point by other means. One way is to fix two of the coordinates and let the third vary. Let $x = -1$ and $y = \pi/2$; this gives the function $g(z) = z - \sin z$. It is easily seen that g has neither a local maximum nor a local minimum at 0; this fact can be deduced from its 3rd order Taylor expansion

$$g(z) = \frac{1}{6}z^3 + R(z).$$

It follows that f has neither a local maximum nor minimum at $(-1,\pi/2,0)$. For more detailed information one can fix any other two coordinates and let the third vary. One can also see this behavior via Taylor expansions of f: the 3rd order Taylor expansion of f at $a = (-1,\pi/2,0)$ is

$$f(x,y,z) = (x+1)z - \frac{1}{2}(y-\pi/2)^2 z + \frac{1}{6}z^3 + R_3(h)$$

where $h = (x+1, y-\pi/2, z)$. Fixing $x = -1$ and $y = \pi/2$, one sees that

$$f(-1,\pi/2,z) = \frac{1}{6}z^3 + R_3(0,0,z)$$

Dividing both sides by z^3 and taking a limit as $z \to 0$, and using the fact that $\lim_{z\to 0} R_3(0,0,z)/z^3 = 0$, one concludes that f takes both positive and negative values in a neighborhood of a.