Math 4604 notation.

In the below let $D \subset \mathbb{R}^n$ be open and $F : D \to \mathbb{R}^m$ be differentiable at a.

Notation.

(1) We define *directional derivatives* $D_v F(a)$ of F at a in the direction of v by

$$D_v F(a) = \lim_{t\to 0} \frac{F(a+tv) - F(a)}{t}.$$

(2) We define *partial derivatives* of F at a by

$$D_i F(a) \equiv D_{e_i} F(a) \equiv \frac{\partial F}{\partial x_i}(a).$$

(3) In case $m = 1$ (so F is real-valued) we define the *gradient vector*

$$\nabla F(a) = \begin{pmatrix} D_1 F(a) & \dots & D_n F(a) \end{pmatrix}.$$

(4) The *differential* $dF_a : \mathbb{R}^n \to \mathbb{R}^m$ of F at a is the (unique) linear map satisfying

$$\lim_{h\to 0} \frac{F(a+h) - F(a) - dF_a(h)}{|h|} = 0.$$

(5) $F'(a)$ is the $m \times n$ matrix of dF_a, called the *derivative matrix* of F at a, and

$$F'(a) = \begin{pmatrix} D_1 F_1(a) & \dots & D_n F_1(a) \\ \vdots & & \vdots \\ D_1 F_m(a) & \dots & D_n F_m(a) \end{pmatrix}$$

We write $F'(a) = (D_j F_i(a))$ for shorthand. Here $F_1, \dots, F_m$ are the component functions of F.

Relationships.

$$\begin{aligned}
&dF_a(h) = F'(a)h = D_h F(a) = h_1 D_1 F(a) + \ldots + h_n D_n F(a) \\
&F'(a) = \nabla F(a)^t = \begin{pmatrix} D_1 F(a) & \dots & D_n F(a) \end{pmatrix} \quad \text{if } m = 1 \\
&F'(a) = D_1 F(a) = \begin{pmatrix} D_1 F_1(a) \\ \vdots \\ D_1 F_m(a) \end{pmatrix} \quad \text{if } n = 1
\end{aligned}$$