# Ramagrog

## Eigenvalues and eigenvectors

# The Dot Product

If $v := (a_1, \ldots, a_n)$ and $w := (b_1, \ldots, b_n)$ are $n$tuples then the dot product of $v$ and $w$, written $\boxed{v \cdot w}$, is defined by:

$$v \cdot w := a_1 b_1 + \cdots + a_n b_n.$$

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*E.g.*: $$(2,3,4) \cdot (5,6,7) = 2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 \\ = 10 + 18 + 28 = 56$$

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Note: $n$tuples are often called vectors.

SKILL: Vector dot product

$$(a_1, \cdots, a_n) \cdot (b_1, \cdots, b_n) := a_1 b_1 + \cdots + a_n b_n.$$

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Game: I pick and tell you an integer $n > 0$.

I pick a secret vector $u \in \mathbb{R}^n$.

Your goal is to find $u$.

You pick and tell me finite sequence

$$v_1, \ldots, v_p \in \mathbb{R}^n$$

and I tell you $u \cdot v_1, \ldots, u \cdot v_p$.

How can you figure out $u$?

I'm thinking of a secret vector $u = (a, b, c, d) \in \mathbb{R}^4$.

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Then $u \cdot (w, x, y, z)$ is equal to

$$aw + bx + cy + dz$$

PLAY GAME . . .

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Then $u \cdot (1, 0, 0, 0) = a$, so you can find the first entry of $u$. The other three entries can be found by asking for

$$u \cdot (0, 1, 0, 0),\ u \cdot (0, 0, 1, 0),\ u \cdot (0, 0, 0, 1).$$

# Linear Relationships

Suppose we regularly measure six quantities

$$u,\ v,\ w,\ x,\ y,\ z.$$

If, after every measurement, we find that

$$y = 2u + 5v + 3w - 9x$$
$$z = 7u - 6v + 4w - 5x,$$

then we can simplify our work.

# Linear Relationships

Suppose we regularly measure six quantities

$$u,\ v,\ w,\ x,\ y,\ z.$$

We say that

$y$ and $z$ **depend linearly** on $u$, $v$, $w$ and $x$

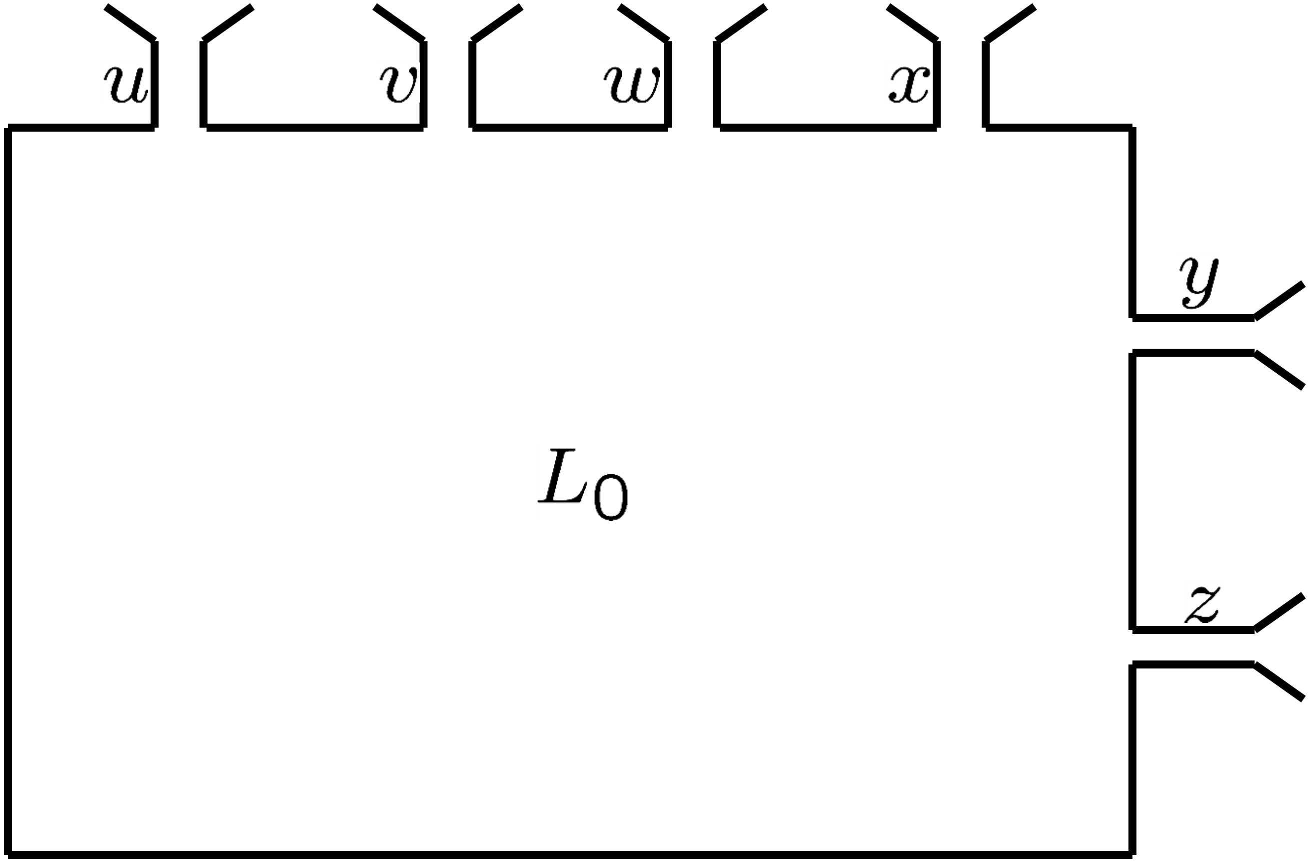
if there are constants

$$a,\ b,\ c,\ d,$$
$$e,\ f,\ g,\ h$$

such that, whenever we do a measurement, we find that

$$\begin{aligned} y &= au + bv + cw + dx \\ z &= eu + fv + gw + hx. \end{aligned}$$

*E.g.*, $y = 2u + 5v + 3w - 9x$
$z = 7u - 6v + 4w - 5x$



$L_0 : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by

$$L_0(u,v,w,x) = (2u + 5v + 3w - 9x, \\ 7u - 6v + 4w - 5x)$$

*E.g.*, $$\begin{aligned} y &= 2u + 5v + 3w - 9x \\ z &= 7u - 6v + 4w - 5x \end{aligned}$$

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Definition: A **matrix** is a two-dimensional rectangular array of numbers.

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**Definition**: A **matrix** is a two-dimensional rectangular array of numbers.

---

*E.g.*, $$M_0 := \begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \end{bmatrix}$$

Dimensions of $M_0$: $2 \times 4$

---

**Key point**: To any $k \times n$ matrix $M$, there is a function $L_M : \mathbb{R}^n \to \mathbb{R}^k$.

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*E.g.*, $$L_{M_0} = L_0$$

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BOTTOM
OF P. 1

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$$L_0(u, v, w, x) = (2u + 5v + 3w - 9x,\ 7u - 6v + 4w - 5x)$$

$$//$$

$$L_{M_0}(u, v, w, x) = ((2, 5, 3, -9) \cdot (u, v, w, x),\ (7, -6, 4, -5) \cdot (u, v, w, x))$$

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$$M_0 := \begin{bmatrix} 2 & 5 & 3 & -9 \\ 7 & -6 & 4 & -5 \end{bmatrix}$$

$$L_{M_0}(u,v,w,x) = ((2,5,3,-9)\cdot(u,v,w,x), \\ (7,-6,4,-5)\cdot(u,v,w,x))$$

$$L_{M_0}(u,v,w,x) \\ = ((2,5,3,-9)\cdot(u,v,w,x),$$

To get the $i$th entry in $L_{M_0}(u,v,w,x)$,

dot the $i$th row of $M_0$ with $(u,v,w,x)$

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To get the $i$th entry in $L_{M_0}(u,v,w,x)$,
dot the $i$th row of $M_0$ with $(u,v,w,x)$

---

$M \in \mathbb{R}^{k\times n}$, $p \in \mathbb{R}^n$

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To get the $i$th entry in $L_M(p)$,
dot the $i$th row of $M$ with $p$

**SKILL: Compute $L_M(p)$.**

Definition: $L_M$ is called the

**linear function corresponding to $M$.**

---

$M \in \mathbb{R}^{k \times n}$, $p \in \mathbb{R}^n$

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TOP OF P. 2

To get the $i$th entry in $L_M(p)$,
dot the $i$th row of $M$ with $p$

SKILL: Compute $L_M(p)$.

*E.g.*: I'm thinking of a secret 3 × 4 matrix

$$M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

---

Definition: $L_M$ is called the

**linear function corresponding to $M$.**

---

Game: I pick and tell you integers $k, n > 0$.

I pick a secret matrix $M \in \mathbb{R}^{k \times n}$.

Your goal is to find $M$.

You pick and tell me finite sequence

$$v_1, \ldots, v_p \in \mathbb{R}^n$$

and I tell you $L_M(v_1), \ldots, L_M(v_p)$.

How can you figure out $M$?

*E.g.*: I'm thinking of a secret 3 × 4 matrix

$$M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$$

---

Then $L_M(w, x, y, z)$ is equal to

$$\begin{aligned} (aw + bx + cy + dz,\\ ew + fx + gy + hz,\\ iw + jx + ky + lz) \end{aligned}$$

PLAY GAME . . .

---

Then $L_M(1, 0, 0, 0) = (a, e, i)$, so you can find the first column of $M$. The other three columns can be found by asking for $L_M(0, 1, 0, 0)$, $L_M(0, 0, 1, 0)$, $L_M(0, 0, 0, 1)$.

Key point to remember:
The entries in the $j$th column of $M$ are the same as the entries in

$$L_M(0,\ldots,0,\underset{\substack{\uparrow\\ \boxed{j\text{th entry}}}}{1},0,\ldots,0).$$

---

Then $L_M(w,x,y,z)$ is equal to

$$\begin{aligned}(aw &+ bx + cy + dz,\\ ew &+ fx + gy + hz,\\ iw &+ jx + ky + lz)\end{aligned}$$

PLAY GAME . . .

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Then $L_M(1,0,0,0) = (a,e,i)$, so you can find the first column of $M$. The other three columns can be found by asking for

$$L_M(0,1,0,0),\ L_M(0,0,1,0),\ L_M(0,0,0,1).$$

## Problem of general interest:

Given $A \in \mathbb{R}^{k\times n}$ and $B \in \mathbb{R}^{n\times q}$,

form $L_A : \mathbb{R}^n \to \mathbb{R}^k$ and $L_B : \mathbb{R}^q \to \mathbb{R}^n$

and then compose:

$$L_A \circ L_B : \mathbb{R}^q \to \mathbb{R}^k$$

$L_B$ then $L_A$

and try to find $C \in \mathbb{R}^{k\times q}$

such that $L_C : \mathbb{R}^q \to \mathbb{R}^k$ is equal to $L_A \circ L_B$.

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E.g.: $k = 2$, $n = 3$, $q = 4$,

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of $C$ "is" $L_C(1,0,0,0)$.

$L_C(1,0,0,0)$ is horizontal,
with commas and parentheses.

The first column of $C$ is vertical,
with no commas and no parentheses.

Same entries, though!

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

$$A = \begin{bmatrix} \boxed{4 \quad -2 \quad 1} \\ \boxed{0 \quad 5 \quad 9} \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of $C$ “is” $L_C(1,0,0,0)$.

$$\begin{array}{ccl} L_A(L_B(1,0,0,0)) & = & L_A((4,0,7)) \\ \| & & \qquad \| \\ L_C(1,0,0,0) & & ((4,0,7)\cdot(4,-2,1) \;, \\ & & \;(4,0,7)\cdot(0,5,9) \quad ) \end{array}$$

DOT PRODUCT
IS COMMUTATIVE

$$C = \begin{bmatrix} (4,0,7)\cdot(4,-2,1) & ??? & ??? & ??? \\ (4,0,7)\cdot(0,5,9) & ??? & ??? & ??? \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

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DOT PRODUCT
IS COMMUTATIVE

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ (0,5,9)\cdot(4,0,7) & ??? & ??? & ??? \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ \boxed{0 \quad\quad 5 \quad\quad 9} \end{bmatrix} \text{ and } B = \left[ \boxed{\begin{matrix} 4 \\ 0 \\ 7 \end{matrix}} \quad \begin{matrix} -2 & 1 & 2 \\ 5 & 9 & 1 \\ 3 & 8 & 2 \end{matrix} \right]$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of $C$ “is” $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ \boxed{\boxed{(0,5,9)} \cdot \boxed{(4,0,7)}} & ??? & ??? & ??? \end{bmatrix}$$

Note: The $\boxed{(2,1) \text{ entry of } C}$

is equal to

the $\boxed{\text{second row of } A}$

dotted against

the $\boxed{\text{first column of } B}$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of $C$ "is" $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} (4,-2,1)\cdot(4,0,7) & ??? & ??? & ??? \\ (0,5,9)\cdot(4,0,7) & ??? & ??? & ??? \end{bmatrix}$$

Note: The $(j,k)$ entry of $C$

is equal to

the $j$th row of $A$

dotted against

the $k$th column of $B$

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } \quad B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of $C$ “is” $L_C(1,0,0,0)$.

$$C = \begin{bmatrix} 23 & ??? & ??? & ??? \\ 63 & ??? & ??? & ??? \end{bmatrix}$$

Note: The $(j,k)$ entry of $C$

is equal to

the $j$th row of $A$

dotted against

the $k$th column of $B$

$$A = \begin{bmatrix} \boxed{4} & \boxed{-2} & \boxed{1} \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & \boxed{1} & 2 \\ 0 & 5 & \boxed{9} & 1 \\ 7 & 3 & \boxed{8} & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

Recall: The first column of $C$ "is" $L_C(1,0,0,0)$.

$$4 - 18 + 8 =$$

$$C = \begin{bmatrix} 23 & -15 & \boxed{-6} & 8 \\ 63 & 52 & 117 & 23 \end{bmatrix}$$

Note: The $(\overset{1}{j}, \overset{3}{k})$ entry of $C$

is equal to

$(4, -2, 1)$ the $j$th row of $A$ (1st)

dotted against

$(1, 9, 8)$ the $k$th column of $B$ (3rd)

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 5 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -2 & 1 & 2 \\ 0 & 5 & 9 & 1 \\ 7 & 3 & 8 & 2 \end{bmatrix}$$

Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

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Def’n: When $L_Z = L_X \circ L_Y$, we say that $Z$ is the **product of** $X$ **by** $Y$,

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Want: $C \in \mathbb{R}^{2\times 4}$ s.t. $L_C = L_A \circ L_B$.

$$\underline{AB = C} = \begin{bmatrix} 23 & -15 & -6 & 8 \\ 63 & 52 & 117 & 23 \end{bmatrix}$$

Def'n: When $L_Z = L_X \circ L_Y$, we say that $Z$ is the **product of** $X$ **by** $Y$, and we write $Z = \boxed{XY}$.

The $(j,k)$ entry of $XY$
is equal to
the $j$th row of $X$
dotted against
the $k$th column of $Y$

# Matrix multiplication

Matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second.

If $A$ is $p \times q$ and $B$ is $q \times r$, then $AB$ is $p \times r$.

Warning:

Matrix multiplication is not commutative:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\neq$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

# Matrix multiplication

Matrices can only be multiplied if the number of columns in the first matrix is the same as the number of rows in the second.

If $A$ is $p \times q$ and $B$ is $q \times r$, then $AB$ is $p \times r$.

Warning:

Matrix multiplication is not commutative.

However, it *is* associative:

$$\forall A \in \mathbb{R}^{p\times q},\ \forall B \in \mathbb{R}^{q\times r},\ \forall C \in \mathbb{R}^{r\times s},$$

$$(L_A \circ L_B) \circ L_C = L_A \circ (L_B \circ L_C),$$

so $(AB)C = A(BC)$.

Let's practice . . .

# Matrix multiplication

$$\boxed{3 \times 4} \longrightarrow \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 6 & 7 & 8 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 5 & 6 \\ -1 & -3 \\ 2 & 0 \end{bmatrix} \longleftarrow \boxed{4 \times 2}$$

$$\begin{bmatrix} 28 + 15 - 2 + 2 & 12 + 18 - 6 + 0 \\ 35 + 30 - 7 + 16 & 15 + 36 - 21 + 0 \\ 14 + 5 + 0 - 2 & 6 + 6 + 0 + 0 \end{bmatrix}$$

$$\boxed{3 \times 2} \longrightarrow \begin{bmatrix} 43 & 24 \\ 74 & 30 \\ 17 & 12 \end{bmatrix}$$

Let’s practice . . .

Next: The world of Ramagrog

SKILL:
Given two matrices, find their product.

*e.g.*:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & -2 & 4 & 6 & -3 \\ 0 & 9 & 2 & -8 & 3 \end{bmatrix} \begin{bmatrix} 6 & 7 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

Discussion:
Shape of answer??
First row of answer??
Last two columns??

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REFERENCE P. 3

# The world of Ramagrog

On Ramagrog, there live
Ramatins and Grogali.

Ramatins and Grogali take
one year to mature. No old age death!

Each spring, each Ramatin produces
14 Ramatins and 28 Grogali.
Each fall, each mature Grogalus eats
6 Ramatins and 11 Grogali.

Current population (just before production):
35 Ramatins and 79 Grogali.

Questions:
How many Ramatins and Grogali
one year from now?
and ten years from now?

Each spring, each Ramatin produces
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Question:
How many Ramatins and Grogali
one year from now?

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$$35 \text{ Ramatins} \longrightarrow \begin{cases} 15 \cdot 35 \text{ Ramatins}, \\ 28 \cdot 35 \text{ Grogali} \end{cases}$$

$$\underbrace{79}_{\text{mature}} \text{ Grogali} \longrightarrow \begin{cases} -\ 6 \cdot 79 \text{ Ramatins}, \\ -11 \cdot 79 \text{ Grogali} \end{cases}$$

$$35\text{ Ramatins} \longrightarrow \begin{cases} 15\cdot 35\text{ Ramatins},\\ 28\cdot 35\text{ Grogali}\end{cases}$$

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$$\left.\begin{array}{l}35\text{ Ramatins},\\ 79\text{ Grogali}\end{array}\right\}$$

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$$\left.\begin{array}{l} 35\ \text{Ramatins},\\ 79\ \text{Grogali} \end{array}\right\} \longrightarrow \begin{cases} 15\cdot 35 - \ \ 6\cdot 79\ \text{Ramatins},\\ 28\cdot 35 - 11\cdot 79\ \text{Grogali} \end{cases}$$

Exercise: Do the arithmetic.

$$\left.\begin{array}{l} r_n\ \text{Ramatins},\\ g_n\ \text{Grogali} \end{array}\right\} \longrightarrow \begin{cases} 15r_n - \ \ 6g_n & \text{Ramatins},\\ 28r_n - 11g_n & \text{Grogali} \end{cases}$$

$$\begin{aligned} r_{n+1} &= 15r_n - \ \ 6g_n\\ g_{n+1} &= 28r_n - 11g_n \end{aligned}$$

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$$B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$\begin{aligned} r_{n+1} &= 15r_n - 6g_n \\ g_{n+1} &= 28r_n - 11g_n \end{aligned}$$

$$\boxed{\begin{matrix} r_{n+1} \\ g_{n+1} \end{matrix}} \begin{matrix} = \\ = \end{matrix} \boxed{\begin{matrix} 15r_n - 6g_n \\ 28r_n - 11g_n \end{matrix}}$$

---

$$\boxed{p_{n+1}} := \begin{bmatrix} \boxed{\begin{matrix} r_{n+1} \\ g_{n+1} \end{matrix}} \end{bmatrix} \qquad B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$p_n := \begin{bmatrix} r_n \\ g_n \end{bmatrix} \qquad \boxed{Bp_n} = \begin{bmatrix} \boxed{\begin{matrix} 15r_n - 6g_n \\ 28r_n - 11g_n \end{matrix}} \end{bmatrix}$$

---

$$p_{n+1} = B\, p_n$$

$$p_{n+1} = B\,p_n \qquad B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \qquad p_n = \begin{bmatrix} r_n \\ g_n \end{bmatrix}$$

Current population
(just before production):

35 Ramatins and 79 Grogali.

$$B := \begin{bmatrix} & -6 \\ 28 & -11 \end{bmatrix}$$

$$p_n := \begin{bmatrix} r_n \\ g_n \end{bmatrix}$$

$$p_{n+1} = B\,p_n$$

$$p_{n+1} = B\, p_n$$

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$$p_n = \begin{bmatrix} r_n \\ g_n \end{bmatrix}$$

Current population
(just before production):
35 Ramatins and 79 Grogali.

$$p_0 = \begin{bmatrix} 35 \\ 79 \end{bmatrix}$$

Question: How many Ramatins and Grogali ten years from now?

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = p_{10} = Bp_9 = B^2 p_8 = B^3 p_7 = \cdots$$

$$= B^{10} p_0$$

Exercise: Compute $B^2$, $(B^2)^2$, $((B^2)^2)^2$.

Compute $B^{10} = [((B^2)^2)^2][B^2]$

Compute $p_{10} = B^{10} p_0$

Cultural note:

Ramatins and Grogali run in herds!

A big herd has 3 Ramatins and 7 Grogali.

A little herd has 1 Ramatin and 2 Grogali.

Current population (just before production):

35 Ramatins and 79 Grogali.

Cultural note:

Current population has

9 big herds and 8 little herds.

---

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$$\begin{array}{ccccccc} \begin{bmatrix}35\\79\end{bmatrix} & = & 9 & \underbrace{\begin{bmatrix}3\\7\end{bmatrix}} & + & 8 & \underbrace{\begin{bmatrix}1\\2\end{bmatrix}} \\ \Vert & & & \overset{\Vert}{\cdot\,\cdot} & & & \overset{\Vert}{\cdot\,\cdot} \\ p_0 & & & h^* & & & h_* \end{array}$$

---

$$p_0 = 9 \quad h^* \quad + \quad 8 \quad h_*$$

$$B := \begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix}$$

$$h^* = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$h_* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Cultural note:

A big herd begets a big herd.
A little herd begets three little herds.

Exercise: Do the arithmetic.

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$B\,h^* = h^*$$

$$\begin{bmatrix} 15 & -6 \\ 28 & -11 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Bh_* = 3\,h_*$$

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9 big herds and 8 little herds.

Cultural note:

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---

Population after 10 years has
9 big herds and $8 \cdot 3^{10}$ little herds.

---

$$\begin{bmatrix} r_{10} \\ g_{10} \end{bmatrix} = 9 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + 8 \cdot 3^{10} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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$$= \begin{bmatrix} 472,419 \\ 944,847 \end{bmatrix}$$

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$$= \begin{bmatrix} 472,419 \\ 944,847 \end{bmatrix}$$

| | | |
|---|---|---|
| $B\,h^* = h^*$ | “$B$ fixes $h^*$.” | “$h^*$ is an eigenvector for $B$ with eigenvalue 1.” |
| $Bh_* = 3\,h_*$ | “$B$ triples $h_*$.” | “$h_*$ is an eigenvector for $B$ with eigenvalue 3.” |

$$p_0 = 9 \quad h^* \quad + \quad 8 \quad h_*$$

“$p_0$ is a linear combination of $h^*$ and $h_*$ with coefficients 9 and 8.”

# The big idea of eigenvectors and eigenvalues:

If you want to apply a matrix many times to a one-column matrix (a.k.a. a "column vector"), the computation becomes much simpler if you can write the column vector as a linear combination of eigenvectors.

---

*e.g.*:

Computing $B^{10}p_0$ looks hard,
but becomes much easier if you note that

$$\begin{array}{cccccc} p_0 & = & 9\,h^* & + & 8\,h_* \\ & & B\,h^* = h^* & & Bh_* = 3\,h_* \end{array}$$

STOP