Paul Cusson’s question

The main results in this note are:

Theorem 30, due to T. Tao,

and Theorem 42, and Theorem 57.

DEFINITION 1. Let \( a, b \in \mathbb{R} \).
Then: \( (a; b) := \{ x \in \mathbb{R} \mid a < x < b \} \), \([a; b] := \{ x \in \mathbb{R} \mid a \leq x < b \} \), \([a; b] := \{ x \in \mathbb{R} \mid a < x \leq b \} \), \([a; b] := \{ x \in \mathbb{R} \mid a \leq x \leq b \} \).

DEFINITION 2. Let \( f \) be a function.
Then \( \mathbb{D}_f \) denotes the domain of \( f \).
Also, \( \mathbb{I}_f := \{ f(x) \mid x \in \mathbb{D}_f \} \) denotes the image of \( f \).

DEFINITION 3. Let \( A \) and \( B \) be sets.
By \( f : A \to B \) we mean: \( f \) is a function and \( \mathbb{D}_f = A \) and \( \mathbb{I}_f \subseteq B \).
By \( f : A \to\to B \) we mean: \( f \) is a function and \( \mathbb{D}_f \subseteq A \) and \( \mathbb{I}_f \subseteq B \).

DEFINITION 4. \( \mathbb{N} := \{1, 2, 3, \ldots\} \) and \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \).
Convention: Any subset of \( \mathbb{R} \) is given the relative topology 

inherited from the standard topology on \( \mathbb{R} \).

NOTE: Any open subset of \( \mathbb{R} \) is locally compact and Hausdorff.

NOTE: Any closed subset of any open subset of \( \mathbb{R} \)

is locally compact and Hausdorff.

THEOREM 5. Let \( W \) be a non\( \emptyset \) bounded open subset of \( \mathbb{R} \).
Let \( U \) be a connected component of \( W \).
Then: \( \exists s, t \in \mathbb{R}\setminus W \text{ s.t. } s < t \text{ and } s,t \in W \).

Proof. Since \( U \) is a connected component of \( W \), we get: \( \emptyset \neq U \subseteq W \).
Since \( W \) is bounded and since \( U \subseteq W \), we get: \( U \) is bounded.
The topological space \( \mathbb{R} \) is locally connected, so,

since \( W \) is open in \( \mathbb{R} \) and
since \( U \) is a connected component of \( W \),
we get: \( U \) is a connected open subset of \( \mathbb{R} \).
Since \( U \) is a non\( \emptyset \) bounded connected open subset of \( \mathbb{R} \),
choose \( s,t \in \mathbb{R} \text{ s.t. } s < t \text{ and } s,t \in W = (s,t) \).
Want: \( s,t \notin W \). \( s,t \notin W \).
Want: \( \{s,t\} \cap W = \emptyset \).
Assume: \( \{s,t\} \cap W \neq \emptyset \). \( \{s,t\} \cap W \neq \emptyset \).
Want: Contradiction.
Choose \( r \in \{s,t\} \cap W \). \( r \in \{s,t\} \cap W \).
Then: \( r \in \{s,t\} \text{ and } r \in W \).
Since $W$ is open in $\mathbb{R}$ and since $r \in W$,

choose $\delta > 0$ s.t. $(r - \delta; r + \delta) \subseteq W$.

Since $r \in \{s, t\}$ and since $\delta > 0$,

we get: $(s; t) \cap (r - \delta; r + \delta) \neq \emptyset$.

Let $I := (r - \delta; r + \delta)$. Then: $I$ is connected and $r \in I \subseteq W$.

Since $r \in I$, we get: $I \neq \emptyset$.

Since $I \subseteq W$ and since $I$ is non-$\emptyset$ and connected,

Let $V$ be the connected component of $W$ s.t. $I \subseteq V$.

We have: $U \cap V \supseteq U \cap I = (s; t) \cap (r - \delta; r + \delta) \neq \emptyset$,

so, since $U$ and $V$ are both connected components of $W$,

we conclude: $U = V$. Then: $r \in I \subseteq V = U$, so $r \in U$.

So, since $r \in \{s, t\}$, we get: $r \in \{s, t\} \cap U$. Then $\{s, t\} \cap U \neq \emptyset$.

However, $\{s, t\} \cap U = \{s, t\} \cap (s; t) = \emptyset$. Contradiction. □

THEOREM 6. Let $c, d, p, r, w \in \mathbb{R}$. Assume: $c < p < w < r < d$.

Let $W$ be an open subset of $(c; d)$. Assume: $w \in W$ and $p, r \notin W$.

Let $U$ be the connected component of $W$ s.t. $w \in U$.

Then there exist $s, t \in [p; r] \setminus W$ s.t. $s < t$ and s.t. $U = (s; t)$.

Proof. We have $w \in U \subseteq W$. Since $w \in W$, we get: $W \neq \emptyset$.

Since $W$ open in $(c; d)$, and since $(c; d)$ is bounded and open in $\mathbb{R}$,

we get: $W$ is a bounded open subset of $\mathbb{R}$.

So, since $U$ is a connected component of $W$, by Theorem 5,

choose $s, t \in \mathbb{R} \setminus W$ s.t. $s < t$ and s.t. $U = (s; t)$.

Want: $s, t \in [p; r]$. Want: $p \leq s < t \leq r$.

Since $U = (s; t)$ and $w \in U$, we get: $(s; w) \subseteq U$.

By hypothesis, $p \notin W$, so, since $(s; w) \subseteq U \subseteq W$, we get: $p \notin (s; w)$.

By hypothesis, $p < w$. Since $p < w$ and $p \notin (s; w)$, we get: $p \leq s$.

By choice of $s$ and $t$, we have: $s < t$. It remains to show: $t \leq r$.

Want: $r \geq t$. Since $U = (s; t)$ and $w \in U$, we get: $(w; t) \subseteq U$.

By hypothesis, $r \notin W$, so, since $(w; t) \subseteq U \subseteq W$, we get: $r \notin (w; t)$.

By hypothesis, $w < r$. Since $r > w$ and $r \notin (w; t)$, we get: $r \geq t$. □

THEOREM 7. Let $a, b \in \mathbb{R}$. Assume $a < b$.

Let $X \subseteq (a; b)$. Let $X' \subseteq X$. Assume $X'$ has non-$\emptyset$ interior in $X$.

Then: $\exists c, d \in [a; b]$ s.t. $c < d$ and s.t. $\emptyset \neq (c; d) \cap X \subseteq X'$.

Proof. Let $W$ denote the interior in $X$ of $X'$. By hypothesis, $W \neq \emptyset$.

Also, $W$ is open in $X$ and $W \subseteq X'$. Since $W \neq \emptyset$, choose $w \in W$.

Since $W$ is open in $X$, choose an open subset $V$ of $\mathbb{R}$ s.t. $W = V \cap X$. 


By hypothesis, \( X \subseteq (a; b) \), so: \( X = (a; b) \cap X \).

Since \( V \) and \( (a; b) \) are open in \( \mathbb{R} \), we get: \( V \cap (a; b) \) is open in \( \mathbb{R} \).

Let \( U := V \cap (a; b) \). Then \( U \) is open in \( \mathbb{R} \).

Also, \( W = V \cap X = V \cap (a; b) \cap X = U \cap X \), so \( W = U \cap X \).

Since \( w \in W = U \cap X \), we get: \( w \in U \) and \( w \in X \).

Since \( w \in U \) and since \( U \) is open in \( \mathbb{R} \),

choose \( c, d \in \mathbb{R} \) s.t. \( c < d \) and s.t. \( w \in (c; d) \subseteq U \).

Since \( (c; d) \subseteq U = V \cap (a; b) \subseteq (a; b) \), we get: \( (c; d) \subseteq (a; b) \).

It follows that \( [c; d] \subseteq [a; b] \). Then \( c, d \in [a; b] \).

It remains to show: \( \emptyset \neq (c; d) \cap X \subseteq X' \).

Since \( w \in (c; d) \) and since \( w \in X \), we get: \( w \in (c; d) \cap X \).

Then \( \emptyset \neq (c; d) \cap X \). Want: \( (c; d) \cap X \subseteq X' \).

Since \( (c; d) \subseteq U \), we get: \( (c; d) \cap X \subseteq U \cap X \).

Recall: \( W \subseteq X' \) and \( W = U \cap X \).

Then: \( (c; d) \cap X \subseteq U \cap X = W \subseteq X' \). \( \square \)

**DEFINITION 8.** \( \forall S \subseteq \mathbb{R} \), let \( \overline{S}^* \) denote the interior in \( \mathbb{R} \) of \( S \).

**DEFINITION 9.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \).

Then: \( \mathbb{D}'_f := \left\{ x \in (\mathbb{D}_f) \cup \lim_{h \to 0} \frac{(f(x + h)) - (f(x))}{h} \text{ exists} \right\} \).

Also, the \textbf{derivative of} \( f \) is the function \( f' : \mathbb{D}'_f \to \mathbb{R} \)

defined by: \( \forall x \in \mathbb{D}'_f \), \( f'(x) = \lim_{h \to 0} \frac{(f(x + h)) - (f(x))}{h} \).

**DEFINITION 10.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( j \in \mathbb{N}_0 \).

Then: \( f^{(j)} \) denotes the \( j \)th derivative of \( f \).

Also, \( \mathbb{D}^{(j)}_f := \mathbb{D}_{f^{(j)}} \) denotes the domain of \( f^{(j)} \).

Note: \( \forall f : \mathbb{R} \rightarrow \mathbb{R} \), \( f^{(0)} = f \) and \( \mathbb{D}^{(0)}_f = \mathbb{D}_f \).

Also, \( \forall f : \mathbb{R} \rightarrow \mathbb{R} \), \( f^{(1)} = f' \) and \( \mathbb{D}^{(1)}_f = \mathbb{D}'_f \).

Also, \( \forall f : \mathbb{R} \rightarrow \mathbb{R} \), \( \mathbb{D}^{(0)}_f \supseteq \mathbb{D}^{(1)}_f \supseteq \mathbb{D}^{(2)}_f \supseteq \mathbb{D}^{(3)}_f \supseteq \cdots \).

In fact, each set is contained in the \textit{interior} in \( \mathbb{R} \) of the preceding one.

**DEFINITION 11.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \).

Then: \( \mathbb{D}^{(\infty)}_f := \mathbb{D}^{(0)}_f \cap \mathbb{D}^{(1)}_f \cap \mathbb{D}^{(2)}_f \cap \mathbb{D}^{(3)}_f \cap \cdots \).
Note that, $f : \mathbb{R} \to \mathbb{R}$, $D_f^{(0)} \cap D_f^{(2)} \cap D_f^{(4)} \cap D_f^{(6)} \cap \cdots = D_f^{(\infty)}$.

Also, $\forall f : \mathbb{R} \to \mathbb{R}$, $\forall j \in \mathbb{N}_0$, $D_f^{(\infty)} = D_f^{(j)}$.

Convention: $0^0 = 1$. Then: $\forall x \in \mathbb{R}, \ x^0 = 1$.

**DEFINITION 12.** Let $f : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N}_0$, $c \in D_f^{(k)}$.

Then: $P_k^{f,c} : \mathbb{R} \to \mathbb{R}$ is defined by:

$$\forall x \in \mathbb{R}, \ P_k^{f,c}(x) = \sum_{i=0}^{k} \left[ (f^{(i)}(c)) \cdot \frac{(x - c)^i}{i!} \right].$$

**DEFINITION 13.** Let $f : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$.

By $f$ is real-analytic at $c$, we mean:

$\exists \delta > 0$ s.t. $P_k^{f,c} \to f$ pointwise on $(c - \delta; c + \delta)$, as $k \to \infty$.

It is well-known that: $\forall f : \mathbb{R} \to \mathbb{R}$, $\forall c \in \mathbb{R}$,

$(f \text{ is real-analytic at } c) \Rightarrow (c \in D_f^{(\infty)})$.

**DEFINITION 14.** Let $f : \mathbb{R} \to \mathbb{R}$, $S \subseteq \mathbb{R}$.

By $f$ is real-analytic on $S$, we mean:

$\forall x \in S$, $f$ is real-analytic at $x$.

**THEOREM 15.** Let $\sigma, \tau : \mathbb{R} \to \mathbb{R}$, $I \subseteq \mathbb{R}$, $q \in I$.

Assume: $I$ is an interval.

Assume: $\sigma$ and $\tau$ are both real-analytic on $I$.

Assume: $\forall j \in \mathbb{N}_0$, $\sigma^{(j)}(q) = \tau^{(j)}(q)$.

Then: $\sigma = \tau$ on $I$.

Theorem 15 is well-known. Its proof is omitted.

**THEOREM 16.** Let $\beta_0, \beta_1, \beta_2, \ldots \in \mathbb{R}$. Let $c \in \mathbb{R}$.

Assume $\{\beta_0, \beta_1, \beta_2, \ldots\}$ is bounded.

Define $\phi : \mathbb{R} \to \mathbb{R}$ by: $\forall x \in \mathbb{R}, \ \phi(x) = \sum_{i=0}^{\infty} \left[ \beta_i \cdot \frac{(x - c)^i}{i!} \right]$.

Then: $\phi$ is real-analytic on $\mathbb{R}$.

Also, $\forall j \in \mathbb{N}_0$, $\forall x \in \mathbb{R}$, $\phi^{(j)}(x) = \sum_{i=0}^{\infty} \left[ \beta_{i+j} \cdot \frac{(x - c)^i}{i!} \right]$.

Theorem 16 is well-known. Its proof is omitted.

**DEFINITION 17.** Let $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$, $M \geq 0$.

By $f$ has $M$-BD at $x$, we mean:
\[ x \in \mathbb{D}^{(x)}_f \quad \text{and} \quad \forall j \in \mathbb{N}_0, \ |f^{(j)}(x)| \leq M. \]

By \( f \) has **M-BED at** \( x \), we mean:
\[ x \in \mathbb{D}^{(x)}_f \quad \text{and} \quad \forall j \in \mathbb{N}_0, \ |f^{(2j)}(x)| \leq M. \]

BD stands for “bounded derivatives”.
BED stands for “bounded even derivatives”.

**DEFINITION 18.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \in \mathbb{R}. \)

By \( f \) has **BD at** \( x \), we mean:
\[ \exists M \geq 0 \quad \text{s.t.} \quad f \text{ has } M\text{-BD at } x. \]

By \( f \) has **BED at** \( x \), we mean:
\[ \exists M \geq 0 \quad \text{s.t.} \quad f \text{ has } M\text{-BED at } x. \]

Note: \( \forall f : \mathbb{R} \rightarrow \mathbb{R}, \forall x \in \mathbb{R}, \)
\( ( f \text{ has BD at } x ) \Rightarrow ( f \text{ has BED at } x ) \Rightarrow ( x \in \mathbb{D}^{(x)}_f ). \)

**DEFINITION 19.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad S \subseteq \mathbb{R}, \quad M \geq 0. \)

By \( f \) has **M-BD on** \( S \), we mean:
\[ \forall x \in S, \quad f \text{ has } M\text{-BD at } x. \]

By \( f \) has **M-BED on** \( S \), we mean:
\[ \forall x \in S, \quad f \text{ has } M\text{-BED at } x. \]

**DEFINITION 20.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad S \subseteq \mathbb{R}. \)

By \( f \) has **PBD on** \( S \), we mean:
\[ \forall x \in S, \quad f \text{ has BD at } x. \]

By \( f \) has **PBED on** \( S \), we mean:
\[ \forall x \in S, \quad f \text{ has BED at } x. \]

By \( f \) has **UBD on** \( S \), we mean:
\[ \exists M \geq 0 \quad \text{s.t.} \quad f \text{ has } M\text{-BD on } S. \]

By \( f \) has **UBED on** \( S \), we mean:
\[ \exists M \geq 0 \quad \text{s.t.} \quad f \text{ has } M\text{-BED on } S. \]

PBD stands for “pointwise bounded derivatives”.
PBED stands for “pointwise bounded even derivatives”.
UBD stands for “uniformly bounded derivatives”.
UBED stands for “uniformly bounded even derivatives”.

**DEFINITION 21.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}. \)

Then \( \overline{\text{BD}_f} := \{ x \in \mathbb{D}^{(x)}_f \mid f \text{ has } BD \text{ at } x \}. \)
DEFINITION 22. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \ c \in \text{BD}_f \).
Then: \( P_{f,c}^{(i)} : \mathbb{R} \rightarrow \mathbb{R} \) is defined by:
\[
\forall x \in \mathbb{R}, \quad P_{f,c}^{(i)}(x) = \sum_{i=0}^{\infty} \left( \frac{f^{(i)}(c)}{i!} \right) \cdot \frac{(x-c)^i}{i!}.
\]

THEOREM 23. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \ c \in \text{BD}_f \), \( g = P_{f,c}^{(i)} \).
Then: \( g \) is real-analytic on \( \mathbb{R} \). Also: \( \forall j \in \mathbb{N}_0, \ f^{(j)}(c) = g^{(j)}(c) \).
Proof. For all \( i \in \mathbb{N}_0 \), let \( \beta_i := f^{(i)}(c) \).
Since \( c \in \text{BD}_f \), we get: \( \{\beta_0, \beta_1, \beta_2, \ldots\} \) is bounded.
Since \( g = P_{f,c}^{(i)} \), we get: \( \forall x \in \mathbb{R}, \ g(x) = \sum_{i=0}^{\infty} \left( \beta_i \cdot \frac{(x-c)^i}{i!} \right) \).
Then, by Theorem 16, we get: \( g \) is real-analytic on \( \mathbb{R} \).
It remains to show: \( \forall j \in \mathbb{N}_0, \ f^{(j)}(c) = g^{(j)}(c) \).
Given \( j \in \mathbb{N}_0 \), want: \( f^{(j)}(c) = g^{(j)}(c) \). Want: \( g^{(j)}(c) = \beta_j \).
By Theorem 16, we get: \( g^{(j)}(c) = \sum_{i=0}^{\infty} \left( \beta_i \cdot \frac{(x-c)^i}{i!} \right) \).
Then \( g^{(j)}(c) = \sum_{i=0}^{\infty} \left( \beta_i \cdot \frac{(x-c)^i}{i!} \right) = \frac{0^j}{0!} + \frac{0^j}{0!} + \cdots + \frac{0^j}{0!} \).
Then \( g^{(j)}(c) = [\beta_j \cdot 1] + \sum_{i=1}^{\infty} (\beta_i + 1) = \beta_j + 0 = \beta_j \). \( \square \)

THEOREM 24. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \ B \subseteq \mathbb{R}, \ c, x \in B, \ M \geq 0 \).
Assume: \( B \) is an interval. Assume: \( f \) has \( M \)-BD on \( B \).
Let \( j \in \mathbb{N}_0 \). Then: \( |f(x) - (P_{f,c}^{(j+1)}(x))| \leq M \cdot \frac{|x-c|^{j+1}}{(j+1)!} \).
Proof. Since \( f \) has \( M \)-BD on \( B \), we get: \( B \subseteq \mathfrak{D}^{(j)}_f \).
By Taylor’s Theorem, choose \( \xi \) strictly between \( c \) and \( x \) s.t.
\[
f(x) = (P_{f,c}^{(j+1)}(x)) + \left( f^{(j+1)}(\xi) \cdot \frac{(x-c)^{j+1}}{(j+1)!} \right).
\]
Then: \( f(x) - (P_{f,c}^{(j+1)}(x)) = f^{(j+1)}(\xi) \cdot \frac{(x-c)^{j+1}}{(j+1)!} \).
Then: \( |f(x) - (P_{f,c}^{(j+1)}(x))| = |f^{(j+1)}(\xi)| \cdot \frac{|x-c|^{j+1}}{(j+1)!} \).
Since \( B \) is an interval and \( c, x \in B \), we get: \( \xi \in B \).
So, since \( f \) has \( M \)-BD on \( B \), we get: \( |f^{(j+1)}(\xi)| \leq M \).
Then: \[|(f(x)) - (P_{j}^{f,c}(x))| \leq M \cdot \frac{|x-c|^{j+1}}{(j+1)!}.\] □

**DEFINITION 25.** Let \( f : \mathbb{R} \to \mathbb{R}, \ x \in \mathbb{R}. \)
By \( f \) has \( \text{UBD near } x \), we mean:
\[\exists \delta > 0 \text{ s.t. } f \text{ has UBD on } (x - \delta; x + \delta).\]

**THEOREM 26.** Let \( f : \mathbb{R} \to \mathbb{R}, \ U \subseteq \mathbb{R}. \)
Assume: \( \forall x \in U, \ f \) has UBD near \( x. \)
Then: \( f \) is real-analytic on \( U. \)

*Proof.* Given \( c \in U, \) we want: \( f \) is real-analytic at \( c. \)

Want: \( \exists \delta > 0 \) s.t. \( P_{j}^{f,c} \to f \) pointwise on \( (c - \delta; c + \delta), \) as \( j \to \infty. \)
Since \( c \in U, \) by hypothesis, \( f \) has UBD near \( c, \) so
choose \( \delta > 0 \) s.t. \( f \) has UBD on \( (c - \delta; c + \delta). \)

Want: \( P_{j}^{f,c} \to f \) pointwise on \( (c - \delta; c + \delta), \) as \( j \to \infty. \)
Let \( B := (c - \delta; c + \delta). \)
Then: \( B \) is an interval and \( c \in B \) and \( f \) has UBD on \( B. \)

Want: \( P_{j}^{f,c} \to f \) pointwise on \( B, \) as \( j \to \infty. \)
Given \( x \in B, \) we want: \( P_{j}^{f,c}(x) \to f(x), \) as \( j \to \infty. \)

Want: \[|(f(x)) - (P_{j}^{f,c}(x))| \to 0, \text{ as } j \to \infty.\]
Since \( f \) has UBD on \( B, \) choose \( M > 0 \) s.t. \( f \) has \( M\)-BD on \( B. \)

Then, by Theorem 24, \( \forall j \in \mathbb{N}_{0}, \) \[|(f(x)) - (P_{j}^{f,c}(x))| \leq M \cdot \frac{|x-c|^{j+1}}{(j+1)!}.\]
So, since \( M \cdot \frac{|x-c|^{j+1}}{(j+1)!} \to 0, \) as \( j \to \infty, \)
we conclude: \[|(f(x)) - (P_{j}^{f,c}(x))| \to 0, \text{ as } j \to \infty.\] □

**THEOREM 27.** Let \( f, g : \mathbb{R} \to \mathbb{R}, \ r, s, t \in \mathbb{R}. \)
Assume: \( s < t \) and \( r \in [s; t]. \)
Assume: \( r \in D^{(\infty)}_{f} \cap D^{(\infty)}_{g} \) and \( (s; t) \subseteq D^{(\infty)}_{f} \cap D^{(\infty)}_{g}. \)
Assume: \( f = g \) on \( (s; t). \)
Then: \( \forall j \in \mathbb{N}_{0}, \ f^{(j)}(r) = g^{(j)}(r). \)

*Proof.* Given \( j \in \mathbb{N}_{0}, \) we want: \( f^{(j)}(r) = g^{(j)}(r). \)
Since \( f = g \) on \( (s; t), \) we get: \( f^{(j)} = g^{(j)} \) on \( (s; t). \)
Let \( \phi := f^{(j)} \) and \( \psi := g^{(j)}. \)
Then: \( \phi = \psi \) on \( (s; t). \)

Want: \( \phi(r) = \psi(r). \)
We have: \( D^{(\infty)}_{\phi} = D^{(\infty)}_{f} \) and \( D^{(\infty)}_{\psi} = D^{(\infty)}_{g}. \)
Then: \( r \in D^{(\infty)}_{\phi} \cap D^{(\infty)}_{\psi} \) and \( (s; t) \subseteq D^{(\infty)}_{\phi} \cap D^{(\infty)}_{\psi}. \)
Since \( r \in \mathbb{D}_\phi^{(\infty)} \cap \mathbb{D}_\psi^{(\infty)} \subset \mathbb{D}_\phi^{(1)} \cap \mathbb{D}_\psi^{(1)}, \) we get: \( \phi \) and \( \psi \) are both differentiable at \( r. \)

Then: \( \phi \) and \( \psi \) are both continuous at \( r. \)

Since \( r \in [s; t], \) choose \( q_1, q_2, q_3, \ldots \in (s; t) \) s.t. \( q_j \to r, \) as \( j \to \infty. \)

By continuity, \( \phi(q_j) \to \phi(r), \) as \( j \to \infty \) and \( \psi(q_j) \to \psi(r), \) as \( j \to \infty. \)

Since \( \phi = \psi \) on \( (s; t), \) we get: \( \forall j \in \mathbb{N}, \) \( \phi(q_j) = \psi(q_j). \)

So, letting \( j \to \infty, \) we get: \( \phi(r) = \psi(r). \) \( \square \)

**THEOREM 28.** Let \( f : \mathbb{R} \to \mathbb{R}, \) \( s, t \in \mathbb{R}, \) \( M \geq 0. \)

Assume: \( s < t. \) Assume: \( \forall x \in (s; t), \) \( f \) has UBD near \( x. \)

Let \( r \in [s; t]. \) Assume: \( f \) has M-BD at \( r. \)

Let \( N := M \cdot e^{t-s}. \) Then: \( f \) has N-BD on \( (s; t). \)

**Proof.** Let \( c := (s + t)/2. \) Then \( c \in (s; t). \)

So, by hypothesis, we get: \( f \) has UBD near \( c. \)

Then \( f \) has BD at \( c. \) Then \( c \in \text{BD}_f. \)

By Theorem 23, \( g \) is real-analytic on \( \mathbb{R}. \)

Then \( \mathbb{D}_g^{(x)} = \mathbb{R}, \) so: \( r \in \mathbb{D}_g^{(\infty)} \) and \( (s; t) \subseteq \mathbb{D}_g^{(\infty)}. \)

By hypothesis, \( f \) has M-BD at \( r, \) so we get: \( r \in \mathbb{D}_f^{(\infty)}. \)

By hypothesis, we have: \( \forall x \in (s; t), \) \( f \) has UBD near \( x. \)

So, by Theorem 26, \( f \) is real-analytic on \( (s; t). \) Then: \( (s; t) \subseteq \mathbb{D}_f^{(\infty)}. \)

Then: \( r \in \mathbb{D}_f^{(\infty)} \cap \mathbb{D}_g^{(\infty)} \) and \( (s; t) \subseteq \mathbb{D}_f^{(\infty)} \cap \mathbb{D}_g^{(\infty)}. \)

By Theorem 23, we get: \( \forall j \in \mathbb{N}_0, \) \( f^{(j)}(c) = g^{(j)}(c). \)

So, since \( c \in (s; t) \) and since \( f \) and \( g \) are both real-analytic on \( (s; t), \)

by Theorem 15, we get: \( f = g \) on \( (s; t). \)

Then, by Theorem 27, we get: \( \forall j \in \mathbb{N}_0, \) \( f^{(j)}(r) = g^{(j)}(r). \)

By hypothesis, \( f \) has M-BD at \( r, \) so \( f \) has BD at \( r. \)

Then \( r \in \text{BD}_f. \)

Let \( h := P_f^{j,r}. \) Then, by Theorem 23, \( h \) is real-analytic on \( \mathbb{R}. \)

Also, by Theorem 23, \( \forall j \in \mathbb{N}_0, \) \( f^{(j)}(r) = h^{(j)}(r). \)

Since \( \forall j \in \mathbb{N}_0, \) \( g^{(j)}(r) = f^{(j)}(r) = h^{(j)}(r). \)

and since \( g \) and \( h \) are both real-analytic on \( \mathbb{R}, \)

by Theorem 15, we get: \( g = h \) on \( \mathbb{R}. \)

So, since \( f = g \) on \( (s; t), \) we get: \( f = h \) on \( (s; t). \)

**It therefore suffices to show:** \( h \) has N-BD on \( (s; t). \)

Given \( u \in (s; t), \) \( \text{want:} \) \( h \) has N-BD at \( u. \)

Given \( j \in \mathbb{N}_0, \) \( \text{want:} \) \( |h^{(j)}(u)| \leq N. \)

By hypothesis, \( r \in [s; t]. \)

Since \( r, u \in [s; t], \) we get: \( |u - r| \leq t - s. \)

Then \( e^{|u-r|} \leq e^{t-s}. \)

So, since \( M \geq 0, \) we get: \( M \cdot e^{|u-r|} \leq M \cdot e^{t-s}. \)
By hypothesis, $f$ has $M$-BD at $r$, so: \[ \forall i \in \mathbb{N}_0, \quad |f^{(i)}(r)| \leq M. \]

Since $h = P^{f_x}_\infty$, we get: \[ \forall x \in \mathbb{R}, \quad h(x) = \sum_{i=0}^{\infty} \left( f^{(i)}(r) \cdot \frac{(x-r)^i}{i!} \right). \]

Then, by Theorem 16, we have: \[ \forall x \in \mathbb{R}, \quad h^{(i)}(x) = \sum_{i=0}^{\infty} \left( f^{(i+j)}(r) \cdot \frac{(x-r)^i}{i!} \right). \]

Then: \[ |h^{(i)}(u)| \leq \sum_{i=0}^{\infty} \left| f^{(i+j)}(r) \cdot \frac{|u-r|^i}{i!} \right| \leq M \cdot \sum_{i=0}^{\infty} \frac{|u-r|^i}{i!} \]

\[ = M \cdot e^{|u-r|} \leq M \cdot e^{t-s} = N. \quad \square \]

**THEOREM 29.** Let $I \subseteq \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$.
Assume: $I$ is a non-empty open interval.
Assume: $\forall x \in I$, $f$ has UBD near $x$. Then: $f$ has UBD on $I$.

**Proof.** Since $I$ is an interval, we get: $I$ is connected.
Since $I$ is a non-empty bounded connected open subset of $\mathbb{R}$,
choose $s, t \in \mathbb{R}$ s.t. $s < t$ and s.t. $I = (s,t)$.
Then: $\forall x \in (s;t)$, $f$ has UBD near $x$.
By Theorem 26, $f$ is real-analytic on $(s,t)$.
Let $r := (s + t)/2$. Then $r \in (s,t)$. Then $r \in I$ and $r \in [s;t]$.
Since $r \in I$, by assumption, $f$ has UBD near $r$.
Then $f$ has BD at $r$. Choose $M \geq 0$ s.t. $f$ has $M$-BD at $r$.
Let $N := M \cdot e^{t-s}$. By Theorem 28, $f$ has $N$-BD on $(s,t)$.
Then $f$ has UBD on $(s,t)$. Then $f$ has UBD on $I$. \quad \square

Theorem 30 and the proof below are both due to T. Tao. See https://mathoverflow.net/questions/413165/does-iterating-the-derivative-infinitely-many-times-give-a-smooth-function-when

**THEOREM 30.** (T. Tao) Let $f : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$.
Assume: $a < b$. Let $I := (a;b)$.
Assume: $f$ has PBD on $I$. Then: $f$ has UBD on $I$.

**Proof.** Let $V := \{ x \in I \mid f$ has UBD near $x \}$. Then $V$ is open in $I$.
By Theorem 29, it suffices to show: $V = I$.
Let $X := I \setminus V$. Then $V = I \setminus X$. Want: $X = \emptyset$. 
Assume: $X \neq \emptyset$.  **Want:** Contradiction.

Since $I = (a; b)$, we get: $I$ is open in $\mathbb{R}$.

Since $V$ is open in $I$ and since $X = I \setminus V$, we get: $X$ is closed in $I$.

Since $X$ is closed in $I$ and since $I$ is open in $\mathbb{R}$, we get: $X$ is locally compact and Hausdorff.

By hypothesis, $f$ has PBD on $I$, so, since $X = I \setminus V \subseteq I$, we get: $f$ has PBD on $X$.

Then: $X \subseteq D_f^{(x)}$. For all $m \in \mathbb{N}$, let $X_m := \{x \in X \mid f$ has $m$-BD at $x\}$.

By continuity, we get: $\forall m \in \mathbb{N}, X_m$ is closed in $X$.

Since $f$ has PBD on $X$, we get: $X_1 \cup X_2 \cup X_3 \cup \cdots = X$.

So, since $X$ is non-$\emptyset$ and locally compact and Hausdorff, by the Baire Category Theorem,

choose $M \in \mathbb{N}$ s.t. $X_M$ has non-$\emptyset$ interior in $X$.

So, since $X = I \setminus V \subseteq I = (a; b)$, by Theorem 7, choose $c, d \in [a; b]$ s.t. $c < d$ and s.t. $\emptyset \neq (c; d) \cap X \subseteq X_M$.

Since $\emptyset \neq (c; d) \cap X$, choose $q \in (c; d) \cap X$.

Then $q \in X_M$. Also, $q \in (c; d)$ and $q \in X$.

Since $q \in (c; d)$ and since $(c; d)$ is open in $\mathbb{R}$, choose $\delta > 0$ s.t. $(q - \delta; q + \delta) \subseteq (c; d)$.

Since $q \in X = I \setminus V$, by definition of $V$,

we get: $f$ does not have UBD near $q$.

Then: $f$ does not have UBD on $(q - \delta; q + \delta)$.

So, since $(q - \delta; q + \delta) \subseteq (c; d)$, we get:

$f$ does not have UBD on $(c; d)$.

Let $K := M \cdot e^{d-c}$. Then $f$ does not have $K$-BD on $(c; d)$.

Choose $p \in (c; d)$ s.t. $f$ does not have $K$-BD at $p$.

Since $c < d$, we get: $e^{d-c} \geq 1$. Then: $K \geq M$.

By definition of $X_M$, $f$ has $M$-BD on $X_M$.

So, since $K \geq M$, we get: $f$ has $K$-BD on $X_M$.

So, since $f$ does not have $K$-BD at $p$, we get: $p \notin X_M$.

Since $I = (a; b)$, we get: $I$ is open in $\mathbb{R}$.

Since $X_M$ is closed in $X$ and since $X$ is closed in $I$,

we get: $X_M$ is closed in $I$. Then: $I \setminus X_M$ is open in $I$.

So, since $I$ is open in $\mathbb{R}$, we get: $I \setminus X_M$ is open in $\mathbb{R}$.

Since $c, d \in [a; b]$, we get: $(c; d) \subseteq (a; b)$.

Since $(c; d) \subseteq (a; b) = I$, we get: $(c; d) \setminus X_M = (c; d) \cap (I \setminus X_M)$.

Let $W := (c; d) \setminus X_M$. Then: $W = (c; d) \cap (I \setminus X_M)$.

Since $(c; d)$ and $I \setminus X_M$ are both open in $\mathbb{R}$,
we get: 
\((c; d) \cap (I \setminus X_M)\) is open in \(\mathbb{R}\). 
Then \(W\) is open in \(\mathbb{R}\).

Since \(p \in (c; d)\) and \(p \notin X_M\), we get: \(p \in W\). 
Then: \(W \neq \emptyset\).

Since \(W = (c; d) \setminus X_M \subseteq (c; d)\), we get: \(W \subseteq (c; d)\).

Then \(W\) is bounded. 
Then \(W\) is a non-empty bounded open subset of \(\mathbb{R}\).

Recall: \((c; d) \cap X \subseteq X_M\). 
Then \([\overline{(c; d) \cap X}] \setminus X_M = \emptyset\).

Then: 
\[W \cap X = \overline{(c; d) \cap X} \setminus X = \overline{(c; d) \cap X} \setminus X_M = \emptyset.\]

Then: \(W \cap X = \emptyset\). 
Also, \(W \subseteq (c; d) \subseteq (a; b) = I\), so \(W \subseteq I\).

Since \(W \subseteq I\) and \(W \cap X = \emptyset\), we get: \(W \subseteq I \setminus X\).

Then \(W \subseteq I \setminus X = V\), 
so, by definition of \(V\),
we get: \(\forall x \in W, \ f\) has UBD near \(x\).

Let \(U\) be the connected component of \(W\) s.t. \(p \in U\).

Then: \(p \in U \subseteq W\). 
Then: \(\forall x \in U, \ f\) has UBD near \(x\).

By Theorem 5, choose \(s, t \in \mathbb{R} \setminus W\) s.t. \(s < t\) and s.t. \(U = (s; t)\).

Then: \(\{s, t\} \subseteq \mathbb{R} \setminus W\). 
Recall: \(W \subseteq (c; d)\).

Then \((s; t) = U \subseteq W \subseteq (c; d)\), 
so \((s; t) \subseteq (c; d)\), 
so \([s; t] \subseteq [c; d]\).

Then: \(s, t \in [c; d]\). 
Then: \(c \leq s < t \leq d\).

Then: \(t - s \leq d - c\). 
Then: \(e^{t-s} \leq e^{d-c}\).

Since \(M \in \mathbb{N}\), we get: \(M > 0\). 
Then: \(M \cdot e^{t-s} \leq M \cdot e^{d-c}\).

Let \(N := M \cdot e^{t-s}\). 
Recall: \(K = M \cdot e^{d-c}\). 
Then \(N \leq K\).

Since \(W = (c; d) \setminus X_M\) and since \(q \in X_M\), we get: \(q \notin W\).

So, since \((s; t) = U \subseteq W\), we get: \(q \notin (s; t)\). 
Recall: \(q \in (c; d)\).

Since \(q \notin (s; t)\) and since \(q \in (c; d)\), we get: \((s; t) \neq (c; d)\).

Since \((s; t) \neq (c; d)\), we get: 
either \(s \neq c\) or \(t \neq d\).

Recall: \(c \leq s < t \leq d\).

Then: \(c \leq s < t \leq d\).

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Then: \(c \leq s < t \leq d\).

Then: \(c \leq s < t \leq d\).

Choose \(r \in \{s, t\} \cap (c; d)\).

Since \(r \in \{s, t\} \subseteq \mathbb{R} \setminus W\), we get: \(r \in \mathbb{R} \setminus W\).

Then: \(r \in (c; d) \setminus W\).

By definition of \(W\), we have: \(W = (c; d) \setminus X_M\).

Since \(r \in (c; d) \setminus W = (c; d) \setminus [(c; d) \setminus X_M] = (c; d) \cap X_M \subseteq X_M\),
by definition of \(X_M\), we get: \(f\) has \(M\)-BD at \(r\).

We have \(r \in \{s, t\} \subseteq [s; t]\), so \(r \in [s; t]\).

Recall: \(\forall x \in U, \ f\) has UBD near \(x\).

Then, by Theorem 28, \(f\) has \(N\)-BD on \((s; t)\).

So, since \(N \leq K\), we get: \(f\) has \(K\)-BD on \((s; t)\).

So, since \(p \in U = (s; t)\), we get: \(f\) has \(K\)-BD at \(p\).

By choice of \(p\), \(f\) does not have \(K\)-BD at \(p\). 
Contradiction. \(\square\)
**THEOREM 31.** Let \( g : \mathbb{R} \rightarrow \mathbb{R}, \ a, b \in \mathbb{R} \), \( M \geq 0 \). Assume: \( a < b \). Let \( I := (a;b) \). Assume: \( I \subseteq \mathbb{D}^{(2)}_g \).

Assume: \(|g| \leq M \) on \( I \) and \(|g''| \leq M \) on \( I \).

Let \( N := M \cdot \left( \frac{6}{b-a} + \frac{b-a}{6} \right) \). Then: \(|g'| \leq N \) on \( I \).

**Proof.** Given \( x \in I \), want: \(|g'(x)| \leq N\).

Let \( \delta := \frac{b-a}{3} \). Then \( \delta > 0 \) and \( \frac{2M}{\delta} + \frac{M\delta}{2} = N \).

Choose \( h \in \{\delta, -\delta\} \) s.t. \( x + h \in I \). Then \(|h| = \delta\).

By Taylor's Theorem, choose \( \xi \) strictly between \( x \) and \( x + h \) s.t.

\[
g(x + h) = (g(x)) + (g'(x)) \cdot h + (g''(\xi)) \cdot \frac{h^2}{2}.
\]

Then:

\[
g'(x) = \frac{(g(x + h)) - (g(x))}{h} - \frac{(g''(\xi)) \cdot h}{2}.
\]

Then:

\[
|g'(x)| \leq \frac{|g(x + h)| + |g(x)|}{|h|} + \frac{|g''(\xi)| \cdot |h|}{2}.
\]

Since \(|g|, |g''| \leq M \) on \( I \) and since \( x, \xi, x + h \in I \), we get:

\[
|g(x)| \leq M \quad \text{and} \quad |g''(\xi)| \leq M \quad \text{and} \quad |g(x + h)| \leq M.
\]

Recall: \(|h| = \delta\). Then: \(|g'(x)| \leq \frac{2M}{\delta} + \frac{M\delta}{2} = N\). \(\square\)

**THEOREM 32.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \ I \subseteq \mathbb{R} \).

Assume: \( I \) is a non\(\emptyset\) bounded open interval.

Assume: \( f \) has UBED on \( I \). Then: \( f \) has UBD on \( I \).

**Proof.** Want: \( \exists N \geq 0 \) s.t. \( f \) has \( N\)-BD on \( I \).

Since \( f \) has UBED on \( I \), choose \( M \geq 0 \) s.t. \( f \) has \( M\)-BED on \( I \).

Since \( I \) is a non\(\emptyset\) bounded open interval,

choose \( a, b \in \mathbb{R} \) s.t. \( a < b \) and s.t. \( I = (a;b) \).

Let \( N := M \cdot \left( \frac{6}{b-a} + \frac{b-a}{6} \right) \). Then \( M \leq N \). Then \( N \geq 0 \).

Want: \( f \) has \( N\)-BD on \( I \). Given \( x \in I \), want: \( f \) has \( N\)-BD at \( x \).

Given \( j \in \mathbb{N}_0 \), want: \(|f^{(j)}(x)| \leq N\).

**Case 1:** \( j \) is even.

**Proof in Case 1:**

Since \( j \) is even, by choice of \( M \), we have: \(|f^{(j)}| \leq M \) on \( I \).

So, since \( x \in I \), we get: \(|f^{(j)}(x)| \leq M \). Then \(|f^{(j)}(x)| \leq M \leq N \).
End of proof in Case 1.

Case 2: \( j \) is odd.

Proof in Case 2:

Since \( j - 1 \) and \( j + 1 \) are even, by the choice of \( M \), we have:

\[
|f^{(j-1)}| \leq M \text{ on } I \quad \text{and} \quad |f^{(j+1)}| \leq M \text{ on } I.
\]

By hypothesis, \( f \) has UBED on \( I \), so: \( I \subseteq \mathbb{D}_f^{(\infty)} \).

Let \( g := f^{(j-1)} \). Then \( I \subseteq \mathbb{D}_g^{(\infty)} = \mathbb{D}_g^{(2)} \subseteq \mathbb{D}_g^{(2)} \), so \( I \subseteq \mathbb{D}_g^{(2)} \).

Also, \( g' = f^{(j)} \) and \( g'' = f^{(j+1)} \).

Then:

\[
|g| \leq M \text{ on } I \quad \text{and} \quad |g''| \leq M \text{ on } I.
\]

By hypothesis, \( f \) has UBED on \( I \), so:

\[
I \subseteq D_{p8qf}.
\]

Let \( h : g^{(j)} \). Then

\[
I \subseteq D_{p8qg} \subseteq D_{p8qg}, \quad \text{so} \quad I \subseteq D_{p2qg}.
\]

Also, \( g' = f^{(j)} \) and \( g'' = f^{(j+1)} \).

Then:

\[
|g'| \leq M \text{ on } I \quad \text{and} \quad |g''| \leq M \text{ on } I.
\]

By Theorem 31, we get:

\[
|g'_{x}| \leq N \text{ on } I.
\]

Then, since \( x \in I \), we get:

\[
|f^{(j)}(x)| = |g'(x)| \leq N.
\]

End of proof in Case 2. \( \square \)

THEOREM 33. Let \( f : \mathbb{R} \to \mathbb{R}, \ c, d \in \mathbb{R} \).

Assume \( c < d \). Let \( J := (c; d) \). Assume \( f \) has PBED on \( J \).

Then \( \exists \text{non}\emptyset \text{ open subintervals } U_1, U_2, U_3, \ldots \text{ of } J \)

s.t. \( \forall i \in \mathbb{N}, \ f \text{ has UBD on } U_i \quad \text{and} \quad s.t. \ U_1 \cup U_2 \cup U_3 \cup \cdots \text{ is dense in } J. \)

Proof. Since \( J \) is second-countable, choose a countable base \( \mathcal{W} \) for \( J \) s.t., \( \forall W \in \mathcal{W}, \ W \neq \emptyset \).

Since \( \mathcal{W} \) is countable, it suffices to prove:

\( \forall W \in \mathcal{W}, \ \exists \text{non}\emptyset \text{ open subinterval } U \text{ of } J \)

s.t. \( U \subseteq W \quad \text{and} \quad s.t. \ f \text{ has UBD on } U \).

Given \( W \in \mathcal{W}, \ \text{want: } \exists \text{non}\emptyset \text{ open subinterval } U \text{ of } J \)

s.t. \( U \subseteq W \quad \text{and} \quad s.t. \ f \text{ has UBD on } U \).

Since \( W \in \mathcal{W}, \ \text{we get: } \ W \neq \emptyset \quad \text{and} \quad W \subseteq J. \)

Since \( W \in \mathcal{W}, \ \text{we get: } \ W \text{ is open in } J. \)

So, since \( J \) is open in \( \mathbb{R}, \ \text{we get: } \ W \text{ is open in } \mathbb{R}. \)

Then: \( W \) is locally compact and Hausdorff.

For all \( m \in \mathbb{N}, \ \text{let } C_m := \{ x \in W \mid f \text{ has } m\text{-BED at } x \}. \)

Since \( f \) has PBED on \( J \) and since \( W \subseteq J \), we get: \( f \) has PBED on \( W \).

Then \( W \subseteq \mathbb{D}_f^{(\infty)} \). So, by continuity, \( \forall m \in \mathbb{N}, \ C_m \text{ is closed in } W. \)

Since \( f \) has PBED on \( W \), we get: \( C_1 \cup C_2 \cup C_3 \cup \cdots = W. \)

So, since \( W \) is non\emptyset and locally compact and Hausdorff,

by the Baire Category Theorem,

choose \( M \in \mathbb{N} \) s.t. \( C_M \text{ has non}\emptyset \text{ interior in } W. \)

Then, since \( W \) is open in \( \mathbb{R}, \ \text{we get: } \ C_M \text{ has non}\emptyset \text{ interior in } \mathbb{R}. \)
So choose \( s, t \in \mathbb{R} \) s.t. \( s < t \) and s.t. \( (s; t) \subseteq C_M \).

Let \( U := (s; t) \). Then: \( U \) is a non\( \emptyset \) open interval and \( U \subseteq C_M \).

Since \( U \subseteq C_M \subseteq W \subseteq J \) and since \( U \) is a non\( \emptyset \) open interval, we get: \( U \) is a non\( \emptyset \) open subinterval of \( J \).

As \( U \subseteq C_M \subseteq W \), it remains only to show: \( f \) has \( \text{UBD} \) on \( U \).

Since \( U \subseteq C_M \), by definition of \( C_M \), we get: \( f \) has \( M\text{-BED} \) on \( U \).

Then \( f \) has \( \text{UBED} \) on \( U \). Then, by Theorem 32, \( f \) has \( \text{UBD} \) on \( U \). □

**DEFINITION 34.** Let \( f : \mathbb{R} \to \mathbb{R} \).

Then \( \text{IBD}_f := (\text{BD}_f)^\circ \) denotes the interior in \( \mathbb{R} \) of \( \text{BD}_f \).

**THEOREM 35.** Let \( f : \mathbb{R} \to \mathbb{R} \), \( c, d \in \mathbb{R} \).

Assume \( c < d \). Let \( J := (c; d) \).

Assume \( f \) has \( \text{PBED} \) on \( J \). Then \( \text{IBD}_f \cap J \) is dense in \( J \).

*Proof.* By Theorem 33, choose non\( \emptyset \) open subintervals \( U_1, U_2, U_3, \ldots \) of \( J \)

\[ \text{s.t. } \forall i \in \mathbb{N}, \ f \text{ has } \text{UBD} \text{ on } U_i \quad \text{and} \quad \text{s.t. } U_1 \cup U_2 \cup U_3 \cup \cdots \text{ is dense in } J. \]

Then: \( \forall i \in \mathbb{N} \), since \( f \) has \( \text{UBD} \) on \( U_i \),

it follows that \( f \) has \( \text{BD} \) on \( U_i \), so \( U_i \subseteq \text{BD}_f \).

Let \( U := U_1 \cup U_2 \cup U_3 \cup \cdots \). Then \( U \subseteq \text{BD}_f \), so \( U^\circ \subseteq (\text{BD}_f)^\circ \).

Since \( \forall i \in \mathbb{N}, U_i \subseteq J \), we get: \( U \subseteq J \).

Since \( \forall i \in \mathbb{N}, U_i \text{ is open in } J \), we get: \( U \text{ is open in } J \).

So, since \( J \) is open in \( \mathbb{R} \), we get: \( U \text{ is open in } \mathbb{R} \). Then \( U^\circ = U \).

Since \( U_1 \cup U_2 \cup U_3 \cup \cdots \text{ is dense in } J \), we get: \( U \text{ is dense in } J \).

Since \( U = U^\circ \subseteq (\text{BD}_f)^\circ = \text{IBD}_f \) and since \( U \subseteq J \),

we get: \( U \subseteq \text{IBD}_f \cap J \).

So, since \( U \text{ is dense in } J \), we get: \( \text{IBD}_f \cap J \text{ is dense in } J \). □

**THEOREM 36.** Let \( \phi : \mathbb{R} \to \mathbb{R} \), \( s, t \in \mathbb{R} \), \( L \geq 0 \). Assume: \( s < t \).

Assume: \( (s; t) \subseteq D^{(2)}_\phi \) and \( \phi \) is continuous both at \( s \) and at \( t \).

Assume: \( \phi^g > 0 \text{ on } (s; t) \). Assume: \( \phi \leq L \text{ on } \{s, t\} \).

Then: \( \phi < L \text{ on } (s; t) \).

Theorem 36 is a special case of the Maximum Principle.

This particular special case follows from the Mean Value Theorem.

We omit the proof.

**THEOREM 37.** Let \( g : \mathbb{R} \to \mathbb{R} \), \( s, t \in \mathbb{R} \), \( L \geq 0 \).

Assume: \( s < t \) and \( t - s \leq 1 \).
Assume: \((s; t) \subseteq \mathbb{D}_g^{(2)}\) and \(g\) is continuous both at \(s\) and at \(t\).

Assume: \(|g| \leq L\) on \((s, t)\). Let \(w \in (s; t)\). Assume \(|g(w)| \geq 2L\).
Then: \(\exists x \in (s; t) \text{ s.t. } |g''(x)| \geq 8L\).

Proof. Choose \(h \in \{g, -g\} \text{ s.t. } |g(w)| = h(w)\). Then \(h(w) \geq 2L\).
Also, \(|h| = |g|\) and \(|h'| = |g'|\) and \(|h''| = |g''|\).
Also, \((s; t) \subseteq \mathbb{D}_h^{(2)}\) and \(h\) is continuous both at \(s\) and at \(t\).

Want: \(\exists x \in (s; t) \text{ s.t. } |h''(x)| \geq 8L\).
Assume: \(|h''| < 8L\) on \((s; t)\). Want: Contradiction.

We have: \(-8L < h'' < 8L\) on \((s; t)\).
Since \(h'' > -8L\) on \((s; t)\), we get: \(8L + h'' > 0\) on \((s; t)\).
Define \(Q : \mathbb{R} \rightarrow \mathbb{R}\) by: \(\forall x \in \mathbb{R}, \ Q(x) = 4L \cdot (x - s) \cdot (x - t)\).
Then: \(Q'' = 8L\) on \(\mathbb{R}\). Then: \((Q + h)'' > 0\) on \((s; t)\).
Let \(\phi := Q + h\). Then \(\phi'' > 0\) on \((s; t)\).
Since \(Q = 0\) on \((s, t)\) and since \(h \leq |h| = |g| \leq L\) on \((s, t)\),
we get: \(Q + h \leq L\) on \((s, t)\). Then: \(\phi \leq L\) on \((s, t)\).
Also, \((s; t) \subseteq \mathbb{D}_h^{(2)}\) and \(\phi\) is continuous both at \(s\) and at \(t\).
Then, by Theorem 36 (Maximum Principle), we get: \(\phi < L\) on \((s; t)\).
By hypothesis, we have: \(w \in (s; t)\). Then \(\phi(w) < L\).
Since \((Q(w)) + (h(w)) = \phi(w) < L\), we get: \(h(w) < L - (Q(w))\).
Let \(c := (s + t)/2\). The minimum value of \(Q\) is \(Q(c)\).
Then \(Q(w) \geq Q(c)\). We calculate: \(Q(c) = -L \cdot (t - s)^2\).
Since \(0 < t - s \leq 1\), we get: \((t - s)^2 \leq 1\).
So, since \(L \geq 0\), we get: \(-L \cdot (t - s)^2 \geq -L\).
Then \(Q(w) \geq Q(c) = -L \cdot (t - s)^2 \geq -L\), so \(-Q(w) \leq L\).
Then \(h(w) < L - (Q(w)) \leq L + L = 2L\), so \(h(w) < 2L\).
Recall, from the start of the proof: \(h(w) \geq 2L\). Contradiction. \(\square\)

**THEOREM 38.** Let \(f : \mathbb{R} \rightarrow \mathbb{R}, \ s, t \in \mathbb{R}, \ M > 0\).
Assume \(s < t\). Assume \(t - s \leq 1\).
Assume \(f\) has \(M\)-BED on \((s, t)\). Assume \(f\) has \(UBED\) on \((s; t)\).
Then \(f\) has \(2M\)-BED on \((s; t)\).

Proof. Given \(p \in (s; t), \ want: f\ has 2M-BED at \(p\).
Given \(f \in \mathbb{N}_0, \ want: |f^{(2j)}(p)| \leq 2M\).
Assume: \(|f^{(2j)}(p)| > 2M\). Want: Contradiction.
Since \(|f^{(2j)}(p)| > 2M\), we get: \(|f^{(2j)}(p)| > 2M\).
For all \(i \in \mathbb{N}_0, \ L_i := 4^i \cdot M\). Then: \(\forall i \in \mathbb{N}_0, \ L_i \geq 0\).
Also, \(L_0 = M\) and \(\forall i \in \mathbb{N}_0, \ L_{i+1} = 4L_i\).
For all \(i \in \mathbb{N}_0, \ let \ B_i := \{q \in (s; t) \text{ s.t. } |f^{(2j+2i)}(q)| \geq 2L_i\}.\)
Claim: \( \forall i \in \mathbb{N}_0, \ B_i \neq \emptyset. \)

Proof of Claim: We have \( |f^{(2j+20)}(p)| = |f^{(2j)}(p)| \geq 2M = 2L_0. \)
Also, \( p \in (s;t) \). Then \( p \in B_0 \). Then \( B_0 \neq \emptyset \).
We proceed by mathematical induction:

Given \( i \in \mathbb{N}_0 \), assume \( B_i \neq \emptyset \), **want:** \( B_{i+1} \neq \emptyset \).
Choose \( w \in B_i \). Then \( w \in (s;t) \) and \( |f^{(2j+2i)}(w)| \geq 2L_i. \)
By hypothesis, \( f \) has \( M\)-BED on \( \{s,t\} \), so \( s, t \in D_f^{(x)}. \)
By hypothesis, \( f \) has \( M\)-BED on \( \{s,t\} \), so \( |f^{(2j+2i)}(w)| \leq M \) on \( \{s,t\} \).
By hypothesis, \( f \) has UBED on \( (s;t) \), so \( (s,t) \subseteq D_f^{(x)}. \)
Let \( g := f^{(2j+2i)} \). Then \( (s,t) \subseteq D_f^{(x)} = D_g^{(x)} \subseteq D_g^{(2)} \), so \( (s,t) \subseteq D_g^{(2)}. \)
Since \( s, t \in D_f^{(x)} = D_f^{(x)} = D_g^{(2)} = D_g^{(1)}, \)
we get: \( g \) is differentiable both at \( s \) and at \( t \).
Then \( g \) is continuous both at \( s \) and at \( t \).
Also, \( |g(w)| = |f^{(2j+2i)}(w)| \geq 2L_i \), so \( |g(w)| \geq 2L_i. \)
Also, \( |g| = |f^{(2j+2i)}| \leq M \) on \( \{s,t\} \), so \( |g| \leq M \) on \( \{s,t\} \).
We have: \( M \leq 4^i \cdot M = L_i \). Then \( |g| \leq L_i \) on \( \{s,t\} \).
By Theorem 37, choose \( x \in (s;t) \) s.t. \( |g''(x)| \geq 8L_i. \)
Since \( g'' = (f^{(2j+2i)})'' = f^{(2j+2i+2)} = f^{(2j+2(i+1))}, \)
we get: \( |f^{(2j+2(i+1))}(x)| = |g''(x)|. \)
Then \( |f^{(2j+2(i+1))}(x)| = |g''(x)| \geq 8L_i = 2 \cdot 4L_i = 2L_{i+1}. \)
Also, \( x \in (s;t) \). Then \( x \in B_{i+1} \). Then \( B_{i+1} \neq \emptyset \).

End of proof of Claim.

By hypothesis, \( f \) has UBED on \( (s;t) \), so \( \choose K \geq 0 \) s.t. \( f \) has \( K\)-BED on \( (s;t) \).
By hypothesis, \( M > 0 \), so choose \( n \in \mathbb{N}_0 \) s.t. \( 2 \cdot 4^n \cdot M > K. \)
By the Claim, \( B_n \neq \emptyset \), so choose \( z \in B_n \).
Then, by definition of \( B_n \), we get: \( z \in (s;t) \) and \( |f^{(2j+2n)}(z)| \geq 2L_n. \)
Then \( |f^{(2j+2n)}(z)| \geq 2L_n = 2 \cdot 4^n \cdot M > K, \) so \( |f^{(2j+2n)}(z)| > K. \)
On the other hand, since \( f \) has \( K\)-BED on \( (s;t) \) and since \( z \in (s;t) \), we get: \( |f^{(2j+2n)}(z)| \leq K. \) Contradiction.

**THEOREM 39.** Let \( c, d \in \mathbb{R} \). Assume: \( c < d \). Let \( J := (c;d). \)
Let \( T \subseteq J \). Assume: \( T \) is finite. Let \( q \in T \).
Then: \( \exists \delta > 0 \) s.t. \( (q - \delta; q) \subseteq J \setminus T. \)
The preceding result is basic. Its proof is left as an exercise.
THEOREM 40. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( c, d \in \mathbb{R} \).
Assume: \( c < d \). Let \( J := (c;d) \). Assume: \( J \subseteq \mathbb{D}_f^{(\infty)} \).
Let \( T := J \setminus \text{BD}_f \). Assume: \( T \neq \emptyset \). Then: \( T \) is infinite.

Proof. Assume: \( T \) is finite. \hspace{1cm} \textbf{Want:} Contradiction.
Since \( T \neq \emptyset \), choose \( q \in T \). Then \( q \in J \) and \( q \notin \text{BD}_f \).
By Theorem 39, choose \( \delta > 0 \) s.t. \( (q-\delta; q) \subseteq J \setminus T \).
Since \( (q-\delta; q) \subseteq J \setminus T \subseteq J \) and since \( q \in J \), we get: \( (q-\delta; q) \subseteq J \).
We have: \( (q-\delta; q) \subseteq J \setminus T = J \setminus (J \setminus \text{BD}_f) = J \setminus \text{BD}_f \subseteq \text{BD}_f \).
so \( (q-\delta; q) \subseteq \text{BD}_f \), so \( f \) has PBD on \( (q-\delta; q) \).
So, by Tao’s Theorem (Theorem 30), we get: \( f \) has UBD on \( (q-\delta; q) \).
Choose \( M \geq 0 \) s.t. \( f \) has M-BD on \( (q-\delta; q) \).
So, since \( (q-\delta; q) \subseteq J \subseteq \mathbb{D}_f^{(\infty)} \), by continuity, \( f \) has M-BD at \( q \).
Then \( f \) has BD at \( q \), so \( q \in \text{BD}_f \). Recall: \( q \notin \text{BD}_f \). Contradiction. \( \square \)

THEOREM 41. Let \( T \subseteq \mathbb{R}, \varepsilon > 0 \).
Assume: \( T \) is bounded and infinite.
Then: \( \exists p, q, r \in T \) s.t. \( p < q < r \) and s.t. \( r - p \leq \varepsilon \).

Proof. Since \( T \) is bounded and infinite, choose a limit point \( x \) of \( T \).
Let \( C := [x - (\varepsilon/2); x + (\varepsilon/2)] \). Then \( C \cap T \) is infinite.
Choose \( p, q, r \in C \cap T \) s.t. \( p < q < r \). \hspace{1cm} \textbf{Want:} \( r - p \leq \varepsilon \).
Since \( p, q, r \in C \cap T \subseteq C = [x - (\varepsilon/2); x + (\varepsilon/2)] \), we get: \( r - p \leq \varepsilon \). \( \square \)

THEOREM 42. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \ a, b \in \mathbb{R} \).
Assume: \( a < b \). Let \( I := (a;b) \).
Assume: \( f \) has PBED on \( I \). Then: \( f \) has PBD on \( I \).

Proof. \hspace{1cm} \textbf{Want:} \( I \subseteq \text{BD}_f \).
Let \( V := I \setminus \text{BD}_f \).
Since \( I \setminus \text{BD}_f \) is open in \( \mathbb{R} \), we get: \( V \) is open in \( I \).
Since \( V \subseteq I \setminus \text{BD}_f \subseteq \text{BD}_f \), \hspace{1cm} \textbf{it suffices to show:} \( I \subseteq V \).
Let \( X := I \setminus V \). \hspace{1cm} \textbf{Want:} \( X = \emptyset \).
Assume \( X \neq \emptyset \). \hspace{1cm} \textbf{Want:} Contradiction.
Since \( V \) is open in \( I \) and since \( X = I \setminus V \), we get: \( X \) is closed in \( I \).
Since \( I = (a;b) \), we get: \( I \) is open in \( \mathbb{R} \).
Since \( X \) is closed in \( I \) and since \( I \) is open in \( \mathbb{R} \),
we get: \( X \) is locally compact and Hausdorff.
By hypothesis, \( f \) has PBED on \( I \), so, since \( X = I \setminus V \subseteq I \),
it follows that: \( f \) has PBED on \( X \). Then: \( X \subseteq \mathbb{D}_f^{(\infty)} \).
For all \( m \in \mathbb{N} \) let \( X_m := \{ x \in X | f \text{ has } m\text{-BED at } x \} \).
Then, by continuity, we get: \( \forall m \in \mathbb{N}, \ X_m \text{ is closed in } X \).
Since \( f \) has PBED on \( X \), we get: \( X_1 \cup X_2 \cup X_3 \cup \cdots = X \).

So, since \( X \) is non\( \emptyset \) and locally compact and Hausdorff,
by the Baire Category Theorem,
choose \( M \in \mathbb{N} \) s.t. \( X_M \) has non\( \emptyset \) interior in \( X \).

So, since \( X = I \setminus V \subseteq I = (a; b) \),
by Theorem 7, choose \( c, d \in [a; b] \)
s.t. \( c < d \) and s.t. \( \emptyset \neq (c; d) \cap X \subseteq X_M \).

Then: \( a < c < d < b \). Then: \( (c; d) \subseteq (a; b) \).

Let \( J := (c; d) \). Then: \( J \) is open in \( \mathbb{R} \), so \( J^\circ = J \).

Also, \( J = (c; d) \subseteq (a; b) = I \), so: \( J \subseteq I \). Then \( J \setminus V = J \cap (I \setminus V) \).

Since \( J \setminus V = J \cap (I \setminus V) = J \cap X = (c; d) \cap X \),
we get: \( J \setminus V = (c; d) \cap X \).

So, since \( \emptyset \neq (c; d) \cap X \subseteq X_M \), we get: \( \emptyset \neq J \setminus V \subseteq X_M \).

Since \( J \setminus V \neq \emptyset \), we get: \( J \notin V \).

Since \( J \notin V = \text{IBD}_f \cap I \) and since \( J \subseteq I \), we get: \( J \notin \text{IBD}_f \).

Since \( J^\circ = J \notin \text{IBD}_f = (\text{BD}_f)^\circ \), we get \( J^\circ \notin (\text{BD}_f)^\circ \), and so \( J \notin \text{BD}_f \).

Then: \( J \setminus \text{BD}_f \neq \emptyset \). Let \( T := J \setminus \text{BD}_f \). Then \( T \neq \emptyset \).

By hypothesis, \( f \) has PBED on \( I \), so, since \( J \subseteq I \),
it follows that: \( f \) has PBED on \( J \). Then \( J \subseteq \mathbb{D}_f^{(\infty)} \).

Then, by Theorem 40, we get: \( T \) is infinite.

Also, \( T = J \setminus \text{BD}_f \subseteq J = (c; d) \), so \( T \subseteq (c; d) \). Then \( T \) is bounded.

By Theorem 41, choose \( p, q, r \in T \) s.t. \( p < q < r \) and s.t. \( r - p \leq 1 \).

Then: \( p, q, r \in T \subseteq (c; d) \). Then: \( a < c < p < q < r < d < b \).

Then: \( [p; r] \subseteq (c; d) \). By Theorem 35, \( \text{IBD}_f \cap J \) is dense in \( J \).

Let \( W := \text{IBD}_f \cap J \). Then: \( W \) is dense in \( J \).

Since \( J \subseteq I \), we get: \( J = I \setminus J \). Then \( W = J \cap \text{IBD}_f \cap I \).

By definition of \( V \), we have: \( V = \text{IBD}_f \cap I \). Then: \( W = J \cap V \).

So, since \( J \setminus V = J \setminus (J \cap V) \), we get: \( J \setminus V = J \setminus W \).

Recall: \( \emptyset \neq J \setminus V \subseteq X_M \).

Since \( J \setminus W = J \setminus V \subseteq X_M \), we get: \( J \setminus W \subseteq X_M \).

We have \( (p; r) \subseteq [p; r] \subseteq (c; d) = J \), so \( (p; r) \subseteq J \).

Then: \( (p; r) \) is an open subset of \( J \).

So, since \( W \) is dense in \( J \), we get: \( W \cap (p; r) \) is dense in \( (p; r) \).

We have \( p, q, r \in T = J \setminus \text{BD}_f \). Then \( p, q, r \notin \text{BD}_f \).

Since \( p < q < r \), we get: \( q \in (p; r) \).

Since \( q \notin \text{BD}_f \), we get: \( f \) does not have BD at \( q \).

So, since \( q \in (p; r) \), we get: \( f \) does not have PBD on \( (p; r) \).

Then \( f \) does not have UBD on \( (p; r) \).

Then, by Theorem 32, \( f \) does not have UBED on \( (p; r) \).
Then: \( f \) does not have 2IM-BED on \((p; r)\).

So, since \((p; r) \subseteq J \subseteq \mathbb{D}_f^{(\infty)}\) and

\[
W \cap (p; r) \text{ is dense in } (p; r), \text{ by continuity,}
\]

we get: \( f \) does not have 2IM-BED on \(W \cap (p; r)\).

Choose \( w \in W \cap (p; r) \) s.t. \( f \) does not have 2IM-BED at \( w \).

Then: \( a \leq c < p < w < r < d \leq b \). Also, \( w \in W \).

By definition of \( W \), we have: \( W = \text{IBD}_f \cup J \).

So, since \( \text{IBD}_f \) is open in \( \mathbb{R} \), we get: \( W \) is an open subset of \( J \).

So, since \( J = (c; d) \), we get: \( W \) is an open subset of \((c; d)\).

Since \( p, r \notin \text{BD}_f \supseteq \text{IBD}_f \supseteq \text{BD}_f \cap J = W \), we get: \( p, r \notin W \).

Let \( U \) be the connected component of \( W \) s.t. \( w \in U \). Then: \( w \in U \subseteq W \).

By Theorem 6, choose \( s, t \in [p; r] \setminus W \) s.t. \( s < t \) and s.t. \( U = (s; t) \).

Then \( p \leq s < t \leq r \). Since \( w \in U = (s; t) \), we get: \( s < w < t \).

Then: \( a \leq c < p \leq s < w < t \leq r < d \leq b \).

Since \( p \leq s < t \leq r \), we get: \( t - s \leq r - p \).

So, since \( r - p \leq 1 \), we get: \( t - s \leq 1 \).

Since \((s; t) = U \subseteq W = \text{IBD}_f \cup J \subseteq \text{IBD}_f \subseteq \text{BD}_f \),

we get: \( f \) has PBD on \((s; t)\).

Then, by Tao’s Theorem (Theorem 30), we get: \( f \) has UBD on \((s; t)\).

Then: \( f \) has UBED on \((s; t)\). Since \( M \in \mathbb{N} \), we get: \( M > 0 \).

Recall: \( J \setminus W \subseteq X_M \) and \( J = (c; d) \) and \([p; r] \subseteq (c; d)\).

Since \( s, t \in [p; r] \setminus W \subseteq (c; d) \setminus W = J \setminus W \subseteq X_M \),

by definition of \( X_M \), we get: \( f \) has M-BED on \((s; t)\).

Then, by Theorem 38, we get: \( f \) has 2M-BED on \((s; t)\).

So, since \( w \in U = (s; t) \), we get: \( f \) has 2M-BED at \( w \).

By choice of \( w \), \( f \) does not have 2M-BED at \( w \). Contradiction. \( \square \)

**DEFINITION 43.** Let \( \mu : \mathbb{R} \rightarrow \mathbb{R}, \quad I \subseteq \mathbb{R} \).

By \( \mu \) is **affine** on \( I \), we mean: \( I \subseteq \mathbb{D}_\mu \) and

\[
\exists m, c \in \mathbb{R} \text{ s.t., } \forall x \in I, \quad \mu(x) = mx + c.
\]

**THEOREM 44.** Let \( \mu : \mathbb{R} \rightarrow \mathbb{R}, \quad a, b \in \mathbb{R} \).

Assume \( a < b \). Let \( I := (a; b) \). Assume: \( I \subseteq \mathbb{D}_\mu \).

Then:

\[
( \mu \text{ is affine on } I ) \iff ( \mu'' = 0 \text{ on } I ) \iff \left( \forall p, q \in I, \forall t \in [0; 1], \mu((1 - t) \cdot p + t \cdot q) = (1 - t) \cdot (\mu(p)) + t \cdot (\mu(q)) \right).
\]

The preceding result is basic. Its proof is left as an exercise.
THEOREM 45. Let \( a, b \in \mathbb{R} \). Assume \( a < b \). Let \( I := (a; b) \).
Let \( \lambda_0, \lambda_1, \lambda_2 \ldots : I \to \mathbb{R} \). Assume: \( \forall j \in \mathbb{N}, \lambda_j \) is affine on \( I \).
Let \( \mu : I \to \mathbb{R} \). Assume: \( \lambda_j \to \mu \) pointwise, as \( j \to \infty \).
Then: \( \mu \) is affine on \( I \).

Proof. Given \( p, q \in I, t \in [0; 1], \) want:
\[
\mu((1-t)p + tq) = (1-t) \cdot (\mu(p)) + t \cdot (\mu(q)).
\]
Since, \( \forall j \in \mathbb{N}_0, \lambda_j \) is affine on \( I \), we get:
\[
\forall j \in \mathbb{N}_0, \lambda_j((1-t)p + tq) = (1-t) \cdot (\lambda_j(p)) + t \cdot (\lambda_j(q)).
\]
So, letting \( j \to \infty \), by pointwise convergence, we get:
\[
\mu((1-t)p + tq) = (1-t) \cdot (\mu(p)) + t \cdot (\mu(q)). \qquad \square
\]

THEOREM 46. Let \( \mu : \mathbb{R} \to \mathbb{R}, I \subseteq \mathbb{R} \).
Assume: \( \mu \) is affine on \( I \). Then: \( \mu \) is Lipschitz on \( I \).

Proof. Choose \( m, c \in \mathbb{R} \) s.t., \( \forall x \in I, \mu(x) = mx + c \).
Want: \( \mu \) is \(|m|\)-Lipschitz on \( I \).
Given \( p, q \in I \), want: \(|(\mu(q)) - (\mu(p))| \leq |m| \cdot |q - p|\).
We have:\( (\mu(q)) - (\mu(p)) = (mq + c) - (mp + c) = m \cdot (q - p) \).
Then: \(|(\mu(q)) - (\mu(p))| = |m \cdot (q - p)| = |m| \cdot |q - p| \).
Then: \(|(\mu(q)) - (\mu(p))| \leq |m| \cdot |q - p| \). \quad \square

THEOREM 47. Let \( \phi : \mathbb{R} \to \mathbb{R}, a, b \in \mathbb{R}, M \geq 0 \).
Assume: \( a < b \). Let \( I := (a; b) \). Assume: \( \phi \) is \( M \)-Lipschitz on \( I \).
Let \( c \in I \). Let \( M' := |\phi(c)| + M \cdot (b - a) \). Then: \(|\phi| \leq M' \) on \( I \).

Proof. Given \( x \in I \), want: \(|\phi(x)| \leq M' \).
Since \( x, x \in I = (a; b) \), we get: \(|x - c| < b - a \).
So, since \( M \geq 0 \), we get: \( M \cdot |x - c| \leq M \cdot (b - a) \).
Since \( \phi \) is \( M \)-Lipschitz on \( I \), we get: \(|(\phi(x)) - (\phi(c))| \leq M \cdot |x - c| \).
Then: \(|\phi(x)| = |[\phi(c)] + [(\phi(x)) - (\phi(c))]| \leq |\phi(c)| + |(\phi(x)) - (\phi(c))| \leq |\phi(c)| + M \cdot |x - c| \leq |\phi(c)| + M \cdot (b - a) = M' \). \quad \square

THEOREM 48. Let \( f : \mathbb{R} \to \mathbb{R}, a, b \in \mathbb{R}, M \geq 0 \).
Assume: \( a < b \). Let \( I := (a; b) \). Assume: \( \phi \) is Lipschitz on \( I \).
Then: \( \phi \) is bounded and continuous on \( I \).

Proof. Since \( \phi \) is Lipschitz on \( I \), we get: \( \phi \) is continuous on \( I \).
It remains to show: \( \phi \) is bounded on \( I \).
Since \( \phi \) is Lipschitz on \( I \), choose \( M \geq 0 \) s.t. \( \phi \) is \( M \)-Lipschitz on \( I \).
Let \( c := (a + b)/2 \). Then \( c \in I \). Let \( M' := |\phi(c)| + M \cdot (b - a) \).
By Theorem 47, we get: \(|\phi| \leq M' \) on \( I \). Then \( \phi \) is bounded on \( I \). \quad \square
DEFINITION 49. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad a, b \in \mathbb{R} \).
Assume: \( a < b \). Let \( I := (a; b) \). Let \( c := (a + b)/2 \).
Assume: \( f \) is bounded and measurable on \( I \).

Then \( \left[ f^\# \right]_I : I \rightarrow \mathbb{R} \) is defined by: \( \forall x \in I, \quad f^\#(x) = \int_c^x f \).

THEOREM 50. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad a, b \in \mathbb{R} \).
Assume: \( a < b \). Let \( I := (a; b) \).
Assume: \( f \) is bounded and continuous on \( I \).
Then: \( f^\# = f \) on \( I \).

Theorem 50 is a case of the Fundamental Theorem of Calculus.

THEOREM 51. Let \( a, b \in \mathbb{R} \). Assume: \( a < b \). Let \( I := (a; b) \).
Let \( f_0, f_1, f_2, \ldots : I \rightarrow \mathbb{R} \) be measurable. Let \( g : I \rightarrow \mathbb{R} \).
Let \( M \geq 0 \). Assume: \( \forall j \in \mathbb{N}_0, \quad |f_j| \leq M \) on \( I \).
Assume: \( f_j \rightarrow g \) pointwise on \( I \), as \( j \rightarrow \infty \).
Then: \( g \) is bounded and measurable on \( I \) and \( (f_j)^\# \rightarrow g^\# \) pointwise on \( I \), as \( j \rightarrow \infty \).

Proof. Since \( \forall j \in \mathbb{N}_0, \quad |f_j| \leq M \) on \( I \)
and since \( f_j \rightarrow g \) pointwise on \( I \), as \( j \rightarrow \infty \),
we get \( |g| \leq M \) on \( I \), so \( g \) is bounded on \( I \).
Since a pointwise limit of measurable functions is measurable,
we get: \( g \) is measurable on \( I \).

It remains to show: \( (f_j)^\# \rightarrow g^\# \) pointwise on \( I \), as \( j \rightarrow \infty \).

Given \( x \in I \), want: \( (f_j)^\#(x) \rightarrow g^\#(x) \), as \( j \rightarrow \infty \).

Let \( c := (a + b)/2 \). Then: \( g^\#(x) = \int_c^x g \).

Also, we have: \( \forall j \in \mathbb{N}_0, \quad (f_j)^\#(x) = \int_c^x f_j \)
Since \( \forall j \in \mathbb{N}_0, \quad |f_j| \leq M \) on \( I \) and
since \( f_j \rightarrow g \) pointwise on \( I \), as \( j \rightarrow \infty \),
by the Dominated Convergence Theorem, we get:

\[
\int_c^x f_j \rightarrow \int_c^x g, \quad \text{as } j \rightarrow \infty.
\]

Then: \( (f_j)^\#(x) \rightarrow g^\#(x) \), as \( j \rightarrow \infty \).

THEOREM 52. Let \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad a, b \in \mathbb{R}, \quad M \geq 0 \).
Assume: \( a < b \). Let \( I := (a; b) \).
Assume: \( f \) is measurable on \( I \). Assume: \( |f| \leq M \) on \( I \).
Then: \( f_I^\# \) is \( M \)-Lipschitz on \( I \).

**Proof.** Given \( s, t \in I \), assume \( s < t \),

want: \( |(f_I^\#(t)) - (f_I^\#(s))| \leq M \cdot (t - s) \).
Since \( s, t \in I \) and since \( I \) is an interval, we get: \( [s; t] \subseteq I \).
Then: \( |f| \leq M \) on \( [s; t] \). Let \( c := (a + b)/2 \).
Then: 
\[
(f_I^\#(t)) - (f_I^\#(s)) = \left( \int_c^t f \right) - \left( \int_c^s f \right) = \int_s^t f.
\]
Then: 
\[
|(f_I^\#(t)) - (f_I^\#(s))| \leq \int_s^t |f|.
\]
So, since \( |f| \leq M \) on \( [s; t] \), we get: 
\[
|(f_I^\#(t)) - (f_I^\#(s))| \leq \int_s^t M.
\]
Then: 
\[
|(f_I^\#(t)) - (f_I^\#(s))| \leq M \cdot (t - s).
\]

**THEOREM 53.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( a, b \in \mathbb{R} \).
Assume \( a < b \). Let \( I := (a; b) \).
Assume: \( f \) is bounded and measurable on \( I \).
Then: \( f_I^\# \) is bounded and continuous on \( I \).

**Proof.** Since \( f \) is bounded on \( I \), choose \( M \geq 0 \) s.t. \( |f| \leq M \) on \( I \).
By Theorem 52, \( f_I^\# \) is \( M \)-Lipschitz on \( I \), so \( f_I^\# \) is Lipschitz on \( I \).
Then, by Theorem 48, \( f_I^\# \) is bounded and continuous on \( I \). \( \square \)

**DEFINITION 54.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( a, b \in \mathbb{R} \).
Assume \( a < b \). Let \( I := (a; b) \).
Assume: \( f \) is bounded and measurable on \( I \).
Then: 
\[
\left[ f_I^\# \right]^\# := (f_I^\#)^\#.
\]
Implicit in Definition 54 is that, by Theorem 53,
\( f_I^\# \) is bounded and continuous on \( I \),
and so \( f_I^\# \) is bounded and measurable on \( I \).

**THEOREM 55.** Let \( g : \mathbb{R} \rightarrow \mathbb{R} \), \( a, b \in \mathbb{R} \).
Assume: \( a < b \). Let \( I := (a; b) \).
Assume: \( g \) is bounded and continuous on \( I \).
Then: 
\[
(g_I^\#)^\# = g \text{ on } I.
\]

**Proof.** By Theorem 50, we get: 
\( (g_I^\#)' = g \) on \( I \).
Let \( h := g_I^\# \). Then \( h' = g \).
Since \( g \) is continuous on \( I \), we get: \( g \) is measurable on \( I \).
Then, by Theorem 53, we get: \( g_I^\# \) is bounded and continuous on \( I \).

So, since \( h = g_I^\# \), we get: \( h \) is bounded and continuous on \( I \).

So, by Theorem 50, we get: \( (h_I^\#)' = h \) on \( I \).

So, since \( h' = g \) on \( I \), we get: \( (h_I^\#)'' = g \) on \( I \).

Then: \( (g_I^\#)'' = ((g_I^\#)')' = (h_I^\#)'' = g \) on \( I \).\( \square \)

**THEOREM 56.** Let \( f : \mathbb{R} \to \mathbb{R} \), \( a, b \in \mathbb{R} \).

Assume: \( a < b \). Let \( I := (a; b) \). Assume: \( I \subseteq \mathbb{D}_I^2 \).

Assume: \( f'' \) is bounded and continuous on \( I \).

Then: \( (f''_I^\#)_i^\# - f \) is affine on \( I \).

**Proof.** Let \( \phi := (f'')_I^\# \).\( \textbf{Want:} \phi - f \) is affine on \( I \).

\( \textbf{Want:} (\phi - f)'' = 0 \) on \( I \). \( \textbf{Want:} \phi'' = f'' \) on \( I \).

Let \( g := f'' \). By hypothesis, \( g \) is bounded and continuous on \( I \).

Then, by Theorem 55, we get: \( (g_I^\#)'' = g \) on \( I \).

Then: \( \phi'' = ((f''_I^\#)_i^\#)' = (g_I^\#)' = g = f'' \) on \( I \).\( \square \)

**THEOREM 57.** Let \( a, b \in \mathbb{R} \). Assume \( a < b \). Let \( I := (a; b) \).

Let \( S := C^\infty(I, \mathbb{R}) \). Define \( L : S \to S \) by: \( \forall h \in S, \ Lh = h'' \).

Let \( f \in S \). Let \( g : I \to \mathbb{R} \). Assume \( f, Lf, L^2f, \ldots \to g \) pointwise on \( I \).

Then: \( g \in S \) and \( Lg = g \).

**Proof.** It suffices to show: \( g'' = g \).

We have: \( \forall j \in \mathbb{N}_0, \ L^j f = f^{(2j)} \).

Then: \( f^{(2j)} \to g \) pointwise on \( I \), as \( j \to \infty \).

It follows that: \( f \) has PBD on \( I \).

Then, by Theorem 42, we get: \( f \) has PBD on \( I \).

Then, by Tao’s Theorem (Theorem 30), we get: \( f \) has UBD on \( I \).

Then: \( f \) has UBED on \( I \). Choose \( M \geq 0 \) s.t. \( f \) has M-BED on \( I \).

Then: \( \forall j \in \mathbb{N}_0, \ |f^{(2j)}| \leq M \) on \( I \).

For all \( j \in \mathbb{N}_0 \), let \( f_j := L^j f \). Then: \( \forall j \in \mathbb{N}_0, \ f_j = f^{(2j)} \).

Then: \( f_j \to g \) pointwise on \( I \), as \( j \to \infty \).

Also, \( \forall j \in \mathbb{N}_0, \ |f_j| \leq M \) on \( I \).

Then, since \( f_j \to g \) pointwise on \( I \), as \( j \to \infty \), by Theorem 51, \( g \) is bounded and measurable on \( I \) and \( (f_j)_i^\# \to g_I^\# \) pointwise on \( I \), as \( j \to \infty \).

By Theorem 52, we get: \( \forall j \in \mathbb{N}_0, \ (f_j)_i^\# \) is \( M \)-Lipschitz on \( I \).

Let \( c := (a + b)/2 \). Then: \( \forall j \in \mathbb{N}_0, \ (f_j)_I^\#(c) = 0 \).

Let \( M' := M \cdot (b - a) \). Then \( M' \geq 0 \).

Also, \( \forall j \in \mathbb{N}_0, \ M' = |(f_j)_I^\#(c)| + M \cdot (b - a) \).
Then, by Theorem 47, we get: \( \forall j \in \mathbb{N}_0, \quad |(f_j)_I^\#| \leq M' \) on \( I \).

Then, since \( (f_j)_I^\# \rightarrow g_I^\# \) pointwise on \( I \), as \( j \rightarrow \infty \), by Theorem 51, \( g_I^\# \) is bounded and measurable on \( I \) and 
\[
(f_j)_I^\# \rightarrow g_I^\# \quad \text{pointwise on } I, \text{ as } j \rightarrow \infty.
\] Re却:
\[
(f_j)_I^\# - f_j \rightarrow g_I^\# - g \quad \text{pointwise on } I, \text{ as } j \rightarrow \infty.
\] 为所有 \( j \in \mathbb{N}_0 \), let \( \lambda_j := (f_j)_I^\# - f_j \). Let \( \mu := g_I^\# - g \).

Then \( \lambda_j \rightarrow \mu \) pointwise on \( I \), as \( j \rightarrow \infty \). Also, \( g = g_I^\# - \mu \).

Since \( f \in S = C^\infty(I, \mathbb{R}) \) and since \( \forall j \in \mathbb{N}_0, \quad f_j^\# = (L^j f)^\# = (f(2^j))^\# = (2^j f)^\# = f_{j+1} \), we conclude:
\[
\forall j \in \mathbb{N}_0, \quad I \subseteq \mathbb{D}_{f_j}^{(2)} \quad \text{and} \quad f_j^\# \text{ is continuous on } I.
\] We have:
\[
\forall j \in \mathbb{N}_0, \quad f_j^\# = L f_j = L L^j f = L^{j+1} f = f_{j+1}.
\] Then:
\[
\forall j \in \mathbb{N}_0, \quad |f_j^\#| = |f_{j+1}| \leq M \text{ on } I.
\] Then:
\[
\forall j \in \mathbb{N}_0, \quad f_j^\# \text{ is bounded on } I.
\] Then, by Theorem 56, we have:
\[
\forall j \in \mathbb{N}_0, \quad (f_j^\#)_I^\# - f_j \text{ is affine on } I.
\] So, since \( \forall j \in \mathbb{N}_0, \quad \lambda_j = (f_j^\#)_I^\# - f_j \),
we get:
\[
\forall j \in \mathbb{N}_0, \quad \lambda_j \text{ is affine on } I.
\] So, since \( \lambda_j \rightarrow \mu \) pointwise on \( I \), as \( j \rightarrow \infty \),
by Theorem 45, we get:
\[
\mu \text{ is affine on } I.
\] So, by Theorem 46, we get:
\[
\mu \text{ is Lipschitz on } I.
\] Then, by Theorem 48, we get:
\[
\mu \text{ is bounded and continuous on } I.
\] Recall:
\[
g_I^\# \text{ is bounded and measurable on } I.
\] So, by Theorem 53,
\[
g_I^\# \text{ is bounded and continuous on } I.
\] Then, since \( g = g_I^\# - \mu \), we get:
\[
g \text{ is bounded and continuous on } I.
\] Then, by Theorem 55, we get:
\[
(g_I^\#)^\# = g.
\] Since \( \mu \) is affine on \( I \), we get:
\[
\mu'' = 0.
\] Then, by subtracting, we get:
\[
(g_I^\# - \mu)^\# = g.
\] So, since \( g = g_I^\# - \mu \), we get:
\[
g'' = g. \quad \square