Central Limit Theorem, Fourier Analysis and Finance

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Applied Coin-Flipping

$N = 10^{10^{100}}$ $\quad$ N coin flips $\quad$ H heads, T tails

Male height (inches): $69 + 5\dfrac{H-T}{\sqrt{N}}$

Probability that: $69 - 5 \le \text{ht} \le 69 + 5$?

$$\cancel{69} - 5 \le \cancel{69} + 5\frac{H-T}{\sqrt{N}} \le \cancel{69} + 5$$

$$-5 \le 5\frac{H-T}{\sqrt{N}} \le 5$$

DIVIDE BY 5

$$-1 \le \frac{H-T}{\sqrt{N}} \le 1$$

Applied Coin-Flipping

$N = 10^{10^{100}}$ $\quad$ N coin flips $\quad$ H heads, T tails

Male height (inches): $69 + 5\frac{H-T}{\sqrt{N}}$

Probability that: $69 - 5 \le \text{ht} \le 69 + 5$?

Probability that: $-1 \le \frac{H-T}{\sqrt{N}} \le 1$?

$$-1 \le \frac{H-T}{\sqrt{N}} \le 1$$

Applied Coin-Flipping

$N = 10^{10^{100}}$ N coin flips H heads T tails

Male height (inches): $69 + 5\dfrac{H-T}{\sqrt{N}}$ square root

Probability that: $69 - 5 \le \text{ht} \le 69 + 5$?

Probability that: $-1 \le \dfrac{H-T}{\sqrt{N}} \le 1$?

Answer: $\approx 68\%$

Grav accel (ft/sec^2): $32 + 10^6\dfrac{H-T}{N}$

NO square root

Probability that: $32 - \boxed{\dfrac{10^9}{\sqrt{N}}} \le \text{acc} \le 32 + \boxed{\dfrac{10^9}{\sqrt{N}}}$?

EXTREMELY small

Probability that: $-1000 \le \dfrac{H-T}{\sqrt{N}} \le 1000$?

EXTREMELY close to 100%

Applied Coin-Flipping

N = number of seconds in 30 days

Current stock price: 1 USD

$$x_+ := \begin{cases} x, & \text{if } x \geq 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

S := stock price
30 days from now

Contract pays: $(S-1)_+$ USD,
30 days from now

Expected payout?

Each second, price changes
either by a factor of 1.000035616
or by a factor of 0.999964386.

50% chance of uptick,
50% chance of downtick.

Applied Coin-Flipping

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $(u^H d^T - 1)_+$,
30 days from now.

Contract pays: $(S-1)_+$ USD,
30 days from now

Expected payout?

Each second, price changes
either by a factor of 1.000035616 $=: u$
or by a factor of 0.999964386. $=: d$

50% chance of uptick,
50% chance of downtick.

Applied Coin-Flipping

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $(u^H d^T - 1)_+$,
30 days from now.

Expected payout?

Computing probabilities is relatively easy,
Expected payout?
computing expected values is generally harder.

Compute the probability that

$$-1 < \frac{H - T}{\sqrt{N}} < 1.$$

$$X := (H - T)/\sqrt{N}$$

Compute the probability that

$$-1 < X < 1.$$

$H_1 :=$ number of heads after first flip

$H_2 :=$ number of heads after second flip

$\vdots$

$H_N :=$ number of heads after Nth flip $= H$

Compute the probability that

$$-1 < \frac{H-T}{\sqrt{N}} < 1.$$

$$\boxed{X := (H-T)/\sqrt{N}}$$

Compute the probability that

$$-1 < \boxed{X} < 1.$$

X is hard . . .

For all integers $j \in [1, N]$,

$H_j :=$ number of heads after jth flip

$T_j :=$ number of tails after jth flip

$D_j := H_j - T_j$

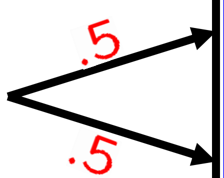
Easier: D_1, $D_1/7$, D_2, D_N

$H = H_N$, $T = T_N$,

$$\boxed{X} = (H_N - T_N)/\sqrt{N}$$
$$= \boxed{D_N/\underline{\sqrt{N}}}$$

$\boxed{D_1} = \overset{0}{H_1} - \overset{1}{T_1}$:

random variable
a variable whose value is determined by random events



1	0.5
-1	0.5

distribution of D_1

distribution of $T_1 - H_1$
is exactly the same

keep the distribution
forget its origin

divide by 7

$D_1 \cancel{= H_1 - T_1}$:

What about $D_1/7$?

Cannot recover the random variable!
Only its distribution.

0.5	1
0.5	−1

z^1 — z

z^{-1}

Fourier transform of the
$\boxed{\text{distribution of } D_1}$
is $\cos t$

$i = \sqrt{-1}$

$T_1 - H_1$ has the same distribution.

Replace z by e^{-it}

Generating ~~function~~:
(expression of z)

Fourier transform:
$\xi\, t$
not time

keep the distribution
forget its origin

$$\cancel{\begin{array}{c} (0.5)z + (0.5)z^{-1} \\ (0.5)e^{-it} + (0.5)e^{it} \\ \| \\ \boxed{\cos t} \end{array}}$$

Repl. t by $t/7$

$0.5 \times [\; e^{it} = \cos t + \cancel{i \sin t} \;] +$

$0.5 \times [\; e^{-it} = \cos t - \cancel{i \sin t} \;]$

Inverse Fourier transform

$D_1/7$:

0.5 — $1/7$ — $z^{1/7}$

0.5 — $-1/7$ — $z^{-1/7}$

What about $D_1/7$?

Replace t by $t/7$.

$i = \sqrt{-1}$

Replace z by e^{-it}

Generating function: $(0.5)z^{1/7} + (0.5)z^{-1/7}$

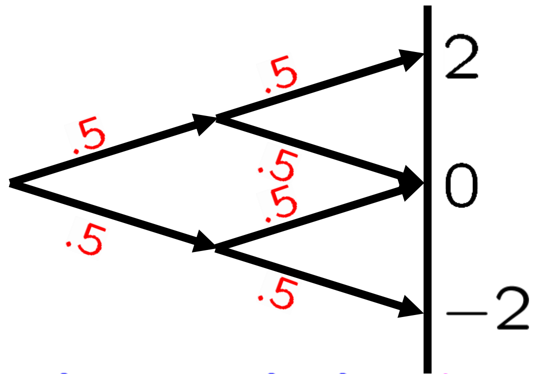
Fourier transform: $(0.5)e^{-it/7} + (0.5)e^{it/7}$

$$\| $$

$$\cos(t/7)$$

$$\boxed{\begin{aligned} e^{it/7} &= \cos(t/7) + i \sin(t/7) \\ e^{-it/7} &= \cos(t/7) - i \sin(t/7) \end{aligned}}$$

$$D_2 = \overset{0}{H_2} - \overset{2}{T_2} :$$



0.25

$0.25 + 0.25 = 0.5$

0.25

forget its origin keep the distribution

$D_2 = H_2 - T_2$:

0.25	2	z^2
0.5	0	$z^0 = 1$
0.25	-2	z^{-2}

forget its origin keep the distribution

Generating function: Watch closely . . . nothing up my sleeve . . .

$$(0.25)z^2 + 0.5 + (0.25)z^{-2}$$

$$= ((0.5)z + (0.5)z^{-1})^2$$

the generating function
of the distribution
of D_1

$i = \sqrt{-1}$

Replace z by e^{-it}

Fourier transform: $(\cos t)^2 = \cos^2 t$

$D_N = \cancel{\boxed{H_N - T_N}}$:

divide by $\sqrt{N}$

NO WAY!!

Goal: $X =$

What about $D_N/\sqrt{N}$?

Replace t by $t/\sqrt{N}$.

Generating function:

NO WAY!!

$= (\underbrace{(0.5)z + (0.5)z^{-1}})^N$

the generating function of the distribution of D_1

$i = \sqrt{-1}$

Replace z by e^{-it}

Fourier transform: $(\cos t)^N = \boxed{\cos^N t}$

$X = D_N/\sqrt{N}$:

NO WAY!!

Goal: $X =$

What about $D_N/\sqrt{N}$?

Replace t by $t/\sqrt{N}$.

Fourier transform: $\cos^N(t/\sqrt{N})$

$X = D_N/\sqrt{N}$:

Generating functions
Fourier transforms

NO WAY!!

Fourier transform: $\cos^N(t/\sqrt{N})$

Fourier transform: $\cos^N(t/\sqrt{N})$

$X = D_N/\sqrt{N}$:

Generating functions
Fourier transforms
Fourier analysis
Spectral theory

Useful?

NO WAY!!

The problem:

Compute the probability that

$$-1 < X < 1.$$

Exercise: $\lim_{n\to\infty} \cos^n(3/\sqrt{n}) = e^{-3^2/2}$

Fourier transform: $\boxed{\cos^N(t/\sqrt{N})}$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

$\boxed{\text{Verify for } t = 3.}$

$$X = D_N/\sqrt{N}:$$

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

Fourier transform: $\cos^N(t/\sqrt{N})$

$$\approx \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = e^{-t^2/2}$$

$\boxed{X} = D_N/\sqrt{N}$:

Fourier transform: $\boxed{\cos^N(t/\sqrt{N})}$

$$\boxed{\approx} \lim_{n\to\infty} \cos^n(t/\sqrt{n}) = \boxed{e^{-t^2/2}}$$

Key idea of Central Limit Theorem:

Let $\boxed{Z}$ have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

in di How to find Z?

Inverse Fourier Transform

The problem:

Compute the probability that

$$-1 < X < 1.$$

Approximately equal to the probability that

$$-1 < Z < 1.$$

Z:

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, \boxed{dx}$$

infinitesimal

x

Do this for *all* $x \in \mathbb{R}$

$\exists$ RV Z with this dist.

NOTES

Mistake:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

$D_2 \in \{2, 0, -2\}$

distribution supported on three points

$D_N \in \{-N, -N+2, \ldots, N-2, N\}$

distribution supported on $N+1$ points

By contrast, the distribution of Z does not have finite support.

Z:

$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$ — x

Do this for *all* $x \in \mathbb{R}$

Problem: Compute the probability that

$$Z = 7$$

Solution: $\displaystyle \int_7^7 \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \;=\; 0$

Z:

$$\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \quad x$$

Do this for *all* $x \in \mathbb{R}$

Problem: Compute the probability that

$$2 < Z < 3$$

Solution:

$$\int_2^3 \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx = [\Phi(x)]_{x=2}^{x=3}$$

$$= \Phi(3) - \Phi(2) = 0.0214$$

$$= 2.14\%$$

Z:

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ | x $\quad z^x$

Do this for *all* $x \in \mathbb{R}$

Generating function:

$$\int_{-\infty}^{\infty} \boxed{z}^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \text{Exercise}$$

Fourier transform:

Verify for $t = 3i$.

$$\int_{-\infty}^{\infty} \boxed{e^{-itx}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

Z:

$X \approx$ $\quad \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \;\Big|\; x \qquad z^x$

Do this for *all* $x \in \mathbb{R}$

Exercise: $$\int_{-\infty}^{\infty} e^{3x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = e^{3^2/2}$$

Fourier transform: $\boxed{\text{Verify for } t = 3i.}$

$$\int_{-\infty}^{\infty} e^{-itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \overset{?}{=} e^{-t^2/2}$$

Key idea of Central Limit Theorem:

Let Z have distr. with Fourier transf. $e^{-t^2/2}$.

Then Z is "close" to X.

Z:

$\approx$

X

$$\boxed{\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx} \; x \qquad z^x$$

probability problems,
then expected value problems

Do this for *all* $x \in \mathbb{R}$

The problem:

Compute the probability that
$-1 < X < 1$.

Approximately equal to the probability that
$-1 < Z < 1$.

Approximate solution:

Berry-Esseen Theorem

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx = [\Phi(x)]_{x=-1}^{x=1}$$
$$= 68.27\%$$

Goal:

Compute the expected value of $f(u^H d^T)$.

equal

Coin-flipping game: Flip a fair coin N times.
If H heads and T tails,
pay $(u^H d^T - 1)_+$,
30 days from now.

$$f(x) = (x - 1)_+$$

$$f(x) = (x-1)_+$$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $f(D_2)$.

$D_2 = H_2 - T_2$:

Probability	D_2	
0.25	2	$f(2)$
0.5	0	$f(0)$
0.25	-2	$f(-2)$

$f :\to g$

works for any function

$$[0.25][f(2)] + [0.5][f(0)] + [0.25][f(-2)] = 0.25$$

Define: $g(x) = 5e^x + x^2$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $g(D_2)$.

$D_2 = H_2 - T_2$:

0.25	2	$g(2)$
0.5	0	$g(0)$
0.25	-2	$g(-2)$

$[0.25][g(2)] + [0.5][g(0)] + [0.25][g(-2)] =$ Exercise

Recall: $\quad f(x) = (x-1)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $f(Z)$.

Z:

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$	x	$f(x)$

Do this for all $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} [f(x)]\, \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$\boxed{f(x) = (x-1)_+}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{[f(x)]}\, e^{-x^2/2}\,dx = \text{exercise}$$

Recall: $f(x) = (x-1)_+$

Goal:

Compute the expected value of $f(u^H d^T)$.

New easier problem:

Compute the expected value of $f(X)$.

Z: $\approx$ X

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ | x | $f(x)$

Do this for all $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} [f(x)]\,\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$\boxed{f(x) = (x-1)_+}$

Approx. Sol'n: $= \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{[f(x)]}\, e^{-x^2/2}\,dx} =$ exercise

write H, T
as expr.s of X

Recall: $f(x) = (x-1)_+$

Goal:

Compute the expected value of $\boxed{f(u^H d^T)}$.

$\| ?$

New easier problem:

Compute the expected value of $\boxed{g(X)}$.

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

write H, T
as expr.s of X

Recall: $f(x) = (x-1)_+$

Goal:
Compute the expected value of $f(u^H d^T)$.

New easier problem:
Compute the expected value of $g(X)$.

$\boxed{X} = \boxed{(H-T)/\sqrt{N}}$ $\quad N = 2,592,000$

$\times\sqrt{N}$ $\quad \times\sqrt{N}$

$H \cancel{+T} = N$ $\quad \cancel{H} + T = N$

$H \cancel{-T} = X\sqrt{N}$ $\quad \cancel{-H} + T = -X\sqrt{N}$

ADD
NEGATE

ADD

$2H = N + X\sqrt{N}$ $\quad 2T = N - X\sqrt{N}$

Approx. Sol'n: $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

Recall: $f(x) = (x-1)_+$

write H, T as expr.s of X

Goal:

Compute the expected value of $\boxed{f(u^H d^T)}$.

$\| ?$

New easier problem:

Compute the expected value of $\boxed{g(X)}$.

$$\boxed{H = N/2 + X\sqrt{N}/2} \qquad \boxed{T = N/2 - X\sqrt{N}/2}$$

$$u^H = u^{N/2} u^{X\sqrt{N}/2} \qquad d^T = d^{N/2} d^{-X\sqrt{N}/2}$$

$$u^H d^T = \quad N := 30 \times 24 \times 60 \times 60 = 2,592,000$$

$$\boxed{2H = N + X\sqrt{N}} \qquad \boxed{2T = N - X\sqrt{N}}$$

Approx. Sol'n: $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$$

Recall: $f(x) = (x-1)_+$

Goal:

Compute the expected value of $\boxed{f(u^H d^T)}$.

$\|$?

New easier problem:

Compute the expected value of $\boxed{g(X)}$.

$$\boxed{H = N/2 + X\sqrt{N}/2} \quad \boxed{T = N/2 - X\sqrt{N}/2}$$

$$u^H = u^{N/2} u^{X\sqrt{N}/2} \qquad d^T = d^{N/2} d^{-X\sqrt{N}/2}$$

$$u^H d^T = \underline{u^{N/2}\ d^{N/2}}\ \ \underline{u^{X\sqrt{N}/2}\ d^{-X\sqrt{N}/2}}$$

$$= \underline{(ud)^{N/2}}\ \ \underline{(u/d)^{\boxed{X}\sqrt{N}/2}}$$

$$= C\ e^{kX}$$

$C := (ud)^{N/2}$

$k := \boxed{\ln}((u/d)^{\sqrt{N}/2})$

Approx. Sol'n: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

$e^k = \boxed{(u/d)^{\sqrt{N}/2}}$

Recall: $f(x) = (x-1)_+$

Goal:

Compute the expected value of $\boxed{f(u^H d^T)}$.

$\|$

New easier problem:

Compute the expected value of $\boxed{g(X)}$.

$$\underline{f(u^H d^T)} = f(Ce^{kX}) = \underline{g(X)}$$

$$\boxed{g(x) := f(Ce^{kx})}$$

$$\begin{aligned}\underline{u^H d^T} &= u^{N/2}\, d^{N/2}\; u^{X\sqrt{N}/2}\, d^{-X\sqrt{N}/2} \\ &= (ud)^{N/2}\; (u/d)^{X\sqrt{N}/2} \\ &= \underline{C\;\; e^{kX}}\end{aligned}$$

$$C := (ud)^{N/2}$$

$$k := \ln((u/d)^{\sqrt{N}/2})$$

Approx. Sol'n: $\displaystyle \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx$

Recall: $f(x) = (x-1)_+$

Goal:

Compute the expected value of $\boxed{f(u^H d^T)}$.

$$\|$$

Restatement of goal:

Compute the expected value of $\boxed{g(X)}$.

$$\boxed{g(x) := f(Ce^{kx})}$$

$$f(u^H d^T) = f(Ce^{kX}) = g(X)$$

$$\begin{aligned} u^H d^T &= u^{N/2}\, d^{N/2}\; u^{X\sqrt{N}/2}\, d^{-X\sqrt{N}/2} \\ &= (ud)^{N/2}\,(u/d)^{X\sqrt{N}/2} \\ &= C\; e^{kX} \end{aligned}$$

$$C := (ud)^{N/2}$$

$$k := \ln((u/d)^{\sqrt{N}/2})$$

Approx. Sol'n: $\boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [g(x)]\, e^{-x^2/2}\, dx}$

Recall: $f(\boxed{x}) = (\boxed{x} - 1)_+$

$\boxed{g(x) := f(\boxed{Ce^{kx}})} = (\boxed{Ce^{kx}} - 1)_+$

$$\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\underset{1.000948567}{\underset{\parallel}{\boxed{C}}} e^{\overset{0.0573390439}{\overset{\parallel}{\boxed{k}}} x} - 1)_+ e^{-x^2/2}\, dx \right]$$

$N = 2{,}592{,}000$

u

1.00010005

0.99989997

d

$C := (ud)^{N/2}$

$k := \ln((u/d)^{\sqrt{N}/2})$

Approx. Sol'n: $\boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{[g(x)]}\, e^{-x^2/2}\, dx}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty} (Ce^{kx} - 1)_+ \, e^{-x^2/2} \, dx$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\underbrace{\boxed{C}}_{\substack{\| \\ 1.000948567}} e^{\overbrace{\boxed{k}}^{\substack{0.0573390439 \\ \|}} x} - 1)_+ \, e^{-x^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \boxed{\int_a^{\infty}} (Ce^{kx} - 1) \boxed{e^{-x^2/2}} \, dx$$

$$Ce^{ka} - 1 = 0$$

$$Ce^{ka} = 1$$

$$e^{ka} = 1/C$$

$$ka = \ln(1/C) = -\ln C$$

$$a = -(\ln C)/k$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(Ce^{kx}-1)_+\, e^{-x^2/2}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\boxed{\int_a^{\infty}}(Ce^{kx}-1)\,\boxed{e^{-x^2/2}}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[C\int_a^{\infty}e^{kx}\,e^{-x^2/2}\,dx-\int_a^{\infty}e^{-x^2/2}\,dx\right]$$

$$a=-(\ln C)/k$$

$$= \frac{1}{\sqrt{2\pi}}\left[C\int_a^\infty e^{kx}\, e^{-x^2/2}\,dx \;-\; \overbrace{\int_a^\infty e^{-x^2/2}\,dx}^{\sqrt{2\pi}\,[\overbrace{1-(\Phi(a))}^{\Phi(-a)}]} \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[C\int_a^\infty e^{kx}\, e^{-x^2/2}\,dx \;-\; \int_a^\infty e^{-x^2/2}\,dx \right]$$

$$\boxed{a = -(\ln C)/k}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\underbrace{C\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)}\right]$$

$$\underbrace{\int_{a-k}^\infty \underbrace{e^{k(x+k)}\, e^{-(x+k)^2/2}}_{\cancel{e^{kx}}\,\boxed{e^{k^2}}\, e^{-x^2/2}\,\boxed{e^{-k^2/2}}\,\cancel{e^{-kx}}}\, dx}$$

$$e^{k^2/2}\underbrace{\int_{a-k}^\infty e^{-x^2/2}\, dx}_{\sqrt{2\pi}\,\Phi(k-a)}$$

$$\boxed{a = -(\ln C)/k}$$

$$= \frac{1}{\sqrt{2\pi}}\left[C\underbrace{\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\,\Phi(-a)} \right]$$

$$\underbrace{\int_{a-k}^\infty \underbrace{e^{k(x+k)}\, e^{-(x+k)^2/2}}_{e^{k^2}\, e^{-x^2/2}\, e^{-k^2/2}}\, dx}$$

$$e^{k^2/2}\underbrace{\qquad\qquad\qquad}_{\sqrt{2\pi}\,\Phi(k-a)}$$

$$\boxed{a = -(\ln C)/k}$$

$$= \frac{1}{\sqrt{2\pi}}\left[C \underbrace{\int_a^\infty e^{kx} e^{-x^2/2}\, dx}_{e^{k^2/2}\sqrt{2\pi}\Phi(k-a)} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\sqrt{2\pi}\Phi(-a)} \right]$$

$$e^{k^2/2}\sqrt{2\pi}\Phi(k-a)$$

$$\boxed{a = -(\ln C)/k}$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}}\left[C \underbrace{\int_a^\infty e^{kx}\, e^{-x^2/2}\, dx}_{e^{k^2/2}\cancel{\sqrt{2\pi}}\,\Phi(k-a)} - \overbrace{\int_a^\infty e^{-x^2/2}\, dx}^{\cancel{\sqrt{2\pi}}\,\Phi(-a)} \right]$$

$$= \overbrace{C\, e^{k^2/2}}^{1.002595363}\left[\Phi(\overbrace{k-a}^{0.073874328})\right] - \left[\Phi(\overbrace{-a}^{0.01653528434})\right]$$

$$= \boxed{0.024214088}$$

☺

$a = -0.01653528434$

$k = 0.0573390439$

$C = 1.000948567$

$$\boxed{a = -(\ln C)/k}$$

SUMMARY:

Coin flipping problems are tractable via CLT, and useful in many applied settings, in particular, finance.