

Points of Density and Continuity in Probability

The main results in this note are:

Theorem 18, Theorem 22, Theorem 24.

DEFINITION 1. Let \mathcal{S} be a set of sets.

$$\text{Then: } \boxed{\bigcup \mathcal{S}} := \begin{cases} \emptyset, & \text{if } \mathcal{S} = \emptyset \\ \bigcup_{S \in \mathcal{S}} S, & \text{if } \mathcal{S} \neq \emptyset. \end{cases}$$

We make a similar convention that an empty sum is equal to 0.

DEFINITION 2. We define $\boxed{\#\emptyset} := 0$.

For any nonempty finite set S ,

$\boxed{\#S}$ denotes the number of elements in S .

For any infinite set S , we define $\boxed{\#S} := \infty$.

DEFINITION 3. Let $\boxed{\mathbb{R}^*} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$.

For all $a, b \in \mathbb{R}^*$, let

$$\boxed{(a; b)} := \{x \in \mathbb{R}^* \mid a < x < b\}, \quad \boxed{[a; b)} := \{x \in \mathbb{R}^* \mid a \leq x < b\},$$

$$\boxed{(a; b]} := \{x \in \mathbb{R}^* \mid a < x \leq b\}, \quad \boxed{[a; b]} := \{x \in \mathbb{R}^* \mid a \leq x \leq b\}.$$

DEFINITION 4. For all $x \in \mathbb{R}^2$, for all $r > 0$, let

$$\boxed{B_x^r} := \{y \in \mathbb{R}^2 \text{ s.t. } |y - x| < r\}.$$

That is: B_x^r is the open disk about x of radius r .

Let $\boxed{\mathcal{B}} := \{B_x^r \mid x \in \mathbb{R}^2, r > 0\}$.

Let $\boxed{\mathcal{T}}$ denote the standard topology on \mathbb{R}^2 ,

so \mathcal{T} is the set of open subsets of \mathbb{R}^2 .

Then: $\forall U \in \mathcal{T}$, U is Lebesgue-measurable. Also, $\mathcal{B} \subseteq \mathcal{T} \setminus \{\emptyset\}$.

DEFINITION 5. Let $x \in \mathbb{R}^2$, $r > 0$, $C := B_x^r$.

Then: $\boxed{\text{rad } C} := r$ and $\boxed{\text{cent } C} := x$ and
 $\forall s > 0$, $\boxed{s \cdot C} := B_x^{s \cdot r}$.

According to the next theorem, if two disks meet,
then the triple of the larger covers the smaller.

THEOREM 6. Let $F, G \in \mathcal{B}$.

Assume: $\text{rad } F \leq \text{rad } G$ and $F \cap G \neq \emptyset$. Then: $3 \cdot G \supseteq F$.

Proof. **Given** $a \in F$, **want:** $a \in 3 \cdot G$.

Since $F \cap G \neq \emptyset$, **choose** $p \in F \cap G$. Then: $p \in F$ and $p \in G$.

Let $x := \text{cent } F$, $y := \text{cent } G$, $r := \text{rad } F$, $s := \text{rad } G$.

Then, by hypothesis, we have: $r \leq s$.

Also, $F = B_x^r$ and $G = B_y^s$ and $3 \cdot G = B_y^{3s}$.

Want: $a \in B_y^{3s}$. **Want:** $|a - y| < 3s$.

Since $a \in F = B_x^r$, we get: $|a - x| < r$.

Since $p \in F = B_x^r$, we get: $|p - x| < r$.

Since $p \in G = B_y^s$, we get: $|p - y| < s$.

Since $r \leq s$, we get: $r + r + s \leq 3s$.

Then $|a - y| \leq |a - x| + |x - p| + |p - y| < r + r + s \leq 3s$. \square

We use “pw-dj” to abbreviate “pairwise-disjoint”.

For any set \mathcal{S} of sets, by $\boxed{\mathcal{S} \text{ is pw-dj}}$,

we mean: $\forall S, T \in \mathcal{S}, (S \neq T) \Rightarrow (S \cap T = \emptyset)$.

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of positive integers.

For any sequence (S_1, S_2, \dots) of sets, by $\boxed{(S_1, S_2, \dots) \text{ is pw-dj}}$,

we mean: $\forall i, j \in \mathbb{N}, (i \neq j) \Rightarrow (S_i \cap S_j = \emptyset)$.

For any $\mathcal{C} \subseteq \mathcal{B}$, for any $s > 0$, **we define:**

$\boxed{s \cdot \mathcal{C}} := \{s \cdot C \mid C \in \mathcal{C}\}$.

THEOREM 7. **Let** $\mathcal{F} \subseteq \mathcal{B}$. *Assume \mathcal{F} is finite.*

Then: $\exists \text{pw-dj } \mathcal{E} \subseteq \mathcal{F} \text{ s.t. } \bigcup(3 \cdot \mathcal{E}) \supseteq \bigcup \mathcal{F}$.

Proof. **Let** $n := \#\mathcal{F}$.

In case $n = 0$, **let** $\mathcal{E} := \emptyset$. We therefore assume $n \geq 1$.

By induction on n , we also assume: $\forall \mathcal{Q} \subseteq \mathcal{B}$,

$(\#\mathcal{Q} < n) \Rightarrow (\exists \text{pw-dj } \mathcal{P} \subseteq \mathcal{Q} \text{ s.t. } \bigcup(3 \cdot \mathcal{P}) \supseteq \bigcup \mathcal{Q})$.

Let $R := \{\text{rad } F \mid F \in \mathcal{F}\}$. Then R is a finite subset of \mathbb{R} .

Let $r := \max R$. Then $r \in R$, so **choose** $G \in \mathcal{F}$ s.t. $\text{rad } G = r$.

Since $G \in \mathcal{F} \subseteq \mathcal{B} \subseteq \mathcal{T} \setminus \{\emptyset\}$, we get: $G \neq \emptyset$.

Let $\mathcal{Q} := \{F \in \mathcal{F} \mid F \cap G = \emptyset\}$. Then $\mathcal{Q} \subseteq \mathcal{F}$ and $G \notin \mathcal{Q}$.

Then $\mathcal{Q} \subseteq \mathcal{F} \setminus \{G\}$, so: $\#\mathcal{Q} \leq \#(\mathcal{F} \setminus \{G\})$.

Since $G \in \mathcal{F}$ and since \mathcal{F} is finite, we get: $\#(\mathcal{F} \setminus \{G\}) < \#\mathcal{F}$.

Since $\#\mathcal{Q} \leq \#(\mathcal{F} \setminus \{G\}) < \#\mathcal{F} = n$, by the induction assumption,

choose a pw-dj $\mathcal{P} \subseteq \mathcal{Q}$ s.t. $\bigcup(3 \cdot \mathcal{P}) \supseteq \bigcup \mathcal{Q}$.

Since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of \mathcal{Q} ,

we get: $\forall P \in \mathcal{P}, P \cap G = \emptyset$.

So, since \mathcal{P} is pw-dj, we get: $\mathcal{P} \cup \{G\}$ is pw-dj.

Since $\mathcal{P} \subseteq \mathcal{Q} \subseteq \mathcal{F}$ and since $G \in \mathcal{F}$, we get: $\mathcal{P} \cup \{G\} \subseteq \mathcal{F}$.

Let $\mathcal{E} := \mathcal{P} \cup \{G\}$. Then: \mathcal{E} is pw-dj and $\mathcal{E} \subseteq \mathcal{F}$.

It remains only to show: $\bigcup(3 \cdot \mathcal{E}) \supseteq \bigcup \mathcal{F}$.

Want: $\forall F \in \mathcal{F}, F \subseteq \bigcup(3 \cdot \mathcal{E})$.

Given $F \in \mathcal{F}$, **want:** $F \subseteq \bigcup(3 \cdot \mathcal{E})$.

Case 1: $F \in \mathcal{Q}$. Proof in Case 1:

Since $\mathcal{P} \subseteq \mathcal{P} \cup \{G\} = \mathcal{E}$, we get $3 \cdot \mathcal{P} \subseteq 3 \cdot \mathcal{E}$, so $\bigcup(3 \cdot \mathcal{P}) \subseteq \bigcup(3 \cdot \mathcal{E})$.

By the choice of \mathcal{P} , we have: $\bigcup(3 \cdot \mathcal{P}) \supseteq \bigcup \mathcal{Q}$.

Since $F \in \mathcal{Q}$, we get: $F \subseteq \bigcup \mathcal{Q}$.

Then: $F \subseteq \bigcup \mathcal{Q} \subseteq \bigcup(3 \cdot \mathcal{P}) \subseteq \bigcup(3 \cdot \mathcal{E})$.

End of proof in Case 1.

Case 2: $F \notin \mathcal{Q}$. Proof in Case 2: Recall: $F \in \mathcal{F}$.

So, by definition of R , we have: $\text{rad } F \in R$. Then $\text{rad } F \leq \max R$.

Since $F \in \mathcal{F}$ and $F \notin \mathcal{Q}$, by definition of \mathcal{Q} , we get: $F \cap G \neq \emptyset$.

So, since $\text{rad } F \leq \max R = r = \text{rad } G$,

by Theorem 6, we get: $3 \cdot G \supseteq F$.

Since $G \in \mathcal{P} \cup \{G\} = \mathcal{E}$, we get $3 \cdot G \in 3 \cdot \mathcal{E}$, so $3 \cdot G \subseteq \bigcup(3 \cdot \mathcal{E})$.

Then: $F \subseteq 3 \cdot G \subseteq \bigcup(3 \cdot \mathcal{E})$.

End of proof in Case 2. □

Let $\boxed{\lambda}$ denote Lebesgue-outer-measure on \mathbb{R}^2 .

THEOREM 8.

Let (A_1, A_2, \dots) *be a sequence of Lebesgue-measurable subsets of* \mathbb{R}^2 .

Then, as $k \rightarrow \infty$, $\lambda(A_1 \cup \dots \cup A_k) \rightarrow \lambda(A_1 \cup A_2 \cup \dots)$.

Proof. For all $k \in \mathbb{N}$, **let** $D_k := A_k \setminus (A_1 \cup \dots \cup A_{k-1})$.

Then, $\forall k \in \mathbb{N}$, D_k is Lebesgue-measurable

and, $\forall k \in \mathbb{N}$, $D_1 \cup \dots \cup D_k = A_1 \cup \dots \cup A_k$

and $D_1 \cup D_2 \cup \dots = A_1 \cup A_2 \cup \dots$

and (D_1, D_2, \dots) is pw-dj.

Since (D_1, D_2, \dots) is pw-dj, by countable-additivity of λ , we get

$$\lambda(D_1 \cup D_2 \cup \dots) = (\lambda(D_1)) + (\lambda(D_2)) + \dots;$$

also, by finite-additivity of λ , we get

$$\forall k \in \mathbb{N}, \lambda(D_1 \cup \dots \cup D_k) = (\lambda(D_1)) + \dots + (\lambda(D_k)).$$

By definition of infinite-summation, we have

$$\text{as } k \rightarrow \infty, (\lambda(D_1)) + \dots + (\lambda(D_k)) \rightarrow (\lambda(D_1)) + (\lambda(D_2)) + \dots.$$

Then: as $k \rightarrow \infty$, $\lambda(D_1 \cup \dots \cup D_k) \rightarrow \lambda(D_1 \cup D_2 \cup \dots)$.
 Then: as $k \rightarrow \infty$, $\lambda(A_1 \cup \dots \cup A_k) \rightarrow \lambda(A_1 \cup A_2 \cup \dots)$. \square

The next result says: for any collection of open disks,
 if its union has finite Lebesgue-measure, then
 \exists finite pw-dj subcollection that covers at least 10% of that union.

THEOREM 9. Let $\mathcal{A} \subseteq \mathcal{B}$. Assume: $\lambda(\bigcup \mathcal{A}) < \infty$.
 Then: \exists finite pw-dj $\mathcal{E} \subseteq \mathcal{A}$ s.t. $\lambda(\bigcup \mathcal{E}) \geq 0.1 \cdot (\lambda(\bigcup \mathcal{A}))$.

Proof. In case $\lambda(\bigcup \mathcal{A}) = 0$, let $\mathcal{E} := \emptyset$.

We therefore assume $\lambda(\bigcup \mathcal{A}) \neq 0$. Then $\lambda(\bigcup \mathcal{A}) > 0$.

By hypothesis, $\lambda(\bigcup \mathcal{A}) < \infty$. Let $c := \lambda(\bigcup \mathcal{A})$.

Then $0 < c < \infty$. Then: $0.9 \cdot c < c$.

Since $\bigcup \mathcal{A}$ is Lindelöf, choose $A_1, A_2, \dots \in \mathcal{A}$
 s.t. $A_1 \cup A_2 \cup \dots = \bigcup \mathcal{A}$.

Since $A_1, A_2, \dots \in \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{T}$, we get:

(A_1, A_2, \dots) is a sequence of Lebesgue-measurable subsets of \mathbb{R}^2 .

So, by Theorem 8, we have:

as $k \rightarrow \infty$, $\lambda(A_1 \cup \dots \cup A_k) \rightarrow \lambda(A_1 \cup A_2 \cup \dots)$.

So, since $0.9 \cdot c < c = \lambda(\bigcup \mathcal{A}) = \lambda(A_1 \cup A_2 \cup \dots)$,

choose $k \in \mathbb{N}$ s.t. $\lambda(A_1 \cup \dots \cup A_k) \geq 0.9 \cdot c$.

Let $\mathcal{F} := \{A_1, \dots, A_k\}$. Then $\lambda(\bigcup \mathcal{F}) \geq 0.9 \cdot c$ and $\mathcal{F} \subseteq \mathcal{A}$.

Also, \mathcal{F} is finite, so, since $\mathcal{F} \subseteq \mathcal{A} \subseteq \mathcal{B}$, by Theorem 7,

choose a pw-dj $\mathcal{E} \subseteq \mathcal{F}$ s.t. $\bigcup(3 \cdot \mathcal{E}) \supseteq \bigcup \mathcal{F}$.

Since $\mathcal{E} \subseteq \mathcal{F}$ and since \mathcal{F} is finite, we get: \mathcal{E} is finite.

Since $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{A}$, it remains only to show: $\lambda(\bigcup \mathcal{E}) \geq 0.1 \cdot (\lambda(\bigcup \mathcal{A}))$.

Since $c = \lambda(\bigcup \mathcal{A})$, we want: $\lambda(\bigcup \mathcal{E}) \geq 0.1 \cdot c$.

We have: $\forall B \in \mathcal{B}$, $\lambda(3 \cdot B) = 9 \cdot (\lambda(B))$.

Since $\bigcup \mathcal{F} \subseteq \bigcup (3 \cdot \mathcal{E})$, by monotonicity and subadditivity of λ ,

$$\lambda(\bigcup \mathcal{F}) \leq \sum_{E \in \mathcal{E}} (\lambda(3 \cdot E)).$$

$$\text{Since } \lambda(\bigcup \mathcal{F}) \leq \sum_{E \in \mathcal{E}} (\lambda(3 \cdot E)) = \sum_{E \in \mathcal{E}} (9 \cdot (\lambda(E))) = 9 \cdot \sum_{E \in \mathcal{E}} (\lambda(E)),$$

$$\text{we get: } (1/9) \cdot (\lambda(\bigcup \mathcal{F})) \leq \sum_{E \in \mathcal{E}} (\lambda(E)).$$

Since $\mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{T}$, we get: $\forall E \in \mathcal{E}$, E is Lebesgue-measurable.

So, since \mathcal{E} is finite and pw-dj, by finite-additivity of λ , we get:

$$\lambda(\bigcup \mathcal{E}) = \sum_{E \in \mathcal{E}} (\lambda(E)).$$

Since $\lambda(\bigcup \mathcal{F}) \geq 0.9 \cdot c$, we get: $(1/9) \cdot (\lambda(\bigcup \mathcal{F})) \geq 0.1 \cdot c$.

Then $\lambda(\bigcup \mathcal{E}) = \sum_{E \in \mathcal{E}} (\lambda(E)) \geq (1/9) \cdot (\lambda(\bigcup \mathcal{F})) \geq 0.1 \cdot c$. \square

Let A and B be sets.

By B is a **superset of A** , we will mean: $B \supseteq A$.

Let \mathcal{B} be a set of sets and let A be a set.

By \mathcal{B} is a **covering of A** , we will mean:

$\bigcup \mathcal{B}$ is a superset of A .

DEFINITION 10. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$.

By \mathcal{V} is a **fine-covering of Q** , we mean:

$$\forall x \in Q, \forall \delta > 0, \exists V \in \mathcal{V} \text{ s.t. } (x \in V) \& (\text{rad } V \leq \delta).$$

NOTE: A fine-covering is a covering, i.e.: $\forall Q \subseteq \mathbb{R}^2, \forall \mathcal{V} \subseteq \mathcal{B}$,
if \mathcal{V} is a fine-covering of Q , then $\bigcup \mathcal{V} \supseteq Q$.

Let Q be a set and let \mathcal{P} be a set of sets. We'll say

\mathcal{P} is **inside Q** if: $\bigcup \mathcal{P} \subseteq Q$.

and \mathcal{P} is **outside Q** if: $(\bigcup \mathcal{P}) \cap Q = \emptyset$.

According to the next theorem,

for any fine-covering $\mathcal{V} \subseteq \mathcal{B}$ of a set $Q \subseteq \mathbb{R}^2$,

for any open $W \subseteq \mathbb{R}^2$,

there is a subset of \mathcal{V} that is

both inside W and a fine-covering of $Q \cap W$.

THEOREM 11. Let $Q, W \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$.

Assume: $W \in \mathcal{T}$ and \mathcal{V} is a fine-covering of Q .

Let $\mathcal{V}' := \{V \in \mathcal{V} \mid V \subseteq W\}$. Then: \mathcal{V}' is a fine-covering of $Q \cap W$.

Proof. **Given** $x \in Q \cap W$, $\delta > 0$,

want: $\exists V \in \mathcal{V}'$ s.t. $(x \in V) \& (\text{rad } V \leq \delta)$.

Since $x \in Q \cap W \subseteq W$ and $W \in \mathcal{T}$, **choose** $\beta > 0$ s.t. $B_x^\beta \subseteq W$.

Let $\alpha := \min\{\beta/2, \delta\}$. Then $\alpha > 0$ and $\alpha \leq \beta/2$ and $\alpha \leq \delta$.

Since $x \in Q \cap W \subseteq Q$ and $\alpha > 0$ and \mathcal{V} is a fine-covering of Q ,

choose $V \in \mathcal{V}$ s.t. $(x \in V) \& (\text{rad } V \leq \alpha)$.

Since $\text{rad } V \leq \alpha \leq \delta$, **it remains only to show:** $V \in \mathcal{V}'$.

By definition of \mathcal{V}' , since $V \in \mathcal{V}$, **we wish to show:** $V \subseteq W$.

Given $v \in V$, **want:** $v \in W$.

Since $B_x^\beta \subseteq W$, **it suffices to show:** $v \in B_x^\beta$. **Want:** $|v - x| < \beta$.

Since $V \in \mathcal{V} \subseteq \mathcal{B}$, **choose** $c \in \mathbb{R}^2$ and $r > 0$ s.t. $V = B_c^r$.

Since $v, x \in V = B_c^r$, we get: $|v - c| < r$ and $|x - c| < r$.

Since $r = \text{rad } B_c^r = \text{rad } V \leq \alpha \leq \beta/2$, we get: $2r \leq \beta$.

Then: $|v - x| \leq |v - c| + |c - x| < r + r = 2r \leq \beta$. \square

According to the next theorem,

for any fine-covering $\mathcal{V} \subseteq \mathcal{B}$ of a set $Q \subseteq \mathbb{R}^2$,

for any open $W \subseteq \mathbb{R}^2$ that is a superset of Q ,

there is a subset of \mathcal{V} that is

both inside W and a fine-covering of Q .

THEOREM 12. Let $W \subseteq \mathbb{R}^2$, $Q \subseteq W$, $\mathcal{V} \subseteq \mathcal{B}$.

Assume: $W \in \mathcal{T}$ and \mathcal{V} is a fine-covering of Q .

Let $\mathcal{V}' := \{V \in \mathcal{V} \mid V \subseteq W\}$. Then: \mathcal{V}' is a fine-covering of Q .

Proof. Since $Q \subseteq W$, we get: $Q \cap W = Q$.

So, by Theorem 11, we get: \mathcal{V}' is a fine-covering of Q . \square

According to the **Carathéodory-condition**,

$\forall Q \subseteq \mathbb{R}^2$, Q is Lebesgue-measurable iff

$$\forall S \subseteq \mathbb{R}^2, \quad \lambda(S) = [\lambda(S \cap Q)] + [\lambda(S \setminus Q)].$$

That is: Q is Lebesgue-measurable iff Q “splits all sets well”.

According to the next theorem,

for any $Q \subseteq \mathbb{R}^2$ of finite Lebesgue-outer-measure,

for any fine-covering \mathcal{V} of Q ,

there is a finite pw-dj subset of \mathcal{V} covering at least 1% of Q .

THEOREM 13. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$.

Assume: \mathcal{V} is a fine-covering of Q . Assume: $\lambda(Q) < \infty$.

Then: \exists finite pw-dj $\mathcal{E} \subseteq \mathcal{V}$ s.t. $\lambda(Q \cap (\cup \mathcal{E})) \geq 0.01 \cdot (\lambda(Q))$.

Idea of proof: In case $\lambda(Q) = 0$, let $\mathcal{E} := \emptyset$, so assume $\lambda(Q) > 0$.

Let $\varepsilon := 0.1 \cdot (\lambda(Q))$. By outer-regularity of λ , **choose** $W \in \mathcal{T}$

s.t. $W \supseteq Q$ and $\lambda(W) \leq (\lambda(Q)) + \varepsilon$.

Then: W approximates Q in measure, to within ε .

By Theorem 12, **choose** a fine-covering $\mathcal{V}' \subseteq \mathcal{V}$, inside W , of Q .

Since $Q \subseteq \bigcup \mathcal{V}' \subseteq W$ and since W approximates Q in measure,
 we conclude that: $\bigcup \mathcal{V}'$ also approximates Q in measure.
 By Theorem 9, **choose** a finite pw-dj $\mathcal{E} \subseteq \mathcal{V}'$ s.t.
 \mathcal{E} covers at least 10% of $\bigcup \mathcal{V}'$.

There are details to check, but,
 assuming our choice of $\varepsilon = 0.1 \cdot (\lambda(Q))$ is small enough, *i.e.*,
 assuming $\bigcup \mathcal{V}'$ approximates Q sufficiently closely in measure,
 then, because \mathcal{E} covers at least 10% of $\bigcup \mathcal{V}'$,
 it will follow that \mathcal{E} covers at least 1% of Q . **QED**

Proof. In case $\lambda(Q) = 0$, **let** $\mathcal{E} := \emptyset$. We therefore assume $\lambda(Q) \neq 0$.
 Then $\lambda(Q) > 0$. By hypothesis, $\lambda(Q) < \infty$. **Let** $b := \lambda(Q)$.

Then $0 < b < \infty$. Then: $1.1 \cdot b > b$.

Since $1.1 \cdot b > b = \lambda(Q)$, by outer-regularity of λ ,

choose $W \in \mathcal{T}$ s.t. $W \supseteq Q$ and $\lambda(W) \leq 1.1 \cdot b$.

Let $\mathcal{V}' := \{V \in \mathcal{V} \mid V \subseteq W\}$. Then $\mathcal{V}' \subseteq \mathcal{V}$. Also, $\bigcup \mathcal{V}' \subseteq W$.

Let $V := \bigcup \mathcal{V}'$. Then: $V \subseteq W$.

So, by monotonicity of λ , we get: $\lambda(V) \leq \lambda(W)$.

Let $c := \lambda(V)$. Then: $c \leq \lambda(W)$.

Since $c \leq \lambda(W) \leq 1.1 \cdot b$, we get: $c \leq 1.1 \cdot b$.

So, since $b < \infty$, we get: $c < \infty$. **Let** $\mathcal{A} := \mathcal{V}'$.

Since $\lambda(\bigcup \mathcal{A}) = \lambda(\bigcup \mathcal{V}') = \lambda(V) = c < \infty$, by Theorem 9,

choose a finite pw-dj $\mathcal{E} \subseteq \mathcal{A}$ s.t. $\lambda(\bigcup \mathcal{E}) \geq 0.1 \cdot (\lambda(\bigcup \mathcal{A}))$.

Then, since $\lambda(\bigcup \mathcal{A}) = \lambda(\bigcup \mathcal{V}') = \lambda(V) = c$, $\lambda(\bigcup \mathcal{E}) \geq 0.1 \cdot c$.

Since $\mathcal{E} \subseteq \mathcal{A} = \mathcal{V}'$, we get: $\mathcal{E} \subseteq \mathcal{V}'$.

So, since $\mathcal{V}' \subseteq \mathcal{V}$, we get: $\mathcal{E} \subseteq \mathcal{V}$.

It remains only to show: $\lambda(Q \cap (\bigcup \mathcal{E})) \geq 0.01 \cdot (\lambda(Q))$.

Since $b = \lambda(Q)$, **we want:** $\lambda(Q \cap (\bigcup \mathcal{E})) \geq 0.01 \cdot b$.

Let $E := \bigcup \mathcal{E}$. **Want:** $\lambda(Q \cap E) \geq 0.01 \cdot b$.

Let $x := \lambda(Q \cap E)$. **Want:** $x \geq 0.01 \cdot b$.

We have: $\forall B \in \mathcal{B}, \lambda(B) < \infty$.

So, since $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{B}$ and \mathcal{E} is finite, we get: $\lambda(\bigcup \mathcal{E}) < \infty$.

So, since $E = \bigcup \mathcal{E}$, we get: $\lambda(E) < \infty$.

Let $y := \lambda(E)$. Then: $y < \infty$.

Since $x = \lambda(Q \cap E) \leq \lambda(E) < \infty$, we get: $x < \infty$.

Since $\mathcal{E} \subseteq \mathcal{V}'$, we get $\bigcup \mathcal{E} \subseteq \bigcup \mathcal{V}'$.

Since $E = \bigcup \mathcal{E} \subseteq \bigcup \mathcal{V}' = V$, we get: $V \cap E = E$.

Then: $\lambda(V \cap E) = \lambda(E)$.

Since $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T}$ and since \mathcal{T} is a topology, we get: $\bigcup \mathcal{E} \in \mathcal{T}$.

Since $E = \bigcup \mathcal{E} \in \mathcal{T}$, it follows that: E is Lebesgue-measurable.

So, by the Carathéodory-condition,

$$\lambda(V) = [\lambda(V \setminus E)] + [\lambda(V \cap E)].$$

So, since $\lambda(V \cap E) = \lambda(E) < \infty$, we get:

$$\lambda(V \setminus E) = [\lambda(V)] - [\lambda(V \cap E)].$$

So, since $c = \lambda(V)$ and $\lambda(V \cap E) = \lambda(E) = y$, we get:

$$\lambda(V \setminus E) = c - y.$$

Since E is Lebesgue-measurable, by the Carathéodory-condition,

$$\lambda(Q) = [\lambda(Q \setminus E)] + [\lambda(Q \cap E)].$$

So, since $\lambda(Q \cap E) \leq \lambda(E) < \infty$, we get:

$$\lambda(Q \setminus E) = [\lambda(Q)] - [\lambda(Q \cap E)].$$

So, since $b = \lambda(Q)$ and $\lambda(Q \cap E) = x$, we get:

$$\lambda(Q \setminus E) = b - x.$$

By Theorem 12, \mathcal{V}' is a fine-covering of Q , so:

$$\bigcup \mathcal{V}' \supseteq Q.$$

Since $V = \bigcup \mathcal{V}' \supseteq Q$, we get:

$$V \setminus E \supseteq Q \setminus E.$$

So, by monotonicity of λ , we get:

$$\lambda(V \setminus E) \geq \lambda(Q \setminus E).$$

So, since $\lambda(V \setminus E) = c - y$ and $\lambda(Q \setminus E) = b - x$, $c - y \geq b - x$.

Recall: $b < \infty$, $c < \infty$, $x < \infty$, $y < \infty$, $c \leq 1.1 \cdot b$.

Since $y = \lambda(E) = \lambda(\bigcup \mathcal{E}) \geq 0.1 \cdot c$,

$$\text{we get: } c - y \leq 0.9 \cdot c.$$

Since $c \leq 1.1 \cdot b$, we get:

$$0.9 \cdot c \leq 0.99 \cdot b.$$

Since

$$b - x \leq c - y \leq 0.9 \cdot c \leq 0.99 \cdot b,$$

we get:

$$x \geq 0.01 \cdot b. \quad \square$$

For any two sets A and B , we define: $\boxed{A \Delta B} := (A \setminus B) \cup (B \setminus A)$.

For any $A, B \subseteq \mathbb{R}^2$, by $\boxed{A \equiv B}$, we mean: $\lambda(A \Delta B) = 0$.

We will read “ \equiv ” as: “ $\boxed{\text{is a.e.-equal to}}$ ”.

For all sets A, B , we have:

$$(A \subseteq B \cup (A \Delta B)) \ \& \ (B \subseteq A \cup (A \Delta B)).$$

So, by monotonicity and subadditivity of λ , we conclude:

$$\forall A, B \subseteq \mathbb{R}^2, \quad (A \equiv B) \Rightarrow (\lambda(A) = \lambda(B)).$$

For any sets A, B, Y, Z , we have:

$$(A \cup Y) \Delta (B \cup Z) \subseteq (A \Delta B) \cup (Y \Delta Z) \quad \text{and}$$

$$(A \cap Y) \Delta (B \cap Z) \subseteq (A \Delta B) \cup (Y \Delta Z) \quad \text{and}$$

$$(A \setminus Y) \Delta (B \setminus Z) \subseteq (A \Delta B) \cup (Y \Delta Z).$$

So, $\forall A, B, Y, Z \subseteq \mathbb{R}^2$, if $A \equiv B$ and $Y \equiv Z$, then:

$$A \cup Y \equiv B \cup Z \quad \text{and} \quad A \cap Y \equiv B \cap Z \quad \text{and} \quad A \setminus Y \equiv B \setminus Z.$$

For all $S \subseteq \mathbb{R}^2$, let \overline{S} denote the closure in \mathbb{R}^2 of S .

NOTE: $\forall x \in \mathbb{R}^2, \forall r > 0$, we have: $\lambda(B_x^r) = \pi r^2 = \lambda(\overline{B_x^r})$.

Then we have: $\forall B \in \mathcal{B}, B \equiv \overline{B}$.

The next result says that

if $Q \subseteq \mathbb{R}^2$ has finite Lebesgue-outer-measure, and
 if $\mathcal{V} \subseteq \mathcal{B}$ is a fine-covering of Q , and
 if, using a finite pw-dj $\mathcal{E} \subseteq \mathcal{V}$, we can cover some portion of Q ,
 then, using a bigger finite pw-dj collection $\mathcal{F} \subseteq \mathcal{V}$,
 we can cover substantially more, by which we mean:
 the UNcovered portion decreases by at least 1%.

THEOREM 14. Let $Q \subseteq \mathbb{R}^2, \mathcal{V} \subseteq \mathcal{B}, \mathcal{E} \subseteq \mathcal{V}$.

Assume: \mathcal{V} is a fine-covering of Q . Assume: $\lambda(Q) < \infty$.

Assume: \mathcal{E} is finite and pw-dj.

Then: \exists finite pw-dj $\mathcal{F} \subseteq \mathcal{V}$ s.t. $\mathcal{E} \subseteq \mathcal{F}$ and s.t.

$$\lambda(Q \setminus (\bigcup \mathcal{F})) \leq 0.99 \cdot (\lambda(Q \setminus (\bigcup \mathcal{E}))).$$

Idea of Proof: Let $S := \bigcup \mathcal{E}$. Then: \mathcal{E} is inside S .

Since \mathcal{E} is a finite set of disks, we get: $\overline{S} \equiv S$.

Then $\mathbb{R}^2 \setminus \overline{S} \equiv \mathbb{R}^2 \setminus S$. Let $W := \mathbb{R}^2 \setminus \overline{S}$.

Then: $W \equiv \mathbb{R}^2 \setminus S$ and W is open in \mathbb{R}^2 .

We have $Q \cap W \equiv Q \cap (\mathbb{R}^2 \setminus S) = Q \setminus S = Q \setminus (\bigcup \mathcal{E})$,

so $Q \cap W \equiv$ (the portion of Q that is uncovered by \mathcal{E}).

Using Theorem 11, choose $\mathcal{V}' \subseteq \mathcal{V}$ s.t.

\mathcal{V}' is a fine-covering of $Q \cap W$ and \mathcal{V}' is inside W .

Then \mathcal{V}' is inside $\mathbb{R}^2 \setminus \overline{S}$, hence outside \overline{S} , hence outside S .

Apply Theorem 13 to get a finite pw-dj subset $\mathcal{E}' \subseteq \mathcal{V}'$ which

covers at least 1% of $Q \cap W$, and, therefore,

covers at least 1% of (the portion of Q that is uncovered by \mathcal{E}).

Then \mathcal{E} is inside S , while \mathcal{E}' is outside S .

Let $\mathcal{F} := \mathcal{E} \cup \mathcal{E}'$. **QED**

Proof. Let $\overline{\mathcal{E}} := \{\overline{E} \mid E \in \mathcal{E}\}$. We have: $\forall B \in \mathcal{B}, B \equiv \overline{B}$.

So, since $\mathcal{E} \subseteq \mathcal{V} \subseteq \mathcal{B}$, we get: $\forall E \in \mathcal{E}, E \equiv \overline{E}$.

So, since \mathcal{E} is finite, we get: $\bigcup \mathcal{E} \equiv \bigcup \overline{\mathcal{E}}$.

Let $S := \bigcup \mathcal{E}$. Since \mathcal{E} is finite, we get: $\overline{S} = \bigcup \overline{\mathcal{E}}$. Then $S \equiv \overline{S}$.

Let $W := \mathbb{R}^2 \setminus \overline{S}$. Since \overline{S} is closed in \mathbb{R}^2 , we get: $W \in \mathcal{T}$.

Let $\mathcal{V}' := \{V \in \mathcal{V} \mid V \subseteq W\}$. Then: $\mathcal{V}' \subseteq \mathcal{V}$ and $\bigcup \mathcal{V}' \subseteq W$.

Also, by Theorem 11, \mathcal{V}' is a fine-covering of $Q \cap W$.

Let $Q' := Q \cap W$. Then \mathcal{V}' is a fine-covering of Q' .

Since $Q' = Q \cap W \subseteq Q$, by monotonicity of λ , we get: $\lambda(Q') \leq \lambda(Q)$.

Since $\lambda(Q') \leq \lambda(Q) < \infty$, by Theorem 13,

choose a finite pw-dj $\mathcal{E}' \subseteq \mathcal{V}'$ s.t. $\lambda(Q' \cap (\bigcup \mathcal{E}')) \geq 0.01 \cdot (\lambda(Q'))$.

Since $\mathcal{E}' \subseteq \mathcal{V}'$, we get: $\bigcup \mathcal{E}' \subseteq \bigcup \mathcal{V}'$. Recall: $\bigcup \mathcal{V}' \subseteq W$.

Since $\bar{S} \supseteq S$, we get: $\mathbb{R}^2 \setminus \bar{S} \subseteq \mathbb{R}^2 \setminus S$. Recall: $S = \bigcup \mathcal{E}$.

Since $\bigcup \mathcal{E}' \subseteq \bigcup \mathcal{V}' \subseteq W = \mathbb{R}^2 \setminus \bar{S} \subseteq \mathbb{R}^2 \setminus S = \mathbb{R}^2 \setminus (\bigcup \mathcal{E})$,

we get: $(\bigcup \mathcal{E}) \cap (\bigcup \mathcal{E}') = \emptyset$.

Then: $\forall E \in \mathcal{E}, \forall E' \in \mathcal{E}', E \cap E' = \emptyset$.

So, since \mathcal{E} and \mathcal{E}' are both pw-dj, we get: $\mathcal{E} \cup \mathcal{E}'$ is pw-dj.

Since \mathcal{E} and \mathcal{E}' are both finite, we conclude: $\mathcal{E} \cup \mathcal{E}'$ is finite.

By hypothesis, $\mathcal{E} \subseteq \mathcal{V}$, so, since $\mathcal{E}' \subseteq \mathcal{V}' \subseteq \mathcal{V}$, we get: $\mathcal{E} \cup \mathcal{E}' \subseteq \mathcal{V}$.

Let $\mathcal{F} := \mathcal{E} \cup \mathcal{E}'$. Then \mathcal{F} is finite and pw-dj. Also, $\mathcal{F} \subseteq \mathcal{V}$.

Since $\mathcal{E} \subseteq \mathcal{E} \cup \mathcal{E}' = \mathcal{F}$,

it remains only to show: $\lambda(Q \setminus (\bigcup \mathcal{F})) \leq 0.99 \cdot (\lambda(Q \setminus (\bigcup \mathcal{E})))$.

Recall: $S = \bigcup \mathcal{E}$. Let $S' := \bigcup \mathcal{E}'$.

Then, since $\bigcup \mathcal{F} = \bigcup (\mathcal{E} \cup \mathcal{E}') = (\bigcup \mathcal{E}) \cup (\bigcup \mathcal{E}') = S \cup S'$,

we want to show: $\lambda(Q \setminus (S \cup S')) \leq 0.99 \cdot (\lambda(Q \setminus S))$.

By hypothesis, $\mathcal{V} \subseteq \mathcal{B}$. So, since $\mathcal{E}' \subseteq \mathcal{V}' \subseteq \mathcal{V}$, we get: $\mathcal{E}' \subseteq \mathcal{B}$.

Since $\mathcal{E}' \subseteq \mathcal{B} \subseteq \mathcal{T}$ and since \mathcal{T} is a topology, we get: $\bigcup \mathcal{E}' \in \mathcal{T}$.

So, since $S' = \bigcup \mathcal{E}'$, we get: $S' \in \mathcal{T}$.

Then S' is Lebesgue-measurable, so, by the Carathéodory-condition,

we get: $\lambda(Q') = [\lambda(Q' \setminus S')] + [\lambda(Q' \cap S')]$.

Let $c := \lambda(Q')$, $a := \lambda(Q' \setminus S')$, $b := \lambda(Q' \cap S')$.

Then: $c = a + b$.

By choice of \mathcal{E}' , we have: $\lambda(Q' \cap (\bigcup \mathcal{E}')) \geq 0.01 \cdot (\lambda(Q'))$.

Then: $\lambda(Q' \cap S') \geq 0.01 \cdot (\lambda(Q'))$.

Then: $b \geq 0.01 \cdot c$.

Recall: $S \equiv \bar{S}$.

Then: $Q \setminus S \equiv Q \setminus \bar{S}$. Recall: $W = \mathbb{R}^2 \setminus \bar{S}$ and $Q' = Q \cap W$.

Since $Q \setminus S \equiv Q \setminus \bar{S} = Q \cap (\mathbb{R}^2 \setminus \bar{S}) = Q \cap W = Q'$, we get:

both $(Q \setminus S) \setminus S' \equiv Q' \setminus S'$ and $\lambda(Q \setminus S) = \lambda(Q')$.

Since $Q \setminus (S \cup S') = (Q \setminus S) \setminus S' \equiv Q' \setminus S'$, we get:

$\lambda(Q \setminus (S \cup S')) = \lambda(Q' \setminus S')$.

Since $\lambda(Q \setminus (S \cup S')) = \lambda(Q' \setminus S') = a$ and since $\lambda(Q \setminus S) = \lambda(Q') = c$,

we want: $a \leq 0.99 \cdot c$.

Recall: $c = a + b$ and $b \geq 0.01 \cdot c$.
 Since $c = a + b \geq a + 0.01 \cdot c$,
 we get: $0.99 \cdot c \geq a$. Then: $a \leq 0.99 \cdot c$. \square

Let $A, B \subseteq \mathbb{R}^2$.

By B is an **a.e.-superset of A** , we will mean: $\lambda(A \setminus B) = 0$.

Let $A, B \subseteq \mathbb{R}^2$, $\varepsilon > 0$.

By B is an **ε -efficient-superset of A** , we will mean:

$$A \subseteq B \quad \text{and} \quad \lambda(B) \leq e^\varepsilon \cdot (\lambda(A)).$$

By B is an **ε -efficient-a.e.-superset of A** , we will mean:

$$\lambda(A \setminus B) = 0 \quad \text{and} \quad \lambda(B) \leq e^\varepsilon \cdot (\lambda(A)).$$

We make the convention that, $\forall c > 0$, $c \cdot \infty = \infty$.

Then: $\forall A \subseteq \mathbb{R}^2$, $\forall \varepsilon \in \mathbb{R}$, $(\lambda(A) = \infty) \Rightarrow (\lambda(\mathbb{R}^2) \leq e^\varepsilon \cdot (\lambda(A)))$.

So, using outer-regularity of λ , we can prove:

Let $A \subseteq \mathbb{R}^2$, $\varepsilon > 0$. Assume: $\lambda(A) > 0$.

Then: $\exists U \in \mathcal{T}$ s.t. U is an ε -efficient-superset of A .

(NOTE: In case $\lambda(A) = \infty$, **let** $U := \mathbb{R}^2$.)

Let \mathcal{B} be a set of subsets of \mathbb{R}^2 , $A \subseteq \mathbb{R}^2$.

By \mathcal{B} is an **a.e.-covering of A** , we will mean:

$$\bigcup \mathcal{B} \text{ is an a.e.-superset of } A.$$

Let \mathcal{B} be a set of subsets of \mathbb{R}^2 , $A \subseteq \mathbb{R}^2$, $\varepsilon > 0$.

By \mathcal{B} is an **ε -efficient-covering of A** , we will mean:

$$\bigcup \mathcal{B} \text{ is an } \varepsilon\text{-efficient-superset of } A.$$

By \mathcal{B} is an **ε -efficient-a.e.-covering of A** , we will mean:

$$\bigcup \mathcal{B} \text{ is an } \varepsilon\text{-efficient-a.e.-superset of } A.$$

DEFINITION 15. Let $S \subseteq \mathbb{R}^2$. By S is **Vitali**, we mean:

$\forall \mathcal{V} \subseteq \mathcal{B}$, if \mathcal{V} is a fine-covering of S ,

then \exists countable pw-dj $\mathcal{D} \subseteq \mathcal{V}$ s.t. $\lambda(S \setminus (\bigcup \mathcal{D})) = 0$.

So, a Vitali set is one for which any fine-covering admits
 a countable pw-dj a.e.-subcovering.

In Theorem 17, below, we will show: any subset of \mathbb{R}^2 is Vitali.

By an **a.e.-partition** of a set $S \subseteq \mathbb{R}^2$, we will mean:
 a pw-dj set of subsets of S that is an a.e.-covering of S .

According to the next theorem, for any $S \subseteq \mathbb{R}^2$,
 for any countable a.e.-partition of S into relatively-open subsets,
 if each subset is Vitali, then S is Vitali.

THEOREM 16. Let $S \subseteq \mathbb{R}^2$, $W_1, W_2, \dots \in \mathcal{T}$.
 Assume: $((W_1, W_2, \dots)$ is pw-dj) & $(\lambda(S \setminus (W_1 \cup W_2 \cup \dots)) = 0)$.
 Assume: $\forall n \in \mathbb{N}$, $S \cap W_n$ is Vitali. Then: S is Vitali.

Proof. Given $\mathcal{V} \subseteq \mathcal{B}$, assume \mathcal{V} is a fine-covering of S ,
want: \exists countable pw-dj $\mathcal{D} \subseteq \mathcal{V}$ s.t. $\lambda(S \setminus (\bigcup \mathcal{D})) = 0$.
 For all $n \in \mathbb{N}$, let $\mathcal{V}_n := \{V \in \mathcal{V} \mid V \subseteq W_n\}$. Then: $\forall n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{V}$.
 Also, by Theorem 11, $\forall n \in \mathbb{N}$, \mathcal{V}_n is a fine-covering of $S \cap W_n$.
 For all $n \in \mathbb{N}$, let $Q_n := S \cap W_n$.
 Then: $\forall n \in \mathbb{N}$, \mathcal{V}_n is a fine-covering of Q_n .
 By hypothesis, we have: $\forall n \in \mathbb{N}$, Q_n is Vitali.
 Then, $\forall n \in \mathbb{N}$, **choose** a countable pw-dj $\mathcal{D}_n \subseteq \mathcal{V}_n$
 s.t. $\lambda(Q_n \setminus (\bigcup \mathcal{D}_n)) = 0$.

Let $\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2 \cup \dots$.
 Since, $\forall n \in \mathbb{N}$, \mathcal{D}_n is countable, we get: \mathcal{D} is countable.
 Since, $\forall n \in \mathbb{N}$, $\mathcal{D}_n \subseteq \mathcal{V}_n \subseteq \mathcal{V}$, we get: $\mathcal{D} \subseteq \mathcal{V}$.
It remains to show: (1) \mathcal{D} is pw-dj and (2) $\lambda(S \setminus (\bigcup \mathcal{D})) = 0$.

Proof of (1): Given $A, B \in \mathcal{D}$, assume $A \neq B$, **want:** $A \cap B = \emptyset$.
 Since $A \in \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \dots$, **choose** $a \in \mathbb{N}$ s.t. $A \in \mathcal{D}_a$.
 Since $B \in \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \dots$, **choose** $b \in \mathbb{N}$ s.t. $B \in \mathcal{D}_b$.
 In case $a = b$, we have $A, B \in \mathcal{D}_a$, and so,
 since \mathcal{D}_a is pw-dj and since $A \neq B$, we get: $A \cap B = \emptyset$.
 We therefore assume that $a \neq b$.
 By hypothesis, (W_1, W_2, \dots) is pw-dj. Then: $W_a \cap W_b = \emptyset$.
 Since $A \in \mathcal{D}_a \subseteq \mathcal{V}_a$, by definition of \mathcal{V}_a , we get: $A \subseteq W_a$.
 Since $B \in \mathcal{D}_b \subseteq \mathcal{V}_b$, by definition of \mathcal{V}_b , we get: $B \subseteq W_b$.
 Then $A \cap B \subseteq W_a \cap W_b = \emptyset$, so $A \cap B = \emptyset$.
End of proof of (1).

Proof of (2): Let $D := \bigcup \mathcal{D}$. **Want:** $\lambda(S \setminus D) = 0$.
 Let $Q := Q_1 \cup Q_2 \cup \dots$.

For all sets X, Y, Z , we have: $X \setminus Z \subseteq (X \setminus Y) \cup (Y \setminus Z)$.

Therefore, $S \setminus D \subseteq (S \setminus Q) \cup (Q \setminus D)$,

It therefore suffices to show: $\lambda(S \setminus Q) = 0 = \lambda(Q \setminus D)$.

By hypothesis, we have: $\lambda(S \setminus (W_1 \cup W_2 \cup \dots)) = 0$.

Let $W := W_1 \cup W_2 \cup \dots$. **Then:** $\lambda(S \setminus W) = 0$.

For all $n \in \mathbb{N}$, by definition of Q_n , we have: $S \cap W_n = Q_n$.

Since $S \cap W = (S \cap W_1) \cup (S \cap W_2) \cup \dots = Q_1 \cup Q_2 \cup \dots = Q$,

we get: $S \setminus (S \cap W) = S \setminus Q$.

For any sets X, Y , by definition of set-subtraction, we have:

$$X \setminus Y = X \setminus (X \cap Y).$$

Since $S \setminus W = S \setminus (S \cap W) = S \setminus Q$, we get: $\lambda(S \setminus W) = \lambda(S \setminus Q)$.

Since $\lambda(S \setminus Q) = \lambda(S \setminus W) = 0$,

it remains only to show: $\lambda(Q \setminus D) = 0$.

Since $Q = Q_1 \cup Q_2 \cup \dots$, we get: $Q \setminus D = (Q_1 \setminus D) \cup (Q_2 \setminus D) \cup \dots$.

It therefore suffices to show: $\forall n \in \mathbb{N}, \lambda(Q_n \setminus D) = 0$.

Given $n \in \mathbb{N}$, **let** $P := Q_n$, **Want:** $\lambda(P \setminus D) = 0$.

By choice of \mathcal{D}_n , we have: $\lambda(Q_n \setminus (\bigcup \mathcal{D}_n)) = 0$.

Let $\mathcal{C} := \mathcal{D}_n$. **Then:** $\lambda(P \setminus (\bigcup \mathcal{C})) = 0$.

Since $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \dots \supseteq \mathcal{D}_n = \mathcal{C}$, we get: $\bigcup \mathcal{D} \supseteq \bigcup \mathcal{C}$.

Since $D = \bigcup \mathcal{D} \supseteq \bigcup \mathcal{C}$, we get: $P \setminus D \subseteq P \setminus (\bigcup \mathcal{C})$.

So, since $\lambda(P \setminus (\bigcup \mathcal{C})) = 0$, we get: $\lambda(P \setminus D) = 0$.

End of proof of (2). □

THEOREM 17. **Let** $S \subseteq \mathbb{R}^2$. **Then:** S is Vitali.

Idea of Proof: Intersecting S with each set of

an a.e.-partition of \mathbb{R}^2 by ϵ open bounded subsets,

we get an a.e.-partition of S into ϵ relatively-open bounded subsets.

By Theorem 16, it suffices to show each relatively-open subset is Vitali.

Given one of these subsets, Q , and a fine-covering of Q ,

we seek a countable pw-dj a.e.-subcovering of Q .

Since Q is bounded, we get: $\lambda(Q) < \infty$.

Starting with the empty set (which covers none of Q),

we use Theorem 14 repeatedly to find

finite coverings of more and more of Q .

Taking the union of these countably-many finite partial coverings,

we arrive at a countable pw-dj a.e.-covering of Q . **QED**

Proof. **Let** $z := (0, 0)$. For all $j \in \mathbb{N}$, **let** $B_j := B_z^j$ and $D_j := \overline{B_j}$.

Let $D_0 := \emptyset$. For all $j \in \mathbb{N}$, **let** $W_j := B_j \setminus D_{j-1}$.

Then: $W_1, W_2, \dots \in \mathcal{T}$. Also, (W_1, W_2, \dots) is pw-dj.
 We have: $\forall j \in \mathbb{N}, \lambda(B_j) = \pi j^2 = \lambda(D_j)$.
 It follows that: $\forall j \in \mathbb{N}, \lambda(D_j \setminus B_j) = 0$.
 So, since $\mathbb{R}^2 \setminus (W_1 \cup W_2 \cup \dots) \subseteq (D_1 \setminus B_1) \cup (D_2 \setminus B_2) \cup \dots$,
 we get: $\lambda(\mathbb{R}^2 \setminus (W_1 \cup W_2 \cup \dots)) = 0$.
 So, since $\mathbb{R}^2 \setminus (W_1 \cup W_2 \cup \dots) \supseteq S \setminus (W_1 \cup W_2 \cup \dots)$
 we get: $\lambda(S \setminus (W_1 \cup W_2 \cup \dots)) = 0$.
 By Theorem 16, **it suffices to show:** $\forall n \in \mathbb{N}, S \cap W_n$ is Vitali.
Given $n \in \mathbb{N}$, **let** $Q := S \cap W_n$, **want:** Q is Vitali.
Given $\mathcal{V} \subseteq \mathcal{B}$, assume \mathcal{V} is a fine-covering of Q ,
want: \exists countable pw-dj $\mathcal{D} \subseteq \mathcal{V}$ s.t. $\lambda(Q \setminus (\bigcup \mathcal{D})) = 0$.
 Since $Q = S \cap W_n \subseteq W_n = B_n \setminus D_{n-1} \subseteq B_n$
 and since $\lambda(B_n) = \pi n^2 < \infty$,
 by monotonicity of λ , we conclude: $\lambda(Q) < \infty$.
Let $\mathcal{E}_0 := \emptyset$. Then $\mathcal{E}_0 \subseteq \mathcal{V}$ and \mathcal{E}_0 is finite and pw-dj.
 By applying Theorem 14 repeatedly, **choose** $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \dots \subseteq \mathcal{V}$
 s.t. $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots$ and
 s.t. $\forall j \in \mathbb{N}, \mathcal{E}_j$ is finite and pw-dj and
 s.t. $\forall j \in \mathbb{N}, \lambda(Q \setminus (\bigcup \mathcal{E}_j)) \leq 0.99 \cdot (\lambda(Q \setminus (\bigcup \mathcal{E}_{j-1})))$.
Let $\mathcal{D} := \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots$. Then $\mathcal{D} \subseteq \mathcal{V}$ and \mathcal{D} is countable.
It remains to show: (1) \mathcal{D} is pw-dj and (2) $\lambda(Q \setminus (\bigcup \mathcal{D})) = 0$.

Proof of (1): **Given** $E, F \in \mathcal{D}$, assume $E \neq F$, **want:** $E \cap F = \emptyset$.
 Since $E \in \mathcal{D} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots$, **choose** $p \in \mathbb{N}$ s.t. $E \in \mathcal{E}_p$.
 Since $F \in \mathcal{D} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots$, **choose** $q \in \mathbb{N}$ s.t. $F \in \mathcal{E}_q$.
Let $r := \max\{p, q\}$. Recall: $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots$. Then $E, F \in \mathcal{E}_r$.
 So, since \mathcal{E}_r is pw-dj and since $E \neq F$, we get: $E \cap F = \emptyset$.
End of proof of (1).

Proof of (2): Recall: $\lambda(Q) < \infty$. **Let** $m := \lambda(Q)$.
 Then: $0 \leq m < \infty$. Then: as $k \rightarrow \infty, (0.99)^k \cdot m \rightarrow 0$.
It therefore suffices to show: $\forall k \in \mathbb{N}, \lambda(Q \setminus (\bigcup \mathcal{D})) \leq (0.99)^k \cdot m$.
Given $k \in \mathbb{N}$, **let** $s := (0.99)^k$, **want:** $\lambda(Q \setminus (\bigcup \mathcal{D})) \leq s \cdot m$.
 Since $\mathcal{E}_0 = \emptyset$, we get $\bigcup \mathcal{E}_0 = \emptyset$, so $Q \setminus (\bigcup \mathcal{E}_0) = Q$.
 We have: $\lambda(Q \setminus (\bigcup \mathcal{E}_k)) \leq (0.99) \cdot (\lambda(Q \setminus (\bigcup \mathcal{E}_{k-1})))$
 $\leq (0.99)^2 \cdot (\lambda(Q \setminus (\bigcup \mathcal{E}_{k-2})))$
 $\leq \dots$
 $\leq (0.99)^k \cdot (\lambda(Q \setminus (\bigcup \mathcal{E}_0)))$

Let $\mathcal{A} := \mathcal{E}_k$. Then: $\lambda(Q \setminus (\bigcup \mathcal{A})) \leq s \cdot (\lambda(Q))$.

Since $\mathcal{D} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \supseteq \mathcal{E}_k = \mathcal{A}$, we get:

$$\bigcup \mathcal{D} \supseteq \bigcup \mathcal{A}.$$

Then: $Q \setminus (\bigcup \mathcal{D}) \subseteq Q \setminus (\bigcup \mathcal{A})$.

So, by monotonicity of λ , we get:

$$\lambda(Q \setminus (\bigcup \mathcal{D})) \leq \lambda(Q \setminus (\bigcup \mathcal{A})).$$

By definition of m , we have: $\lambda(Q) = m$.

Then: $\lambda(Q \setminus (\bigcup \mathcal{D})) \leq \lambda(Q \setminus (\bigcup \mathcal{A})) \leq s \cdot (\lambda(Q)) = s \cdot m$.

End of proof of (2). \square

According to the next theorem, for any $Q \subseteq \mathbb{R}^2$,
for any fine-covering of Q , for any $\varepsilon > 0$,
there is a countable pw-dj ε -efficient-a.e.-subcovering of Q .
The set Q need not be Lebesgue-measurable.

THEOREM 18. Let $Q \subseteq \mathbb{R}^2$, $\mathcal{V} \subseteq \mathcal{B}$, $\varepsilon > 0$.

Assume: \mathcal{V} is a fine-covering of Q .

Then: \exists countable pw-dj $\mathcal{C} \subseteq \mathcal{V}$ s.t.

$$(\lambda(Q \setminus (\bigcup \mathcal{C})) = 0) \quad \& \quad (\lambda(\bigcup \mathcal{C}) \leq e^\varepsilon \cdot (\lambda(Q))).$$

Proof. In case $\lambda(Q) = 0$, let $\mathcal{C} := \emptyset$. We therefore assume $\lambda(Q) > 0$.

By outer-regularity of λ , choose $W \in \mathcal{T}$ s.t.

$$\text{both } W \supseteq Q \text{ and } \lambda(W) \leq e^\varepsilon \cdot (\lambda(Q)).$$

Let $\mathcal{V}' := \{V \in \mathcal{V} \mid V \subseteq W\}$. Then: $\mathcal{V}' \subseteq \mathcal{V}$ and $\bigcup \mathcal{V}' \subseteq W$.

By Theorem 12, \mathcal{V}' is a fine-covering of Q .

So, since, by Theorem 17, Q is Vitali,

$$\text{choose a countable pw-dj } \mathcal{C} \subseteq \mathcal{V}' \text{ s.t. } \lambda(Q \setminus (\bigcup \mathcal{C})) = 0.$$

Since $\mathcal{C} \subseteq \mathcal{V}' \subseteq \mathcal{V}$, **it remains only to show:** $\lambda(\bigcup \mathcal{C}) \leq e^\varepsilon \cdot (\lambda(Q))$.

Since $\mathcal{C} \subseteq \mathcal{V}'$, we get: $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{V}'$.

Since $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{V}' \subseteq W$, by monotonicity of λ , we get: $\lambda(\bigcup \mathcal{C}) \leq \lambda(W)$.

Then: $\lambda(\bigcup \mathcal{C}) \leq \lambda(W) \leq e^\varepsilon \cdot (\lambda(Q))$. \square

DEFINITION 19. Let $Q \subseteq \mathbb{R}^2$, $\varepsilon > 0$.

$$\text{Then: } \boxed{\mathcal{I}_Q^\varepsilon} := \{B \in \mathcal{B} \mid \lambda(B) > e^\varepsilon \cdot (\lambda(Q \cap B))\}.$$

Then $\mathcal{I}_Q^\varepsilon$ is the set of all

disks B that are NOT ε -efficient in covering $Q \cap B$.

The letter “ \mathcal{I} ” stands for “inefficient”.

By Theorem 18, every fine-covering has some efficiency.

The next theorem is based on the contrapositive:

Since $\mathcal{I}_Q^\varepsilon$ has no efficiency, it cannot be a fine-covering.

THEOREM 20. Let $Q \subseteq \mathbb{R}^2$, $\varepsilon > 0$. Assume: $\lambda(Q) > 0$.
Then: $\mathcal{I}_Q^\varepsilon$ is not a fine-covering of Q .

Idea of proof:

Assume, for a contradiction, that: $\mathcal{I}_Q^\varepsilon$ is a fine-covering of Q .

By Theorem 18, choose

a countable pw-dj ε -efficient-a.e.-subcovering, \mathcal{C} , of Q .

Since \mathcal{C} is an a.e.-covering of Q , we get: $Q \cap (\bigcup \mathcal{C}) \equiv Q$.

Since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon$, we get: each $C \in \mathcal{C}$ is ε -inefficient at covering $Q \cap C$.

Summing, we find that: \mathcal{C} is ε -inefficient at covering $Q \cap (\bigcup \mathcal{C})$.

So, since $Q \cap (\bigcup \mathcal{C}) \equiv Q$, \mathcal{C} is ε -inefficient at a.e.-covering Q .

This contradicts the choice of \mathcal{C} .

QED

Proof. Assume $\mathcal{I}_Q^\varepsilon$ is a fine-covering of Q . **Want:** Contradiction.

By Theorem 18, choose a countable pw-dj $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon$ s.t.

$$(\lambda(Q \setminus (\bigcup \mathcal{C})) = 0) \ \& \ (\lambda(\bigcup \mathcal{C}) \leq e^\varepsilon \cdot (\lambda(Q))).$$

Since $\lambda(Q \setminus (\bigcup \mathcal{C})) = 0 < \lambda(Q)$, we get: $\lambda(Q \setminus (\bigcup \mathcal{C})) \neq \lambda(Q)$.

Then $Q \setminus (\bigcup \mathcal{C}) \neq Q$, so $\bigcup \mathcal{C} \neq \emptyset$. Then $\mathcal{C} \neq \emptyset$.

Since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon \subseteq \mathcal{B} \subseteq \mathcal{T}$ and since \mathcal{T} is a topology, we get: $\bigcup \mathcal{C} \in \mathcal{T}$.

Let $A := \bigcup \mathcal{C}$. Then $A \in \mathcal{T}$. Then A is Lebesgue-measurable.

So, by the Carathéodory-condition, we get:

$$\lambda(Q) = [\lambda(Q \cap A)] + [\lambda(Q \setminus A)].$$

So, since $\lambda(Q \setminus A) = \lambda(Q \setminus (\bigcup \mathcal{C})) = 0$,

$$\text{we get: } \lambda(Q) = \lambda(Q \cap A).$$

Since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon \subseteq \mathcal{B} \subseteq \mathcal{T}$, we conclude:

$$\forall C \in \mathcal{C}, \quad C \text{ is Lebesgue-measurable.}$$

So, since \mathcal{C} is countable and pw-dj,

$$\text{by countable-additivity of } \lambda, \quad \lambda(\bigcup \mathcal{C}) = \sum_{C \in \mathcal{C}} (\lambda(C)).$$

$$\text{Since } Q \cap A = Q \cap (\bigcup \mathcal{C}) = Q \cap (\bigcup_{C \in \mathcal{C}} C) = \bigcup_{C \in \mathcal{C}} (Q \cap C),$$

$$\text{by countable-subadditivity of } \lambda, \quad \lambda(Q \cap A) \leq \sum_{C \in \mathcal{C}} (\lambda(Q \cap C)).$$

$$\text{So, since } \lambda(Q) = \lambda(Q \cap A), \quad \lambda(Q) \leq \sum_{C \in \mathcal{C}} (\lambda(Q \cap C)).$$

$$\text{By choice of } \mathcal{C}, \quad \lambda(\bigcup \mathcal{C}) \leq e^\varepsilon \cdot (\lambda(Q)).$$

Since $\sum_{C \in \mathcal{C}} (\lambda(C)) = \lambda(\bigcup C) \leq e^\varepsilon \cdot (\lambda(Q)) \leq e^\varepsilon \cdot \sum_{C \in \mathcal{C}} (\lambda(Q \cap C))$,

we get: $\sum_{C \in \mathcal{C}} (\lambda(C)) \leq e^\varepsilon \cdot \sum_{C \in \mathcal{C}} (\lambda(Q \cap C))$.

On the other hand, since $\mathcal{C} \subseteq \mathcal{I}_Q^\varepsilon$, by definition of $\mathcal{I}_Q^\varepsilon$, we get:

$$\forall C \in \mathcal{C}, \quad \lambda(C) > e^\varepsilon \cdot (\lambda(Q \cap C)).$$

So, since $\mathcal{C} \neq \emptyset$, summing these inequalities gives:

$$\sum_{C \in \mathcal{C}} (\lambda(C)) > e^\varepsilon \cdot \sum_{C \in \mathcal{C}} (\lambda(Q \cap C)). \quad \text{Contradiction.} \quad \square$$

DEFINITION 21. For every $X \subseteq \mathbb{R}^2$, we define:

$$\boxed{\text{DP}_X} := \left\{ x \in X \mid \lim_{r \rightarrow 0^+} \frac{\lambda(X \cap B_x^r)}{\lambda(B_x^r)} = 1 \right\}.$$

Elements of DP_X are called “ X -density points”.

According to the next theorem,

every subset of \mathbb{R}^2 is comprised a.e. of density points.

The same result can be proved, similarly, in any Euclidean space.

Interestingly, the subset need not be Lebesgue-measurable.

THEOREM 22. Let $X \subseteq \mathbb{R}^2$. Then: $\lambda(X \setminus \text{DP}_X) = 0$.

Sketch of proof:

For all $j \in \mathbb{N}$, let $S_j := \left\{ x \in X \mid \liminf_{r \rightarrow 0^+} \frac{\lambda(X \cap B_x^r)}{\lambda(B_x^r)} \geq \frac{j}{j+1} \right\}$.

Then $\text{DP}_X = S_1 \cap S_2 \cap \dots$, so $X \setminus \text{DP}_X = (X \setminus S_1) \cup (X \setminus S_2) \cup \dots$.

It therefore suffices to show, given $j \in \mathbb{N}$, that $\lambda(X \setminus S_j) = 0$.

Let $Q := X \setminus S_j$ and assume, for a contradiction, that $\lambda(Q) > 0$.

Let $\varepsilon := \ln((j+1)/j)$. Then $e^{-\varepsilon} = j/(j+1)$ and $\varepsilon > 0$.

Since $Q \subseteq X$, by monotonicity of λ , we get:

$$\forall x \in \mathbb{R}^2, \forall r > 0, \quad \lambda(Q \cap B_x^r) \leq \lambda(X \cap B_x^r).$$

For all $x \in Q$, since $x \notin S_j$, we get: $\liminf_{r \rightarrow 0^+} \frac{\lambda(X \cap B_x^r)}{\lambda(B_x^r)} < \frac{j}{j+1}$.

For all $x \in Q$, we have

$$\liminf_{r \rightarrow 0^+} \frac{\lambda(Q \cap B_x^r)}{\lambda(B_x^r)} \leq \liminf_{r \rightarrow 0^+} \frac{\lambda(X \cap B_x^r)}{\lambda(B_x^r)} < \frac{j}{j+1} = e^{-\varepsilon},$$

so, for some sequence of positive reals $r_1, r_2, \dots \rightarrow 0$, we have

$$\forall i \in \mathbb{N}, \quad \frac{\lambda(Q \cap B_x^{r_i})}{\lambda(B_x^{r_i})} < e^{-\varepsilon},$$

and so $\forall i \in \mathbb{N}, \lambda(B_x^{r_i}) > e^\varepsilon \cdot (\lambda(Q \cap B_x^{r_i})),$
and so $\forall i \in \mathbb{N}, B_x^{r_i} \in \mathcal{I}_Q^\varepsilon.$

Then $\mathcal{I}_Q^\varepsilon$ covers each point of Q by balls of arbitrarily small radii.
Then $\mathcal{I}_Q^\varepsilon$ is a fine-covering of Q , contradicting Theorem 20. **QED**

Proof. We wish to show: for λ -a.e. $x \in X, x \in \text{DP}_X.$

Define $F : \mathbb{R}^2 \times (0; \infty) \rightarrow [0; 1]$ by:

$$\forall x \in \mathbb{R}^2, \forall r > 0, \quad F(x, r) = \frac{\lambda(X \cap B_x^r)}{\lambda(B_x^r)}.$$

We wish to show: for λ -a.e. $x \in X, \lim_{r \rightarrow 0^+} (F(x, r)) = 1.$

Define $\phi, \psi : X \rightarrow [0; 1]$ by: $\forall x \in X,$

$$\phi(x) = \liminf_{r \rightarrow 0^+} (F(x, r)) \quad \text{and} \quad \psi(x) = \limsup_{r \rightarrow 0^+} (F(x, r)).$$

We wish to show: for λ -a.e. $x \in X, \phi(x) = 1 = \psi(x).$

We have: $\forall x \in X, \phi(x) \leq \psi(x) \leq 1.$

Therefore, **it suffices to show:** for λ -a.e. $x \in X, \phi(x) \geq 1.$

Let $P := \{x \in X \mid \phi(x) < 1\}.$ **Want:** $\lambda(P) = 0.$

For all $j \in \mathbb{N},$ **let** $P_j := \{x \in X \mid \phi(x) < j/(j+1)\}.$

Since $P = P_1 \cup P_2 \cup \dots,$ **it suffices to show:** $\forall j \in \mathbb{N}, \lambda(P_j) = 0.$

Given $j \in \mathbb{N},$ **let** $Q := P_j,$ **want:** $\lambda(Q) = 0.$

Assume $\lambda(Q) > 0,$ **want:** contradiction.

Let $\varepsilon := \ln((j+1)/j).$ Then: $e^{-\varepsilon} = j/(j+1)$ and $\varepsilon > 0.$

So, since $Q = P_j = \{x \in X \mid \phi(x) < j/(j+1)\},$

we get: $Q = \{x \in X \mid \phi(x) < e^{-\varepsilon}\}.$ Note that $Q \subseteq X.$

By Theorem 20, $\mathcal{I}_Q^\varepsilon$ is not a fine-covering of $Q.$ **Let** $\mathcal{W} := \mathcal{I}_Q^\varepsilon.$

Then \mathcal{W} is not a fine-covering of $Q,$ so **choose** $x \in Q$ and $\delta > 0$ s.t.

$$\forall W \in \mathcal{W}, \quad (x \in W) \Rightarrow (\text{rad } W > \delta).$$

Since $x \in Q,$ we get: $\phi(x) < e^{-\varepsilon}.$

Since $\liminf_{r \rightarrow 0^+} (F(x, r)) = \phi(x) < e^{-\varepsilon},$

choose $r \in (0; \delta)$ s.t. $F(x, r) < e^{-\varepsilon}.$ **Let** $W := B_x^r.$

Since $r \in (0; \delta),$ we have $r > 0,$ so: $\pi r^2 > 0.$

So, since $\lambda(W) = \lambda(B_x^r) = \pi r^2,$ we get: $\lambda(W) > 0.$

Since $Q \subseteq X,$ we get: $Q \cap W \subseteq X \cap W.$

So, by monotonicity of $\lambda,$ we get: $\lambda(Q \cap W) \leq \lambda(X \cap W).$

Since $\frac{\lambda(Q \cap W)}{\lambda(W)} \leq \frac{\lambda(X \cap W)}{\lambda(W)} = \frac{\lambda(X \cap B_x^r)}{\lambda(B_x^r)} = F(x, r) < e^{-\varepsilon},$

we get $\lambda(Q \cap W) < e^{-\varepsilon} \cdot (\lambda(W)),$

so $e^\varepsilon \cdot (\lambda(Q \cap W)) < \lambda(W),$

so $\lambda(W) > e^\varepsilon \cdot (\lambda(Q \cap W))$,
 so, since $W = B_x^r \in \mathcal{B}$, by definition of $\mathcal{I}_Q^\varepsilon$, we conclude: $W \in \mathcal{I}_Q^\varepsilon$.
 Since $W \in \mathcal{I}_Q^\varepsilon = \mathcal{W}$ and since $x \in B_x^r = W$, by choice of x and δ ,
 we get: $\text{rad } W > \delta$.
 On the other hand, since $\text{rad } W = \text{rad } B_x^r = r \in (0; \delta)$,
 we get: $\text{rad } W < \delta$. Contradiction. \square

For any function f , let \mathbb{D}_f denote the domain of f .

For any set S , for any function f , we define:

$$f^*S := \{x \in \mathbb{D}_f \mid f(x) \in S\}.$$

Note: $\forall \text{set } S, \forall \text{function } f$, we have: $f^*S \subseteq \mathbb{D}_f$.

DEFINITION 23. Let $X \subseteq \mathbb{R}^2$, let $x \in X$ and let $f : X \rightarrow \mathbb{R}$.
 Then, for all $\varepsilon > 0$, for all $r > 0$, we define:

$$A_x^r(f, \varepsilon) := \{u \in X \cap B_x^r \text{ s.t. } |(f(u)) - (f(x))| < \varepsilon\}.$$

We say f is CiOP at x if: $\forall \varepsilon > 0, \lim_{r \rightarrow 0^+} \frac{\lambda(A_x^r(f, \varepsilon))}{\lambda(B_x^r)} = 1$.

Here, ‘‘CiOP’’ stands for: ‘‘continuous-in-outer-probability’’.
 Every function, measurable or not, is CiOP a.e.:

THEOREM 24. Let $X \subseteq \mathbb{R}^2, f : X \rightarrow \mathbb{R}$.

Then: for λ -a.e. $x \in X$, f is CiOP at x .

Here, we assume that the domain of f is a subset of \mathbb{R}^2
 and that the image of f is a subset of \mathbb{R} ,

but the result could be proved for any two Euclidean spaces.
 Interestingly, neither X nor f need be Lebesgue-measurable.

Proof. Let Y_1, Y_2, \dots be a countable base for the topology on \mathbb{R} .

For all $j \in \mathbb{N}$, let $X_j := f^*Y_j$.

By Theorem 22, we have: $\forall j \in \mathbb{N}, \lambda(X_j \setminus \text{DP}_{X_j}) = 0$.

For all $j \in \mathbb{N}$, let $D_j := \text{DP}_{X_j}$.

Then: $\forall j \in \mathbb{N}, \lambda(X_j \setminus D_j) = 0$.

For all $j \in \mathbb{N}$, let $Z_j := X_j \setminus D_j$.

Then: $\forall j \in \mathbb{N}, \lambda(Z_j) = 0$.

Let $Z := Z_1 \cup Z_2 \cup \dots$. Then: $\lambda(Z) = 0$.

It therefore suffices to show: $\forall x \in X \setminus Z, f$ is CiOP at x .

Given $x \in X \setminus Z$, given $\varepsilon > 0$, want: $\lim_{r \rightarrow 0^+} \frac{\lambda(A_x^r(f, \varepsilon))}{\lambda(B_x^r)} = 1$.

Let $y := f(x)$. We have: $y \in (y - \varepsilon; y + \varepsilon)$.

So, since Y_1, Y_2, \dots is a base for the topology on \mathbb{R} ,

choose $j \in \mathbb{N}$ s.t. $y \in Y_j \subseteq (y - \varepsilon; y + \varepsilon)$.

Since $f(x) = y \in Y_j$, we get: $x \in f^*Y_j$.

Since $x \in X \setminus Z$, we get: $x \in X$ and $x \notin Z$.

Since $x \notin Z = Z_1 \cup Z_2 \cup \dots \supseteq Z_j$, we get: $x \notin Z_j$.

So, since $x \in f^*Y_j = X_j$, we get: $x \in X_j \setminus Z_j$.

Since $D_j = \text{DP}_{X_j} \subseteq X_j$ and $Z_j = X_j \setminus D_j$, we get: $X_j \setminus Z_j = D_j$.

Since $x \in X_j \setminus Z_j = D_j = \text{DP}_{X_j}$, we get: $\lim_{r \rightarrow 0^+} \frac{\lambda(X_j \cap B_x^r)}{\lambda(B_x^r)} = 1$.

So, by the Squeeze Theorem, **it suffices to show:**

$$\forall r > 0, \quad \frac{\lambda(X_j \cap B_x^r)}{\lambda(B_x^r)} \leq \frac{\lambda(A_x^r(f, \varepsilon))}{\lambda(B_x^r)} \leq 1.$$

Given $r > 0$, **want:** $\lambda(X_j \cap B_x^r) \leq \lambda(A_x^r(f, \varepsilon)) \leq \lambda(B_x^r)$.

By monotonicity of λ ,

it suffices to show: $X_j \cap B_x^r \subseteq A_x^r(f, \varepsilon) \subseteq B_x^r$.

By definition of $A_x^r(f, \varepsilon)$, $A_x^r(f, \varepsilon) \subseteq X \cap B_x^r$.

Then: $A_x^r(f, \varepsilon) \subseteq B_x^r$.

It remains to show: $X_j \cap B_x^r \subseteq A_x^r(f, \varepsilon)$.

Given $u \in X_j \cap B_x^r$, **want:** $u \in A_x^r(f, \varepsilon)$.

Since $u \in X_j \cap B_x^r$, we get: $u \in X_j$ and $u \in B_x^r$.

Since $u \in X_j = f^*Y_j \subseteq \mathbb{D}_f = X$ and $u \in B_x^r$, we get: $u \in X \cap B_x^r$.

So, by definition of $A_x^r(f, \varepsilon)$, **we want:** $|(f(u)) - (f(x))| < \varepsilon$.

Since $u \in X_j = f^*Y_j$, we get: $f(u) \in Y_j$.

By the choice of j , we have: $Y_j \subseteq (y - \varepsilon; y + \varepsilon)$.

Since $f(u) \in Y_j \subseteq (y - \varepsilon; y + \varepsilon)$, we get: $|(f(u)) - y| < \varepsilon$.

By definition of y , $y = f(x)$. Then: $|(f(u)) - (f(x))| < \varepsilon$. \square