Cents and Citizens

The main results in this note are:

MORE LATER

DEFINITION 1. *We define* $\boxed{\#\varnothing} := 0$.
For any non$\varnothing$ *finite set* S,

$\boxed{\#S}$ *denotes the number of elements in* S.

For any infinite set S, *we define* $\boxed{\#S} := \infty$.

For any sets A, B, let $\boxed{B^A}$ denote
the set of all functions $A \to B$.
For any function f, let $\boxed{\mathbb{D}_f}$ denote the domain of f.
For any function f, let $\boxed{\mathbb{I}_f} := \{f(x) \,|\, x \in \mathbb{D}_f\}$ denote the image of f.
For any fucntion f, for any set S, let

$\boxed{f_* S} := \{f(x) \,|\, x \in S \bigcap \mathbb{D}_f\}$ and let

$\boxed{f^* S} := \{x \in \mathbb{D}_f \,|\, f(x) \in S\}$.

For any function f, for any $x \in \mathbb{D}_f$, let $\boxed{f_x} := f(x)$.
Let $\boxed{\mathbb{R}^*} := \mathbb{R} \bigcup \{\infty, -\infty\}$.

DEFINITION 2. *Let* $a, b \in \mathbb{R}^*$.
Then: $\boxed{(a;b)} := \{x \in \mathbb{R}^* \,|\, a < x < b\}$, $\boxed{[a;b)} := \{x \in \mathbb{R}^* \,|\, a \leqslant x < b\}$,
$\boxed{(a;b]} := \{x \in \mathbb{R}^* \,|\, a < x \leqslant b\}$, $\boxed{[a;b]} := \{x \in \mathbb{R}^* \,|\, a \leqslant x \leqslant b\}$.

Let $\boxed{\mathbb{Z}^*} := \mathbb{Z} \bigcup \{\infty, -\infty\}$.

DEFINITION 3. *Let* $a, b \in \mathbb{R}^*$.
Then: $\boxed{(a..b)} := (a;b) \bigcap \mathbb{Z}^*$, $\boxed{[a..b)} := [a;b) \bigcap \mathbb{Z}^*$,
$\boxed{(a..b]} := (a;b] \bigcap \mathbb{Z}^*$, $\boxed{[a..b]} := [a;b] \bigcap \mathbb{Z}^*$.

Let $\boxed{\mathbb{N}} := [1..\infty)$ and let $\boxed{\mathbb{N}_0} := [0..\infty)$.
For any set S, for any $m \in \mathbb{N}$, let $\boxed{S^m} := S^{[1..m]}$.
For any set S, a $\boxed{\textbf{sequence}}$ in S is an element of $S^{\mathbb{N}}$.

For any topological space X, $\boxed{\mathcal{T}_X}$ denotes the set of open subsets of X.
Give $\mathbb{R}^*$ its standard topology.
For any topological space X, for any $W \subseteq X$,
give W the relative topology inherited from X.
For any finite set X, give X the discrete topology.

For any topological space X,
give X the Borel structure generated by $\mathcal{T}_X$.
For any Borel space X, for any $W \subseteq X$,
give W the relative Borel structure inherited from X.
For any countable set X, give X the discrete Borel structure.

DEFINITION 4. *Let X be a Borel space. Then:*
$\boxed{\mathcal{B}_X}$ *denotes the set of Borel subsets of X,*
$\boxed{\mathcal{M}_X}$ *denotes the set countably-additive functions $\mathcal{B}_X \to [0;\infty]$,*
$\boxed{\mathcal{FM}_X} := \{\mu \in \mathcal{M}_X \,|\, \mu(X) < \infty\}$,
$\boxed{\mathcal{PM}_X} := \{\mu \in \mathcal{M}_X \,|\, \mu(X) = 1\}$ *and*
$\boxed{\mathcal{BF}_X}$ *denotes the set of Borel bounded functions $X \to \mathbb{R}$.*

NOTE: For any countable set X, $\mathcal{BF}_X = \mathbb{R}^X$.

DEFINITION 5. *Let Ω be a finite non$\varnothing$ set.*
By an $\boxed{\Omega\text{-}MC}$, *we mean: a function $E : \Omega \times \Omega \to [0;1]$ s.t.*

$$\forall \phi \in \Omega, \quad \sum_{\psi \in \Omega} [\, E(\psi,\phi)\,] \;=\; 1.$$

In the preceding definition, "MC" stands for Markov-chain.
The set Ω is the set of "states", and
the quantity $E(\psi,\phi)$ should be thought of as
the probability of transitioning from state ϕ to ψ.
Since the state ϕ must transition to *some* state,
these probabilities should sum to 1 over ψ.

DEFINITION 6. *Let Ω be a finite non$\varnothing$ set, E and Ω-MC.*
For all $m \in \mathbb{N}$, let
$\boxed{\mathrm{Ch}_m E} := \{\omega \in \Omega^{[0..m]} \,|\, \forall j \in [1..m],\ E(\omega_j, \omega_{j-1}) > 0\}$.
For all $m \in \mathbb{N}$, let $\boxed{\mathrm{Cyc}_m E} := \{\omega \in \mathrm{Ch}_m E \,|\, \omega_0 = \omega_m\}$.
Let $\boxed{\mathrm{Per}_E} := \{m \in \mathbb{N} \,|\, \mathrm{Cyc}_m E \neq \varnothing\}$.

Elements of $\mathrm{Ch}_m E$ are called "chains in E".
Elements of $\mathrm{Cyc}_m E$ are called "cycles in E".

DEFINITION 7. *Let Ω be a finite non$\varnothing$ set, E an Ω-MC.*
By E is $\boxed{\textbf{symmetric}}$, *we mean:*

$\forall \phi, \psi \in \Omega, \quad E(\phi, \psi) = E(\psi, \phi).$

By E is $\boxed{\textbf{irreducible}}$*, we mean:*

$\forall \phi, \psi \in \Omega, \quad \exists \omega \in \mathrm{Ch}_m E$ *s.t.* $(\omega_0 = \phi)\,\&\,(\omega_m = \psi).$

By E is $\boxed{\textbf{aperiodic}}$*, we mean:* $\gcd \mathrm{Per}_E = 1.$

By E is $\boxed{\textbf{odd-periodic}}$*, we mean:* $\{1, 3, 5, 7, \ldots\} \bigcap \mathrm{Per}_E \neq \varnothing.$

THEOREM 8. *Let Ω be a finite non$\varnothing$ set, E an Ω-MC.*
Then: (i) $(\#\Omega = 1) \Rightarrow (E$ *is aperiodic*$)$
and (ii) $((E$ *is symmetric*$)\,\&\,(\#\Omega \geqslant 2)) \Rightarrow (2 \in \mathrm{Per}_E)$
and (iii) $((2 \in \mathrm{Per}_E)\,\&\,(E$ *is odd-periodic*$)) \Rightarrow (E$ *is aperiodic*$)$.
and (iv) $((E$ *is symmetric*$)\,\&\,(E$ *is odd-periodic*$)) \Rightarrow (E$ *is aperiodic*$)$.

Proof is omitted.

DEFINITION 9. *Let Ω be a finite non$\varnothing$ set, E, F Ω-MCs.*
Then the Ω-MC $\boxed{E * F}$ *is defined by:* $\forall \phi, \psi \in \Omega,$

$$(E * F)(\psi, \phi) = \sum_{\chi \in \Omega} [\,(E(\psi, \chi)) \cdot (E(\chi, \phi))\,].$$

For any Ω-MC E and any $m \in \mathbb{N}$, we define

$\boxed{*^m E} := E * E * \cdots * E \qquad (m \text{ times}).$

DEFINITION 10. *Let Ω be a finite non$\varnothing$ set, E an Ω-MC.*
Let $\nu \in \mathcal{PM}_\Omega$. Then $\boxed{E * \nu} \in \mathcal{PM}_\Omega$ *is defined by:*

$$\forall \phi \in \Omega, \quad (E * \nu)\{\phi\} = \sum_{\psi \in \Omega} (E(\phi, \psi)) \cdot (\nu\{\psi\}).$$

DEFINITION 11. *Let Ω be a finite non$\varnothing$ set, E an Ω-MC.*
Then: $\boxed{\mathcal{PM}_\Omega^E} := \{\nu \in \mathcal{PM}_\Omega \,|\, E * \nu = \nu\}.$

Elements of $\mathcal{PM}_\Omega^E$ are called "E-invariant probability measures on Ω".

According to the next result, for an irreducible E,
there can be at most one E-invariant probability measure:

THEOREM 12. *Let Ω be a finite non$\varnothing$ set, E an Ω-MC.*
Assume E is irreducible. Then $\#\mathcal{PM}_\Omega^E \leqslant 1.$

Proof omitted.

DEFINITION 13. *Let Ω be a finite non$\varnothing$ set.*
Then $\boxed{\nu_\Omega} \in \mathcal{PM}_\Omega$ *is defined by:* $\quad \forall \omega \in \Omega, \quad \nu_\omega\{\omega\} = \dfrac{1}{\#\Omega}.$

That is, ν_Ω gives equal probability to each state in the state-space Ω.

THEOREM 14. *Let Ω be a finite non$\emptyset$ set, E an Ω-MC. Assume E is symmetric. Then $\nu_\Omega \in \mathcal{PM}_\Omega^E$.*

Proof omitted.

THEOREM 15. *Let Ω be a finite non$\emptyset$ set, E an Ω-MC. Assume E is symmetric and irreducible. Then $\mathcal{PM}_\Omega^E = \{\nu_\Omega\}$.*

Proof omitted.