The main results in this note are:

DEFINITION 1. We define \( \#\emptyset := 0 \).
For any nonempty finite set \( S \),
\( \#S \) denotes the number of elements in \( S \).
For any infinite set \( S \), we define \( \#S := \infty \).

For any sets \( A, B \), let \( B^A \) denote the set of all functions \( A \rightarrow B \).
For any function \( f \), let \( \mathbb{D}_f \) denote the domain of \( f \).
For any function \( f \), let \( \mathbb{I}_f := \{ f(x) \mid x \in \mathbb{D}_f \} \) denote the image of \( f \).
For any function \( f \), for any set \( S \), let
\[
\begin{align*}
\mathbf{f}^\ast S &:= \{ f(x) \mid x \in S \cap \mathbb{D}_f \} \\
\mathbf{f}^\ast_\ast S &:= \{ x \in \mathbb{D}_f \mid f(x) \in S \}.
\end{align*}
\]

For any function \( f \), for any \( x \in \mathbb{D}_f \), let \( f_x := f(x) \).
Let \( \mathbb{R}^* := \mathbb{R} \cup \{ \infty, -\infty \} \).

DEFINITION 2. Let \( a, b \in \mathbb{R}^* \).
Then:
\[
\begin{align*}
(a; b) &:= \{ x \in \mathbb{R}^* \mid a < x < b \}, \\
[a; b) &:= \{ x \in \mathbb{R}^* \mid a \leq x < b \}, \\
(a; b] &:= \{ x \in \mathbb{R}^* \mid a < x \leq b \}, \\
[a; b] &:= \{ x \in \mathbb{R}^* \mid a \leq x \leq b \}.
\end{align*}
\]
Let \( \mathbb{Z}^* := \mathbb{Z} \cup \{ \infty, -\infty \} \).

DEFINITION 3. Let \( a, b \in \mathbb{R}^* \).
Then:
\[
\begin{align*}
(a..b) &:= (a; b) \cap \mathbb{Z}^*, \\
(a; b] &:= [a; b) \cap \mathbb{Z}^*, \\
(a..b] &:= (a; b] \cap \mathbb{Z}^*, \\
[a..b] &:= [a; b] \cap \mathbb{Z}^*.
\end{align*}
\]
Let \( \mathbb{N} := [1..\infty) \) and let \( \mathbb{N}_0 := [0..\infty) \).

For any set \( S \), for any \( m \in \mathbb{N} \), let \( S^m := S^{[1..m]} \).
For any set \( S \), a sequence in \( S \) is an element of \( S^\mathbb{N} \).

For any topological space \( X \), \( \mathcal{T}_X \) denotes the set of open subsets of \( X \).
Give \( \mathbb{R}^* \) its standard topology.
For any topological space \( X \), for any \( W \subseteq X \),
give \( W \) the relative topology inherited from \( X \).
For any finite set \( X \), give \( X \) the discrete topology.
For any topological space \( X \),
give \( X \) the Borel structure generated by \( \mathcal{T}_X \).
For any Borel space \( X \), for any \( W \subseteq X \),
give \( W \) the relative Borel structure inherited from \( X \).
For any countable set \( X \), give \( X \) the discrete Borel structure.

**DEFINITION 4.** Let \( X \) be a Borel space. Then:
\[
\begin{align*}
\mathcal{B}_X & \text{ denotes the set of Borel subsets of } X, \\
\mathcal{M}_X & \text{ denotes the set countably-additive functions } \mathcal{B}_X \rightarrow [0; \infty], \\
\mathcal{F}M_X & := \{ \mu \in \mathcal{M}_X \mid \mu(X) < \infty \}, \\
\mathcal{P}M_X & := \{ \mu \in \mathcal{M}_X \mid \mu(X) = 1 \} \text{ and} \\
\mathcal{B}F_X & \text{ denotes the set of Borel bounded functions } X \rightarrow \mathbb{R}.
\end{align*}
\]

**NOTE:** For any countable set \( X \), \( \mathcal{B}F_X = \mathbb{R}^X \).

**DEFINITION 5.** Let \( \Omega \) be a finite non\( \emptyset \) set.
By an \( \Omega\text{-MC} \), we mean: a function \( E : \Omega \times \Omega \rightarrow [0; 1] \) s.t.
\[
\forall \phi \in \Omega, \quad \sum_{\psi \in \Omega} [ E(\psi, \phi) ] = 1.
\]

In the preceding definition, “MC” stands for Markov-chain.
The set \( \Omega \) is the set of “states”, and
the quantity \( E(\psi, \phi) \) should be thought of as
the probability of transitioning from state \( \phi \) to \( \psi \).
Since the state \( \phi \) must transition to some state,
these probabilities should sum to 1 over \( \psi \).

**DEFINITION 6.** Let \( \Omega \) be a finite non\( \emptyset \) set, \( E \) and \( \Omega\text{-MC} \).
For all \( m \in \mathbb{N} \), let
\[
\text{Ch}_m E := \{ \omega \in \Omega^{[0..m]} \mid \forall j \in [1..m], \ E(\omega_j, \omega_{j-1}) > 0 \}.
\]

For all \( m \in \mathbb{N} \), let \( \text{Cyc}_m E := \{ \omega \in \text{Ch}_m E \mid \omega_0 = \omega_m \} \).
Let \( \text{Per}_E := \{ m \in \mathbb{N} \mid \text{Cyc}_m E \neq \emptyset \} \).

Elements of \( \text{Ch}_m E \) are called “chains in \( E \)”.
Elements of \( \text{Cyc}_m E \) are called “cycles in \( E \)”.

**DEFINITION 7.** Let \( \Omega \) be a finite non\( \emptyset \) set, \( E \) an \( \Omega\text{-MC} \).
By \( E \) is symmetric, we mean:
\forall \phi, \psi \in \Omega, \ E(\phi, \psi) = E(\psi, \phi).

By \( E \) is \underline{irreducible}, we mean:
\[ \forall \phi, \psi \in \Omega, \exists \omega \in \text{Ch}_m E \text{ s.t. } (\omega_0 = \phi) \& (\omega_3 = \psi). \]

By \( E \) is \underline{aperiodic}, we mean: \( \gcd \text{ Per}_E = 1 \).

By \( E \) is \underline{odd-periodic}, we mean: \( \{1, 3, 5, 7, \ldots\} \cap \text{Per}_E \neq \emptyset \).

**THEOREM 8.** Let \( \Omega \) be a finite non-\( \emptyset \) set, \( E \) an \( \Omega \)-MC. Then: (i) \( (\#\Omega = 1) \Rightarrow (E \text{ is aperiodic}) \)

and (ii) \( ((E \text{ is symmetric}) \& (\#\Omega \geq 2)) \Rightarrow (2 \in \text{Per}_E) \)

and (iii) \( ((2 \in \text{Per}_E) \& (E \text{ is odd-periodic}) \Rightarrow (E \text{ is aperiodic}) \)

and (iv) \( ((E \text{ is symmetric}) \& (E \text{ is odd-periodic}) \Rightarrow (E \text{ is aperiodic}) \).

Proof is omitted.

**DEFINITION 9.** Let \( \Omega \) be a finite non-\( \emptyset \) set, \( E, F \) \( \Omega \)-MCs.

Then the \( \Omega \)-MC \( \boxed{E \ast F} \) is defined by: \( \forall \phi, \psi \in \Omega, \)
\[ (E \ast F)(\psi, \phi) = \sum_{\chi \in \Omega} \left( [E(\psi, \chi) \cdot (E(\chi, \phi))] \right). \]

For any \( \Omega \)-MC \( E \) and any \( m \in \mathbb{N} \), we define
\[ \boxed{E^m} := E \ast E \ast \ldots \ast E \quad (m \text{ times}). \]

**DEFINITION 10.** Let \( \Omega \) be a finite non-\( \emptyset \) set, \( E \) an \( \Omega \)-MC.

Let \( \nu \in \mathcal{PM}_\Omega \). Then \( \boxed{E \ast \nu} \in \mathcal{PM}_\Omega \) is defined by:
\[ \forall \phi \in \Omega, \ (E \ast \nu)\{\phi\} = \sum_{\psi \in \Omega} (E(\phi, \psi)) \cdot (\nu\{\psi\}). \]

**DEFINITION 11.** Let \( \Omega \) be a finite non-\( \emptyset \) set, \( E \) an \( \Omega \)-MC.

Then: \( \boxed{\mathcal{PM}_\Omega^{E}} := \{\nu \in \mathcal{PM}_\Omega \mid E \ast \nu = \nu\}. \)

Elements of \( \mathcal{PM}_\Omega^{E} \) are called “\( E \)-invariant probability measures on \( \Omega \)”.

According to the next result, for an irreducible \( E \),

there can be at most one \( E \)-invariant probability measure:

**THEOREM 12.** Let \( \Omega \) be a finite non-\( \emptyset \) set, \( E \) an \( \Omega \)-MC.

Assume \( E \) is irreducible. Then \( \#\mathcal{PM}_\Omega^{E} \leq 1 \).

Proof omitted.

**DEFINITION 13.** Let \( \Omega \) be a finite non-\( \emptyset \) set.

Then \( \boxed{\nu_\Omega} \in \mathcal{PM}_\Omega \) is defined by: \( \forall \omega \in \Omega, \ \nu_\omega\{\omega\} = \frac{1}{\#\Omega}. \)
That is, $\nu_\Omega$ gives equal probability to each state in the state-space $\Omega$.

**THEOREM 14.** Let $\Omega$ be a finite non-$\emptyset$ set, $E$ an $\Omega$-MC. Assume $E$ is symmetric. Then $\nu_\Omega \in \mathcal{P}M^E_\Omega$.

Proof omitted.

**THEOREM 15.** Let $\Omega$ be a finite non-$\emptyset$ set, $E$ an $\Omega$-MC. Assume $E$ is symmetric and irreducible. Then $\mathcal{P}M^E_\Omega = \{\nu_\Omega\}$.

Proof omitted.