Professors and Grants

1. Introduction

This note is intended as a compliment and complement to
B. Zhang's very enjoyable "Coconuts and Islanders",
which motivates the Boltzmann distribution in the case where
every nonnegative integer is a possible energy-level.
Here, our initial focus is, instead, on Boltzmann distributions where
0 and 1 and 10 are the only possible energy-levels.
Taking our cue from "Coconuts and Islanders", we motivate by story.

From §3 to §13, we analyze **three systems** for
dispensing grant money to N professors.
Congress allocates N dollars to award to the N professors,
so the average award is: \$1. The grant rules stipulate: each professor receives \$0 or \$1 or \$10.
Each professor is identified by a number, from 1 to N.
By a **dispensation**, we mean a full complement of awards,
with a specific amount (\$0 or \$1 or \$10) to Professor#1,
a specific amount (\$0 or \$1 or \$10) to Professor#2,
etc., up to and including Professor#N,
such that the total of the awards is the \$$N$ allocated by Congress.

The **first system** (see §3) for awarding grants is very simple:
There are many possible dispensations, and, among all of them,
one is selected randomly,
giving equal probability to each possible dispensation.
The **main problem** is to figure out:
Using this first system, for a given professor,
what is the probability of being awarded \$0? \$1? \$10?

Later (see §5), we describe
second and third probabilistic award systems.
Each of these systems depends on three parameters p, q, r
satisfying $p, q, r > 0$ and $p + q + r = 1 = q + 10r$.
The **second system** uses
an iid system of random-variables, $X_1, \ldots, X_N$
such that, $\forall \ell$, $\Pr[X_\ell = 0] = p$,

$$\Pr[X_\ell = 1] = q,$$
$$\Pr[X_\ell = 10] = r.$$

For all ℓ, the second system awards X_ℓ dollars to Professor$\#\ell$.
The total dollar payout $X_1 + \cdots + X_N$ is then random;
if $X_1 = \cdots = X_N = 0$, it could be as small as 0 dollars,
and if $X_1 = \cdots = X_N = 10$, it could be as large as $10N$ dollars.
The **third system** is obtained from the second
by conditioning on the event $X_1 + \cdots + X_N = N$,
so that the total payout is exactly the $\$N$ allocated by Congress.

KEY POINT: With exactly the right choice of p, q, r,
the first and third systems are shown to be equivalent.
In §6 and §7, we show that this parameter choice is Boltzmann,
meaning: (p, q, r) is, for some real number β,
a scalar multiple of $(e^{-0\cdot\beta}, e^{-1\cdot\beta}, e^{-10\cdot\beta})$.
That is, $\exists \beta, C \in \mathbb{R}$ s.t. $(p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta})$.

The second and third systems are
accessible by basic tools of probability theory,
while the above "main problem" involves the first system.
However, once we know the first and third systems are equivalent,
we can bring these probabilistic tools to bear on the main problem.
Thanks to J. Steif, for pointing out to me that
the Discrete Local Limit Theorem, which is described in §10,
is the right tool for the main problem, which is solved in §13.

Boltzmann distributions are often motivated by entropy, but,
from our perspective,
what's special about $(p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta})$ is:
For any $i, j, k \geqslant 0$, we have
$p^i q^j r^k = C^{i+j+k} \cdot e^{-\beta\cdot(j+10k)}$,
so $p^i q^j r^k$ depends only on: $i + j + k$ and $j + 10k$.
In the third system of grant awards,
there exists a normalizing constant $S > 0$ s.t.,
for any dispensation in which
i professors receive \$ 0,
j professors receive \$ 1,
k professors receive \$10,

the probability of that dispensation is $p^i q^j r^k / S$,
which is equal to $C^{i+j+k} \cdot e^{-\beta \cdot (j+10k)} / S$.
That proabability, then, depends only on
$i + j + k$, which is the number of professors,
and $j + 10k$, which is the total dollar payout.
So, since the number of professors is equal to N
and the total dollar payout is also equal to N,
we conclude: each award-dispensation has probability $C^N \cdot e^{-\beta \cdot N} / S$,
so they are all equally likely, which exactly describes the first system.
Therefore, under the Boltzmann assumption,
the first and third systems are equivalent.

In §15, we expose the inequitablity of the first system.
In fact, assuming N is sufficiently large, we show, in §15, that:
with probability $> 99\%$, over half of the professors receive \$0.
Thanks to V. Reiner for suggesting
applying Chebyshev's inequality to a sum of indicator variables,
to transition from individual statistics to population statistics.
In §16 and §17 and §18, we extend the theory to handle cases where
the award-sets are arbitrary finite sets of rational numbers,
not necessarily equal to $\{0, 1, 10\}$.
In §19, we show that
irrational award amounts can lead to non-Boltzmann statistics.
In §20 and §21 and §22, we extend our earlier results to include
degenerate energy-levels, with a finite set of states.
In §23 through §27, we extend these results further to include
cases that involve a countably infinite set of states.

Thanks to C. Prouty for help with many calculations.
For some of his Python code, see §28.

2. Some notation

A box around an expression indicates that it is global,
meaning that it is fixed (or "bound") to the end of these notes.
Unboxed variables are freed at the end of each section, if not earlier.

Let $\boxed{\mathbb{R}^*} := \{-\infty\} \bigcup \mathbb{R} \bigcup \{\infty\}$, $\boxed{\mathbb{Z}^*} := \{-\infty\} \bigcup \mathbb{Z} \bigcup \{\infty\}$.
For any $s, t \in \mathbb{R}^*$, **let**

$$\boxed{(s;t)} := \{x \in \mathbb{R}^* \mid s < x < t\}, \quad \boxed{[s;t)} := \{x \in \mathbb{R}^* \mid s \leqslant x < t\},$$
$$\boxed{(s;t]} := \{x \in \mathbb{R}^* \mid s < x \leqslant t\}, \quad \boxed{[s;t]} := \{x \in \mathbb{R}^* \mid s \leqslant x \leqslant t\}.$$

For any $s, t \in \mathbb{R}^*$, **let** $\boxed{(s..t)} := (s;t) \bigcap \mathbb{Z}^*$, $\boxed{[s..t)} := [s;t) \bigcap \mathbb{Z}^*$,
$\boxed{(s..t]} := (s;t] \bigcap \mathbb{Z}^*$, $\boxed{[s..t]} := [s;t] \bigcap \mathbb{Z}^*$.

Let $\boxed{\mathbb{N}} := [1..\infty)$ be the set of positive integers.
For any finite set F, **let** $\boxed{\#F}$ be the number of elements in F.
For any infinite set F, **let** $\boxed{\#F} := \infty$. Then $\#\mathbb{Z} = \infty = \#\mathbb{R}$.
For any set F, we have: $\#F \in [0..\infty]$.
For all $t \in \mathbb{R}$, **let** $\boxed{\lfloor t \rfloor} := \max\{n \in \mathbb{N} \mid n \leqslant t\}$ be the $\boxed{\textbf{floor} \text{ of } t}$.
For any sets S, T, for any function $f : S \to T$,
the $\boxed{\textbf{image} \text{ of } f}$ is: $\boxed{\mathbb{I}_f} := \{ f(x) \mid x \in S \} \subseteq T$.
For any sets S, T, for any function $f : S \to T$,
for any set A, we **define**: $\boxed{f^* A} := \{x \in S \mid f(x) \in A\}$.
By convention, in these notes, we define $\boxed{0^0} := 1$.
By "C^ω" we mean: "real-analytic".
Choose an element of $\{z \in \mathbb{C} \mid z^2 = -1\}$ and denote it by $\boxed{\sqrt{-1}}$.
Define $\boxed{\Re} : \mathbb{C} \to \mathbb{R}$ and $\boxed{\Im} : \mathbb{C} \to \mathbb{R}$ by:
$\forall x, y \in \mathbb{R}$, $\Re(x + y\sqrt{-1}) = x$ and $\Im(x + y\sqrt{-1}) = y$.
For convenience of notation, we define:
$\forall z \in \mathbb{C}$, $\boxed{\Re_z} := \Re(z)$ and $\boxed{\Im_z} := \Im(z)$.

3. First system of grant awards

Let $\boxed{N} \in \mathbb{N}$. Think of N as large.
Whenever we need to formulate and prove
precise mathematical statements about N,
we will "pass to the thermodynamic limit", which means:
we replace N by a variable $n \in \mathbb{N}$, and let $n \to \infty$.
((Alternatively, within nonstandard analysis,
N could be defined as an *infinite* integer,
and the various approximations involving N,
could be defined as equality-modulo-infinitesimals.))
Suppose there are N professors, numbered 1 to N,
who apply, once per year, to the GFA (Grant Funding Agency),
seeking funding for the very important work they are doing.
Each year, Congress authorizes $\$N$ for the GFA to dispense

to the N professors.

The GFA has the rule: every award is 0 or 1 or 10 dollars.

The set of grant-dispensations is represented by:

$$\boxed{\Omega} := \left\{ \omega : [1..N] \to \{0,1,10\} \;\middle|\; \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}.$$

The GFA has set aside $\#\Omega$ pieces of paper,
and has written down all possible dispensations,
one on each piece of paper.

So, for example, there is a piece of paper that says:
Professors 1 to N each get \$1.

Another piece of paper says:
Professors 1 to $N-10$ each get \$1 and
Professors $N-9$ to $N-1$ each get \$0 and
Professor N gets \$10.

Since N is large, it follows that $\#\Omega$ is large, and so
there are many, many, many other pieces of paper.

Each year, a GFA bureaucrat
places all the pieces of paper in a big bin,
then selects one at random and
makes the awards as indicated on that piece of paper.

Under this **first system** of awarding grants, we have:
$\forall \omega \in \Omega$, the probability that
the selected grant-dispensation is ω
is equal to $1/(\#\Omega)$.

Suppose I am one of the professors. Here is our **main problem**:
Calculate my probability of getting \$0.
Then calculate my probability of getting \$1.
Then calculate my probability of getting \$10.

Approximate answers are acceptable.

In §5 to §13 of this note,
we reformulate and then solve this problem.

Spoiler: It's a Boltzmann distribution, approximately.

4. Particles and energy

Recall that $N \in \mathbb{N}$. Think of N as large.

Suppose there are N particles, numbered 1 to N,
each of which has a certain amount of energy.

Suppose the total energy is N, dispensed among the N particles.

Suppose physicists have somehow determined that, for any particle,
its possible energy-levels are: 0 or 1 or 10.
Recall: $\Omega = \left\{ \omega : [1..N] \to \{0, 1, 10\} \;\middle|\; \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}$.
Then Ω represents the set of energy-dispensations.
Assume that physicists have somehow determined
that this system of particles has a random energy-dispensation
and that all energy-dispensations in Ω are equally probable.
That is, physicists tell us:
$\forall \omega \in \Omega$, the probability that
the energy-dispensation is ω
is equal to $1 / (\#\Omega)$.
The equal probability of all energy-dispensations
is a recurring theme in microcanonical-ensemble thermodynamics,
and can often be motivated through
rules of random energy transfer between random pairs of particles.
For examples of this, either see §20 below or
search for "Coconuts and Islanders" by B. Zhang,
and, in particular, see the work leading up to
the last paragraph of §3.2 therein.
In §20 below,
instead of particles exchanging energy,
there are professors exchanging dollars,
but the principle is exactly the same.
In Zhang's exposition,
instead of particles exchanging energy,
there are islanders exchanging coconuts,
but the principle is exactly the same.

Returing to our N particles, pick any one of them.
Problem: Calculate its probability of having energy-level 0.
Then calculate its probability of having energy-level 1.
Then calculate its probability of having energy-level 10.
Approximate answers are acceptable.
Spoiler: It's a Boltzmann distribution, approximately.

Except for terminology, this problem is the same as
the main problem (end of §3) about professors and grants.

We will go back to professors and grants.
Mathematically it makes no difference, but it's more fun.

5. Second and third systems of grant awards

In an effort to go paperless, the GFA changes to a new system:
In this **second system**, instead of all those pieces of paper,
the GFA chooses $p, q, r > 0$ s.t. $p + q + r = 1$,
and then, for each of the N professors,
awards \$ 0 with probability p,
\$ 1 with probability q,
\$10 with probability r.
No professor's award depends in any way on any other professor's;
the awards are independent.
The expected payout, for any professor, is $p \cdot 0 + q \cdot 1 + r \cdot 10$ dollars.
Under this second system,
there is no guarantee that the total payout will be \$$N$,
which is a difficulty that we will discuss later.
However, recognizing that the average award is *intended* to be \$1,
the GFA chooses the numbers p, q, r subject to the constraint that

$$p \cdot 0 + q \cdot 1 + r \cdot 10 = 1, \quad \textit{i.e.}, \quad q + 10r = 1.$$

For each function $\omega : [1..N] \to \{0, 1, 10\}$, **let**

$$\boxed{i_\omega} := \#\{\ \ell \in [1..N] \mid \omega(\ell) = 0\ \},$$
$$\boxed{j_\omega} := \#\{\ \ell \in [1..N] \mid \omega(\ell) = 1\ \},$$
$$\boxed{k_\omega} := \#\{\ \ell \in [1..N] \mid \omega(\ell) = 10\ \};$$

that is, i_ω is the number of professors awarded \$ 0 and
j_ω is the number of professors awarded \$ 1 and
k_ω is the number of professors awarded \$10.
Then, $\forall \omega : [1..N] \to \{0, 1, 10\}$, we have:
the total number of awards is $i_\omega + j_\omega + k_\omega$
and the total dollar payout is $i_\omega \cdot 0 + j_\omega \cdot 1 + k_\omega \cdot 10$,
i.e., $j_\omega + 10k_\omega$.
Then, $\forall \omega : [1..N] \to \{0, 1, 10\}$, we have:

$$i_\omega + j_\omega + k_\omega = N \quad \text{and} \quad j_\omega + 10k_\omega = \textstyle\sum_{\ell=1}^N [\omega(\ell)].$$

Recall: $\Omega = \left\{ \omega : [1..N] \to \{0, 1, 10\} \;\middle|\; \sum_{\ell=1}^N [\omega(\ell)] = N \right\}$.
That is, Ω is the set of all payout functions

$$\omega : [1..N] \to \{0, 1, 10\}$$

s.t. the total dollar payout is N.

Then: $\forall \omega : [1..N] \to \{0, 1, 10\}$, we have:

$$\omega \in \Omega \quad \Leftrightarrow \quad j_\omega + 10k_\omega = N.$$

For every $i, j, k \in [0..N]$,
if $i + j + k = N$ and $j + 10k = N$,
then $\exists \omega \in \Omega$ s.t. $(i, j, k) = (i_\omega, j_\omega, k_\omega)$;
indeed, one such $\omega : [1..N] \to \{0, 1, 10\}$ is described by:
$\omega = 0$ on $[1..i]$, $\omega = 1$ on $(i..i + j]$, $\omega = 10$ on $(i + j..N]$.
Let $\boxed{A} := \{(i_\omega, j_\omega, k_\omega) \,|\, \omega \in \Omega\}$.
Then A is the set of all (i, j, k) s.t. $i, j, k \in [0..N]$ and
$i + j + k = N$ and $j + 10k = N$.
Under the second system,
each \$ 0 award happens with probability p and
each \$ 1 award happens with probability q and
each \$10 award happens with probability r.
So, $\forall \omega : [1..N] \to \{0, 1, 10\}$, under the second system,
the probability that the grant-dispensation is equal to ω
is $p^{i_\omega} q^{j_\omega} r^{k_\omega}$.
Let $S := \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega}$.
Then S is the probability (using the second system) that $\omega \in \Omega$,
i.e., the probability that the total payout is exactly N dollars.
Assuming N is large, it turns out that S is close to zero.
So, under this second system,
the probability of paying out exactly N dollars
is very small.
Congress only allocates \$$N$ per year for the N professors.
So, using this second system, each year,
with probability $1 - S \approx 1$, the GFA will run a surplus or a deficit.
On the other hand, since $q + 10r = 1$, we see that,
each year, the expected payout is \$1 per professor,
so, each year, the expected total payout is \$$N$.
So these surpluses and deficits should, over time, cancel one another.
Unfortunately, Congress is a paragon of fiscal responsibility, and,
as soon as it finds out about the GFA's second system,
it insists that the GFA never again underspend or overspend.
So the GFA changes its system one more time, as follows.
Under its **third system**, each year,
before announcing any of the awards publicly,
the GFA writes out, in an *internal* memo,

a *tentative* proposal of awards that,
independently, for each of the N professors,
awards \$ 0 with probability p,
\$ 1 with probability q,
\$10 with probability r.

If the memo's total award payout is NOT equal to $\$N$,
the GFA deems the memo as unacceptable,
deletes it, and starts over, making memo after memo,
until an acceptable one (meaning payout exactly $\$N$) appears.

Each memo has a probability S of being acceptable, so, each year,
the GFA will likely need to repeat the memo process many times
to get to a memo with total payout exactly equal to $\$N$.

However, as soon as that happens,
the GFA uses that first acceptable memo,
and publicizes its dispensation of awards.

Mathematically, we are conditioning on the event $\omega \in \Omega$.

So, using the third system, the probability that $\omega \notin \Omega$ is 0.

Also, for this third system, $\forall \omega \in \Omega$, the probability of ω is $p^{i_\omega} q^{j_\omega} r^{k_\omega} / S$.

The sum of these probabilities is 1:

$$\sum_{\omega \in \Omega} \frac{p^{i_\omega} q^{j_\omega} r^{k_\omega}}{S} = \frac{1}{S} \cdot \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega} = \frac{1}{S} \cdot S = 1.$$

This third system is not necessarily equivalent to the first, because
in the first system, all the probabilities were $1 / (\#\Omega)$,
whereas, in the third system, they are $p^{i_\omega} q^{j_\omega} r^{k_\omega} / S$.

So a **new question** arises:
Is it possible to choose $p, q, r > 0$ in such a way that

$$p + q + r = 1 \quad \text{and} \quad q + 10r = 1 \quad \text{and}$$
$$\forall \omega \in \Omega, \quad p^{i_\omega} q^{j_\omega} r^{k_\omega} / S = 1 / (\#\Omega) \qquad ?$$

If yes, then, using that (p, q, r),
the first and third systems are equivalent.

We will see that the answer to this new question, in fact, *is* yes.

In the next two sections, assuming $N \geqslant 10$,
we will show how to compute the only (p, q, r) that works.

Spoiler: It's a Boltzmann distribution, exactly.

6. Computing p, q, r à la Boltzmann

Recall (§3): $\Omega \;=\; \left\{ \omega : [1..N] \to \{0, 1, 10\} \;\middle|\; \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}$.
As in the preceding section, **let** $p, q, r > 0$, $S := \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega}$.
We assume: $p + q + r = 1$ and $q + 10r = 1$.
We also assume: $\forall \omega \in \Omega,\;\; p^{i_\omega} q^{j_\omega} r^{k_\omega} / S \;=\; 1 / (\#\Omega)$.
We will prove that, if $N \geqslant 10$, then
there is at most one (p, q, r) that satisfies these conditions,
specifically, $(p, q, r) = \dfrac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$.

Define the dot product, $\odot$, on $\mathbb{R}^3$, by:
$\forall x, y, z, X, Y, Z \in \mathbb{R},\;\; (x, y, z) \odot (X, Y, Z) = xX + yY + zZ$.
For all $u \in \mathbb{R}^3$, **let** $u^\perp \;:=\; \{ v \in \mathbb{R}^3 \mid u \odot v = 0 \}$;
then $u^\perp$ is a vector subspace of $\mathbb{R}^3$.
Also, $\forall u \in \mathbb{R}^3,\;\; u \in u^{\perp\perp}$.
For all $U \subseteq \mathbb{R}^3$, **let** $U^\perp \;:=\; \{ v \in \mathbb{R}^3 \mid \forall u \in U,\; u \odot v = 0 \}$;
then $U^\perp$ is a vector subspace of $\mathbb{R}^3$.
Also, $\forall t \in \mathbb{R}^3, \forall U \subseteq \mathbb{R}^3,\qquad (\, t \in U \,) \;\Rightarrow\; (\, t^\perp \supseteq U^\perp \,)$.
Also, $\forall T, U \subseteq \mathbb{R}^3,\qquad (\, T \subseteq U \,) \;\Rightarrow\; (\, T^\perp \supseteq U^\perp \,)$.
For all $u, v \in \mathbb{R}^3$, **let** $\langle u, v \rangle_{\text{span}}$ denote the $\mathbb{R}$-span of $\{u, v\}$, *i.e.*,
let $\langle u, v \rangle_{\text{span}} \;:=\; \{ su + tv \mid s, t \in \mathbb{R} \}$;
then $\langle u, v \rangle_{\text{span}}$ is a vector subspace of $\mathbb{R}^3$.
Recall (§5): $A \;=\; \{(i_\omega, j_\omega, k_\omega) \,|\, \omega \in \Omega\}$.
Recall (§5): A is the set of all (i, j, k) s.t. $i, j, k \in [0..N]$ and
$i + j + k = N$ and $j + 10k = N$.
Then: A is the set of all (i, j, k) s.t. $i, j, k \in [0..N]$ and
$(1, 1, 1) \odot (i, j, k) = N$ and $(0, 1, 10) \odot (i, j, k) = N$.
For all $a, b \in A$, we have
$(1, 1, 1) \odot a = N = (1, 1, 1) \odot b$ and
$(0, 1, 10) \odot a = N = (0, 1, 10) \odot b$,
so we get
$(1, 1, 1) \odot (a - b) = 0$ and $(0, 1, 10) \odot (a - b) = 0$,
so $a - b \in (1, 1, 1)^\perp \bigcap (0, 1, 10)^\perp$.
Let $V := (1, 1, 1)^\perp \bigcap (0, 1, 10)^\perp$.
Then: $\forall a, b \in A,\;\; a - b \in V$.
Let $D := \{a - b \,|\, a, b \in A\}$. Then $D \subseteq V$.
Also, we have: $V \subseteq (1, 1, 1)^\perp$ and $V \subseteq (0, 1, 10)^\perp$.
Then: $V^\perp \supseteq (1, 1, 1)^{\perp\perp}$ and $V^\perp \supseteq (0, 1, 10)^{\perp\perp}$.

Since $(1,1,1) \in (1,1,1)^{\perp\perp} \subseteq V^\perp$ and $(0,1,10) \in (0,1,10)^{\perp\perp} \subseteq V^\perp$,
we get: $\langle\, (1,1,1)\,,\, (0,1,10)\,\rangle_{\text{span}} \subseteq V^\perp$.

Let $W := \langle\, (1,1,1)\,,\, (0,1,10)\,\rangle_{\text{span}}$. Then: $W \subseteq V^\perp$.

Assume $N \geqslant 10$. **Let** $a_1 := (0,N,0)$, $a_2 := (9, N-10, 1)$.

Then $a_1, a_2 \in A$. **Let** $d_1 := a_2 - a_1$. Then $d_1 \in D$.

Since $d_1 \neq (0,0,0)$, we get: $\dim d_1^\perp = 2$.

Since $W = \langle (1,1,1),(0,1,10)\rangle_{\text{span}}$, we get: $\dim W = 2$.

Since $d_1 \in D \subseteq V$ and $W \subseteq V^\perp$, we get: $d_1^\perp \supseteq D^\perp \supseteq V^\perp \supseteq W$.

So, since $\dim d_1^\perp = 2 = \dim W$, we get: $d_1^\perp = D^\perp = V^\perp = W$.

Then $D^\perp = W$. Recall: $\forall \omega \in \Omega$, $p^{i_\omega} q^{j_\omega} r^{k_\omega} / S \;=\; 1/(\#\Omega)$.

So, since $A = \{(i_\omega, j_\omega, k_\omega)\,|\, \omega \in \Omega\}$, we get:

$$\forall (i,j,k) \in A, \qquad p^i q^j r^k \,/\, S \;=\; 1/(\#\Omega).$$

Equivalently, $\forall (i,j,k) \in A$,

$$i \cdot (\ln p) + j \cdot (\ln q) + k \cdot (\ln r) - (\ln S) \;=\; -(\ln(\#\Omega)).$$

Equivalently, $\forall (i,j,k) \in A$,

$$(i,j,k) \odot (\, \ln p\,,\, \ln q\,,\, \ln r\,) = (\ln S) - (\ln(\#\Omega)).$$

Then: $\forall a, b \in A$,

$$a \odot (\, \ln p\,,\, \ln q\,,\, \ln r\,) = (\ln S) - (\ln(\#\Omega)) = b \odot (\, \ln p\,,\, \ln q\,,\, \ln r\,),$$

so we get: $(a-b) \odot (\, \ln p\,,\, \ln q\,,\, \ln r\,) = 0$.

Then: $\forall d \in D$, $d \odot (\, \ln p\,,\, \ln q\,,\, \ln r\,) = 0$.

Then: $(\, \ln p\,,\, \ln q\,,\, \ln r\,) \in D^\perp$.

Since $(\, \ln p\,,\, \ln q\,,\, \ln r\,) \in D^\perp = W = \langle (1,1,1),(0,1,10)\rangle_{\text{span}}$,
choose a real number $C > 0$ and $\beta \in \mathbb{R}$ s.t.

$$(\, \ln p\,,\, \ln q\,,\, \ln r\,) \;=\; (\ln C) \cdot (1,1,1) - \beta \cdot (0,1,10).$$

Then $(\, \ln p\,,\, \ln q\,,\, \ln r\,) \;=\; (\; \ln C\;,\; (\ln C) - \beta\;,\; (\ln C) - 10\beta\;)$.

Then $(p,q,r) \;=\; (C, Ce^{-\beta}, Ce^{-10\beta})$.

Then $(p,q,r) \;=\; C \cdot (1, e^{-\beta}, e^{-10\beta})$.

So, since $p+q+r=1$, we get: $C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1$.

Then $C = \dfrac{1}{1 + e^{-\beta} + e^{-10\beta}}$. Then $(p,q,r) \;=\; \dfrac{(1, e^{-\beta}, e^{-10\beta})}{1 + e^{-\beta} + e^{-10\beta}}$.

So, since $q + 10r = 1$, we get: $\dfrac{e^{-\beta} + 10e^{-10\beta}}{1 + e^{-\beta} + e^{-10\beta}} = 1$.

Then $e^{-\beta} + 10e^{-10\beta} = 1 + e^{-\beta} + e^{-10\beta}$. Then $9e^{-10\beta} = 1$.

Then $e^{-10\beta} = 9^{-1}$. Then $e^{-\beta} = 9^{-1/10}$. Then $(p,q,r) \;=\; \dfrac{(\,1\,,\, 9^{-1/10}\,,\, 9^{-1})}{1 \;+\; 9^{-1/10} \;+\; 9^{-1}}$.

So this is the only (p,q,r) that can possibly work.
In the next section, we show that it *does* work.

7. Showing the Boltzmann p, q, r work

In this section, we prove
the converse of the result from the preceding section.

That is, we **let** $(p, q, r) := \dfrac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$ and $S := \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega}$,
and **we wish to show**: $p + q + r = 1$ and $q + 10r = 1$ and
$\forall \omega \in \Omega, \quad p^{i_\omega} q^{j_\omega} r^{k_\omega} / S = 1 / (\#\Omega)$.

Let $\beta := (\ln 9)/10$. Then $e^{-\beta} = 9^{-1/10}$. Then $e^{-10\beta} = 9^{-1}$.
Then $(p, q, r) = \dfrac{(1, e^{-\beta}, e^{-10\beta})}{1 + e^{-\beta} + e^{-10\beta}}$. **Let** $C := \dfrac{1}{1 + e^{-\beta} + e^{-10\beta}}$.
Then $(p, q, r) = C \cdot (1, e^{-\beta}, e^{-10\beta})$. Then $(p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta})$.
Let $K := C^N \cdot e^{-\beta \cdot N}$.
Recall (§3): $\Omega = \left\{ \omega : [1..N] \to \{0, 1, 10\} \;\middle|\; \sum_{\ell=1}^{N} [\omega(\ell)] = N \right\}$.

Claim: $\forall \omega \in \Omega, \quad p^{i_\omega} q^{j_\omega} r^{k_\omega} = K$.
Proof of Claim: **Given** $\omega \in \Omega$, **want:** $p^{i_\omega} q^{j_\omega} r^{k_\omega} = K$.
Recall (§5): $i_\omega + j_\omega + k_\omega = N$ and $j_\omega + 10k_\omega = \sum_{\ell=1}^{N} [\omega(\ell)]$.
By definition of Ω, since $\omega \in \Omega$, we get: $\sum_{\ell=1}^{N} [\omega(\ell)] = N$.
Then: $j_\omega + 10k_\omega = N$. Recall: $(p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta})$.
Then:
$$\begin{aligned} p^{i_\omega} q^{j_\omega} r^{k_\omega} &= C^{i_\omega} \cdot (Ce^{-\beta})^{j_\omega} \cdot (Ce^{-10\beta})^{k_\omega} \\ &= C^{i_\omega + j_\omega + k_\omega} \cdot e^{-\beta \cdot (j_\omega + 10k_\omega)} = C^N \cdot e^{-\beta \cdot N} = K. \end{aligned}$$
End of proof of Claim.

By definition of S, we have: $S = \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega}$.
So, by the Claim, we get: $S = (\#\Omega) \cdot K$. Then $K/S = 1/(\#\Omega)$.
We have $10/9 = 1 + (1/9)$. That is, $10 \cdot 9^{-1} = 1 + 9^{-1}$.
So, since $e^{-10\beta} = 9^{-1}$, we get: $10e^{-10\beta} = 1 + e^{-10\beta}$.
Then: $e^{-\beta} + 10e^{-10\beta} = 1 + e^{-\beta} + e^{-10\beta}$.
Recall: $(p, q, r) = C \cdot (1, e^{-\beta}, e^{-10\beta})$.
By definition of C, we get: $C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1$.
Since $p + q + r = C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1$
and since
$$\begin{aligned} q + 10r &= C \cdot (e^{-\beta} + 10e^{-10\beta}) \\ &= C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1, \end{aligned}$$
it remains only to show: $\forall \omega \in \Omega, \quad p^{i_\omega} q^{j_\omega} r^{k_\omega} / S = 1 / (\#\Omega)$.
Given $\omega \in \Omega$, **want:** $p^{i_\omega} q^{j_\omega} r^{k_\omega} / S = 1 / (\#\Omega)$.
By the Claim, we get: $p^{i_\omega} q^{j_\omega} r^{k_\omega} = K$.

Recall: $K/S = 1/(\#\Omega)$.

Then: $p^{i_\omega} q^{j_\omega} r^{k_\omega}/S = K/S = 1/(\#\Omega)$.

8. Unordered summation

The theorems in this section are all basic. We omit proofs.

In the next definition, "SP" stands for "semi-positive".

DEFINITION 8.1. Let *I be a set,* $a : I \to [0;\infty]$. *For all $i \in I$,* **let** $a_i := a(i)$.
Let $\mathcal{F} := \{F \subseteq I \,|\, \#F < \infty\}$.
Then the $\boxed{\textit{SP-sum, over } i \in I, \textit{ of } a_i}$ *is:*

$$\boxed{\sum_{i\in I}^{\mathrm{SP}} a_i} := \sup_{F\in\mathcal{F}} \left[\sum_{i\in F} a_i \right] \in [0;\infty].$$

For any set I, we have: $\#I := \sum_{i\in I}^{\mathrm{SP}} 1$.

DEFINITION 8.2. Let *I be a set,* $a : I \to \mathbb{C}$.
For all $i \in I$, **let** $a_i := a(i)$.
Let $\mathcal{F} := \{F \subseteq I \,|\, \#F < \infty\}$, *directed by inclusion.*
By $\boxed{a \textit{ is summable in } \mathbb{C}}$, *we mean:*

$\exists s \in \mathbb{C}$ *s.t., as $F \to \infty$ in $\mathcal{F}$,* $\sum_{i\in F} a_i \to s$.

By the definition of limit (over a directed set),

"as $F \to \infty$ in $\mathcal{F}$, $\sum_{i\in F} a_i \to s$"

means

"$\forall$real $\delta > 0$, $\exists F_0 \in \mathcal{F}$ s.t., $\forall F \in \mathcal{F}$,
$(F \supseteq F_0) \Rightarrow (|(\sum_{i\in F} a_i) - s| < \delta)$".

THEOREM 8.3. Let *I be a set,* $a : I \to \mathbb{C}$.
For all $i \in I$, **let** $a_i := a(i)$.
Then: (*a is summable in $\mathbb{C}$*) $\Leftrightarrow$ ($\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$).

DEFINITION 8.4. Let *I be a set,* $a : I \to \mathbb{C}$.
For all $i \in I$, **let** $a_i := a(i)$. *Assume:* $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$.
Let $\mathcal{F} := \{F \subseteq I \,|\, \#F < \infty\}$, *directed by inclusion.*
Then the $\boxed{\textit{sum, over } i \in I, \textit{ of } a_i}$ *is:*

$$\boxed{\sum_{i\in I} a_i} := \left(\text{the limit, as } F \to \infty \text{ in } \mathcal{F}, \text{ of } \sum_{i\in F} a_i \right).$$

THEOREM 8.5. **Let** *$k \in \mathbb{Z}$, $I := [k..\infty)$, $a : I \to [0;\infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$. Then: $\sum_{i\in I}^{\mathrm{SP}} a_i = \sum_{i=k}^{\infty} a_i$.*

THEOREM 8.6. **Let** *$k \in \mathbb{Z}$, $I := [k..\infty)$, $a : I \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$. Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$.*
Then: $\sum_{i\in I} a_i = \sum_{i=k}^{\infty} a_i$.

THEOREM 8.7. **Let** *I be a set, $a : I \to [0;\infty)$.*
For all $i \in I$, **let** *$a_i := a(i)$.*
Assume: $\sum_{i\in I}^{\mathrm{SP}} a_i < \infty$. Then: $\sum_{i\in I} a_i = \sum_{i\in I}^{\mathrm{SP}} a_i$.

We have a Cauchy-Schwarz result:

THEOREM 8.8. **Let** *I be a set, $a : I \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$.*
Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$. Then: $|\sum_{i\in I} a_i| \leqslant \sum_{i\in I}^{\mathrm{SP}} |a_i|$.

THEOREM 8.9. **Let** *I be a set, $a : I \to [0;\infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$.* **Let** *$J, K \subseteq I$. Assume: $J \subseteq K$.*
Then: $\sum_{i\in J}^{\mathrm{SP}} a_i \leqslant \sum_{i\in K}^{\mathrm{SP}} a_i$.

THEOREM 8.10. **Let** *I be a set, $a, b : I \to [0;\infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$ and $b_i := b(i)$.*
Assume: $\forall i \in \mathbb{I}$, $a_i \leqslant b_i$. Then: $\sum_{i\in I}^{\mathrm{SP}} a_i \leqslant \sum_{i\in I} b_i$.

THEOREM 8.11. **Let** *I be a set, $a : I \to [0;\infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$.* **Let** *$J, K \subseteq I$. Assume: $J \bigcap K = \varnothing$.*
Then: $(\sum_{i\in J}^{\mathrm{SP}} a_i) + (\sum_{i\in K}^{\mathrm{SP}} a_i) = \sum_{i\in J\bigcup K}^{\mathrm{SP}} a_i$.

THEOREM 8.12. **Let** *I be a set, $a : I \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$. Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$.*
Let *$J, K \subseteq I$. Assume: $J \bigcap K = \varnothing$.*
Then: $(\sum_{i\in J} a_i) + (\sum_{i\in K} a_i) = \sum_{i\in J\bigcup K} a_i$.

THEOREM 8.13. **Let** *I be a set, $c \in [0;\infty]$, $a : I \to [0;\infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$. Then: $\sum_{i\in I}^{\mathrm{SP}} [c \cdot a_i] = c \cdot \sum_{i\in I}^{\mathrm{SP}} a_i$.*

THEOREM 8.14. **Let** *I be a set, $c \in \mathbb{C}$, $a : I \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$.*
Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$. Then: $\sum_{i\in I}^{\mathrm{SP}} |c \cdot a_i| < \infty$.
Also, $\sum_{i\in I} [c \cdot a_i] = c \cdot \sum_{i\in I} a_i$.

THEOREM 8.15. **Let** *I be a set, $a, b : I \to [0;\infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$ and $b_i := b(i)$.*
Then: $\sum_{i\in I}^{\mathrm{SP}} [a_i + b_i] = (\sum_{i\in I}^{\mathrm{SP}} a_i) + (\sum_{i\in I}^{\mathrm{SP}} b_i)$.

THEOREM 8.16. Let *I be a set, $a, b : I \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$ and $b_i := b(i)$.*
Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$ and $\sum_{i\in I}^{\mathrm{SP}} |b_i| < \infty$.
Then: $\sum_{i\in I}^{\mathrm{SP}} |a_i + b_i| < \infty$.
Also, $\sum_{i\in I} [a_i + b_i] \;=\; (\sum_{i\in I} a_i) + (\sum_{i\in I} b_i)$.

THEOREM 8.17. Let *I, J be sets, $f : I \to J$, $a : I \to [0; \infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$.*
Then: $\sum_{j\in J}^{\mathrm{SP}} [\sum_{i\in f^*\{j\}}^{\mathrm{SP}} a_i] \;=\; \sum_{i\in I}^{\mathrm{SP}} a_i$.

THEOREM 8.18. Let *I, J be sets, $f : I \to J$, $a : I \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$. Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$.*
Then: $\forall j \in J, \quad \sum_{i\in f^*\{j\}}^{\mathrm{SP}} |a_i| \quad < \quad \infty$.
Also, $\sum_{j\in J}^{\mathrm{SP}} \mid \sum_{i\in f^*\{j\}} \; a_i \mid \quad < \quad \infty$.
Also, $\sum_{j\in J} [\; \sum_{i\in f^*\{j\}} \; a_i \;] \quad = \quad \sum_{i\in I} a_i$.

THEOREM 8.19. Let *I, J be sets, $a : I \to [0; \infty]$, $b : J \to [0; \infty]$.*
For all $i \in I$, **let** *$a_i := a(i)$. For all $j \in J$,* **let** *$b_j := b(j)$.*
Then: $(\sum_{i\in I}^{\mathrm{SP}} a_i) \cdot (\sum_{j\in J}^{\mathrm{SP}} b_j) = \sum_{i\in I}^{\mathrm{SP}} [\sum_{j\in J}^{\mathrm{SP}} [a_i \cdot b_j]]$.

THEOREM 8.20. Let *I, J be sets, $a : I \to \mathbb{C}$, $b : J \to \mathbb{C}$.*
For all $i \in I$, **let** *$a_i := a(i)$. For all $j \in J$,* **let** *$b_j := b(j)$.*
Assume: $\sum_{i\in I}^{\mathrm{SP}} |a_i| < \infty$ and $\sum_{j\in J}^{\mathrm{SP}} |b_j| < \infty$.
Then: $\forall j \in J, \quad \sum_{j\in J}^{\mathrm{SP}} |a_i \cdot b_j| \;<\; \infty$.
Also, $\sum_{i\in I}^{\mathrm{SP}} \mid \sum_{j\in J}^{\mathrm{SP}} [a_i \cdot b_j] \mid \;<\; \infty$.
Also, $(\sum_{i\in I} a_i) \cdot (\sum_{j\in J} b(j)) = \sum_{i\in I} [\sum_{j\in J} [a_i \cdot b_j]]$.

9. Countable measure theory

By convention, in this note,
any countable set is given its discrete Borel structure.

Let Θ be a countable set. **Let** $\mathcal{B}$ be the set of subsets of Θ.
A $\boxed{\textbf{measure}}$ on Θ is a function $\mu : \mathcal{B} \to [0; \infty]$
such that, $\forall$pairwise-disjoint $\Theta_1, \Theta_2, \ldots \subseteq \Theta$, we have:

$$\mu(\Theta_1 \bigcup \Theta_2 \bigcup \cdots) = (\mu(\Theta_1)) + (\mu(\Theta_2)) + \cdots.$$

Recall (§8): the notation $\sum^{\mathrm{SP}}$.
A measure μ on a countable set Θ
is completely determined by

the function $t \mapsto \mu\{t\} : \Theta \to [0;\infty]$,
because: $\forall \Theta_0 \subseteq \Theta$, we have $\mu(\Theta_0) = \sum_{t\in\Theta_0}^{\mathrm{SP}} [\,\mu\{t\}\,]$.

DEFINITION 9.1. Let Θ *be a countable set.*
Then $\boxed{\mathcal{M}_\Theta}$ *denotes the set of measures on* Θ,
and $\boxed{\mathcal{FM}_\Theta} := \{\mu \in \mathcal{M}_\Theta \,|\, \mu(\Theta) < \infty\}$,
and $\boxed{\mathcal{FM}_\Theta^\times} := \{\mu \in \mathcal{M}_\Theta \,|\, 0 < \mu(\Theta) < \infty\}$,
and $\boxed{\mathcal{P}_\Theta} := \{\mu \in \mathcal{M}_\Theta \,|\, \mu(\Theta) = 1\}$.

Then $\mathcal{M}_\Theta$ is the set of measures on Θ
and $\mathcal{FM}_\Theta$ is the set of finite measures on Θ
and $\mathcal{FM}_\Theta^\times$ is the set of nonzero finite measures on Θ
and $\mathcal{P}_\Theta$ is the set of probability measures on Θ.
The only measure on $\varnothing$ is the zero measure.
Therefore: $\mathcal{FM}_\varnothing^\times = \varnothing = \mathcal{P}_\varnothing$.

DEFINITION 9.2. Let Θ *be a countable set,* $\mu \in \mathcal{FM}_\Theta$.
Let $n \in \mathbb{N}$. *Then* $\boxed{\mu^n} \in \mathcal{FM}_{\Theta^n}$ **is defined by:**
$\forall x \in \Theta^n$, $\mu^n\{x\} = (\mu\{x_1\}) \cdots (\mu\{x_n\})$.

The following is a basic fact, whose proof we omit:
Let Θ be a countable set, $\mu \in \mathcal{FM}_\Theta$, $n \in [2..\infty)$.
Let $Z \subseteq \Theta^n$, $X \subseteq \Theta^{n-1}$, $Y \subseteq \Theta$. Assume that:
under the standard bijection $\Theta^n \longleftrightarrow \Theta^{n-1} \times \Theta$,
we have: $Z \longleftrightarrow X \times Y$.
Then: $\mu^n(Z) = (\mu^{n-1}(X)) \cdot (\mu(Y))$.
It is common to identify Z with $X \times Y$, in which case we have:
$$\mu^n(X \times Y) = (\mu^{n-1}(X)) \cdot (\mu(Y)).$$

We also omit proof of:
Let Θ be a countable set, $\mu \in \mathcal{FM}_\Theta$, $n \in \mathbb{N}$.
Then: $\mu^n(\Theta^n) = (\mu(\Theta))^n$.
In particular, $(\,\mu \in \mathcal{P}_\Theta\,) \Rightarrow (\,\mu^n \in \mathcal{P}_{\Theta^n}\,)$.

The countable sets that are of interest in this note
all carry the discrete topology. We therefore define:

DEFINITION 9.3. Let Θ *be a countable set,* $\mu \in \mathcal{M}_\Theta$.
Then the $\boxed{\textbf{support } \textit{of } \mu}$ *is:* $\boxed{S_\mu} := \{\, t \in \Theta \mid \mu\{t\} \neq 0 \,\}$.

DEFINITION 9.4. Let $\Theta \subseteq \mathbb{R}$ *be countable,* $\mu \in \mathcal{M}_\Theta$.
Let $\rho \geqslant 1$ *be real.* *Then:* $\boxed{|\mu|_\rho} := (\sum_{t\in\Theta}^{\mathrm{SP}} [|t|^\rho \cdot (\mu\{t\})])^{1/\rho}$.

Note: $\forall$countable $\Theta \subseteq \mathbb{R}$, $\forall \mu \in \mathcal{FM}_\Theta$,
if $\#S_\mu < \infty$, then: $\forall$real $\rho \geqslant 1$, $|\mu|_\rho < \infty$.

DEFINITION 9.5. Let $\Theta \subseteq \mathbb{R}$ *be countable.*
Let $\mu \in \mathcal{P}_\Theta$. *Assume:* $|\mu|_1 < \infty$.
Then the $\boxed{\textbf{mean } \textit{of } \mu}$ *is:* $\boxed{M_\mu} := \sum_{t\in\Theta} [\, t \cdot (\mu\{t\})]$.
Also, the $\boxed{\textbf{variance } \textit{of } \mu}$ *is:* $\boxed{V_\mu} := \sum_{t\in\Theta}^{\mathrm{SP}} [(t - M_\mu)^2 \cdot (\mu\{t\})]$.

Let $\Theta \subseteq \mathbb{R}$ be countable, $\mu \in \mathcal{P}_\Theta$. Assume: $|\mu|_1 < \infty$.
Then, by subadditivity of absolute value, we get $|M_\mu| \leqslant |\mu|_1$.
In particular, $|M_\mu| < \infty$, *i.e.*, $-\infty < M_\mu < \infty$.
Also, by expanding the square in the formula for V_μ,
we get $V_\mu = |\mu|_2^2 - M_\mu^2$.
In particular, $(\, V_\mu < \infty \,) \Leftrightarrow (\, |\mu|_2 < \infty \,)$.

Let $\Theta \subseteq \mathbb{R}$ be countable and **let** X be a Θ-valued random-variable.
Let μ denote the distribution on Θ of X,
i.e., **define** $\mu \in \mathcal{P}_\Theta$ by: $\forall t \in \Theta$, $\mu\{t\} = \Pr[X = t]$.
Then, $\forall$real $\rho \geqslant 1$, we have: $|\mu|_\rho$ is the L^ρ-norm of X.
Then, $\forall$real $\rho \geqslant 1$, we have: $(\, |\mu|_\rho < \infty \,) \Leftrightarrow (\, X \text{ is } L^\rho \,)$.
In particular, $(\, |\mu|_1 < \infty \,) \Leftrightarrow (\, X \text{ is } L^1 \,)$.
Also, if X is L^1, then $M_\mu = \mathrm{E}[X]$ and $V_\mu = \mathrm{Var}[X]$.
That is, if X is L^1, then
M_μ is the mean (aka expected value, aka average value) of X
and V_μ is the variance of X.

THEOREM 9.6. Let $\Theta \subseteq \mathbb{R}$ *be countable,* $\mu \in \mathcal{P}_\Theta$.
Assume: $|\mu|_1 < \infty$. *Then:* $(\, V_\mu > 0 \,) \Leftrightarrow (\, \#S_\mu \geqslant 2 \,)$.

The preceding result is a measure-theoretic analogue of the statement:
An L^1 random-variable is has positive variance
iff it is not deterministic.
We omit proof.

Because $\forall t \in \mathbb{Z}$, $|t| \leqslant t^2$, we conclude:
for any $\mathbb{Z}$-valued random-variable X, $\mathrm{E}[\,|X|\,] \leqslant \mathrm{E}[\,X^2\,]$.
It follows that for any $\mathbb{Z}$-valued L^2 random-variable X, we have:

$$X \text{ is } L^1, \quad \text{and so} \quad \mathrm{E}[X] \text{ is defined and finite.}$$

Because $\forall t \in \mathbb{Z}$, $|t| \leqslant t^2$, we conclude:
$\forall \Theta \subseteq \mathbb{Z}$, $\forall \mu \in \mathcal{M}_\Theta$, $|\mu|_1 \leqslant |\mu|_2^2$;
it follows that if $|\mu|_2 < \infty$, then
$|\mu|_1 < \infty$, and so M_μ is defined and finite.

DEFINITION 9.7. **Let** Θ *be a countable set.*
Let $\mu_1, \mu_2, \ldots \in \mathcal{P}_\Theta$ *and* **let** $\lambda \in \mathcal{P}_\Theta$.
By $\boxed{\mu_1, \mu_2, \ldots \to \lambda}$, *we mean:* $\forall \Theta_0 \subseteq \Theta$, $\mu_1(\Theta_0), \mu_2(\Theta_0), \ldots \to \lambda(\Theta_0)$.

Recall (§2): $\forall$function f, the notation: $\mathbb{I}_f$.
Recall (§2): $\forall$function f, $\forall$set A, the notation: f^*A.

For any countable set S, for any set T,
for any function $f : S \to T$, for any $\mu \in \mathcal{M}_S$,
we **define** $\boxed{f_*\mu} \in \mathcal{M}_{\mathbb{I}_f}$ by: $\forall A \subseteq \mathbb{I}_f$, $(f_*\mu)(A) = \mu(f^*A)$.

Let S be a countable set, T a set, $f : S \to T$. **Let** $n \in \mathbb{N}$.
Define $f^n : S^n \to T^n$ by: $\forall x \in S^n$, $f^n(x) = (\, f(x_1)\,, \ldots, \, f(x_n)\,)$.
Then: $(f^n)_*(\mu^n) = (f_*\mu)^n$.

For any nonempty countable set Θ, for any $\mu \in \mathcal{FM}_\Theta^\times$,
let $\boxed{\mathcal{N}(\mu)} := \dfrac{\mu}{\mu(\Theta)} \in \mathcal{P}_\Theta$; then $\forall \Theta_0 \subseteq \Theta$, $(\mathcal{N}(\mu))(\Theta_0) = \dfrac{\mu(\Theta_0)}{\mu(\Theta)}$,
and $\mathcal{N}(\mu)$ is called the $\boxed{\textbf{normalization of } \mu}$.

Let $\widehat{\Theta}$ be a countable set. **Let** $\mu \in \mathcal{M}_{\widehat{\Theta}}$. **Let** $\Theta \subseteq \widehat{\Theta}$.
Then the $\boxed{\textbf{restriction of } \mu \textbf{ to } \Theta}$, denoted $\boxed{\mu|\Theta} \in \mathcal{M}_\Theta$,
is **defined by:** $\forall \Theta_0 \subseteq \Theta$, $(\mu|\Theta)(\Theta_0) = \mu(\Theta_0)$.
NOTE: We have $(\mu|\Theta)(\Theta) = \mu(\Theta)$. So, if $0 < \mu(\Theta) < \infty$, then:

$$\mu|\Theta \in \mathcal{FM}_\Theta^\times \qquad \text{and} \qquad \mathcal{N}(\mu|\Theta) = \frac{\mu|\Theta}{\mu(\Theta)}$$

$$\text{and} \qquad \forall \Theta_0 \subseteq \Theta, \quad (\mathcal{N}(\mu|\Theta))(\Theta_0) = \frac{\mu(\Theta_0)}{\mu(\Theta)}.$$

DEFINITION 9.8. **Let** F *be a nonempty finite set.*
Then we **define** $\boxed{\nu_F} \in \mathcal{P}_F$ *by:* $\forall f \in F$, $\nu_F\{f\} = 1/(\#F)$.

Also, we **define** $\boxed{\nu_\varnothing} : \{\varnothing\} \to \{-1\}$ by: $\nu_\varnothing(\varnothing) = -1$.

THEOREM 9.9. Let F *be a nonempty finite set.* **Let** $\theta \in \mathcal{P}_F$.
Assume: $\forall f, g \in F,\ \ \theta\{f\} = \theta\{g\}$. *Then:* $\theta = \nu_F$.

Proof. Since F is nonempty, **choose** $g_0 \in F$. **Let** $b := \theta\{g_0\}$.
Then: $\forall f \in F,\ \ \theta\{f\} = b$. Then: $\sum_{f \in F} (\theta\{f\}) = (\#F) \cdot b$.
Since $\theta \in \mathcal{P}_F$, we get: $\theta(F) = 1$.
Since $(\#F) \cdot b = \sum_{f \in F} (\theta\{f\}) = \theta(F) = 1$, we get: $b = 1/(\#F)$.
Since $\forall f \in F,\ \ \theta\{f\} = b = 1/(\#F) = \nu_F\{f\}$, we get: $\theta = \nu_F$. $\square$

10. The Discrete Local Limit Theorem

DEFINITION 10.1. Let $E \subseteq \mathbb{Z}$.
By E *is* $\boxed{\textbf{residue-constrained}}$, *we mean:*
$\exists m \in [2..\infty),\ \exists n \in \mathbb{Z}$ *s.t.* $E \subseteq m\mathbb{Z} + n$.
By E *is* $\boxed{\textbf{residue-unconstrained}}$, *we mean:*
E *is not residue-constrained.*

Since $\varnothing \subseteq 2 \cdot \mathbb{Z} + 1$, we get: $\varnothing$ is residue-constrained.
For all $b \in \mathbb{Z}$, since $\{b\} \subseteq 2 \cdot \mathbb{Z} + b$, we get: $\{b\}$ is residue-constrained.
Then: $\forall$residue-unconstrained $E \subseteq \mathbb{Z}$, $\#E \geqslant 2$.
We have: $\{0, 3, 9\} \subseteq 3\mathbb{Z} + 0$ and $\{2, 5, 11\} \subseteq 3\mathbb{Z} + 2$,
so $\{0, 3, 9\}$ and $\{2, 5, 11\}$ are both residue-constrained.
Here is a test for residue-unconstrainedness:
Let $E \subseteq \mathbb{Z}$. Assume $\#E \geqslant 2$. **Let** $\varepsilon_0 \in E$.
Then: (E is residue-unconstrained) iff ($\gcd(E - \varepsilon_0) = 1$).
By this test, we see that:
$\{0, 1, 10\}$ and $\{2, 4, 8, 9\}$ and $\{3, 9, 13, 18\}$ are all residue-unconstrained.

DEFINITION 10.2. *For all* $\alpha \in \mathbb{R}$, *for all real* $v > 0$,
define $\boxed{\Phi_\alpha^v} : \mathbb{R} \to (0; \infty)$ *by:* $\forall t \in \mathbb{R},\ \ \Phi_\alpha^v(t) = \dfrac{\exp(\ -(t - \alpha)^2\ /\ (2v)\ )}{\sqrt{2\pi v}}$.

Note: Φ_α^v is a PDF of a normal variable with mean α and variance v.

The next result is a version of the Discrete Local Limit Theorem;
this version is stated in probability-theoretic terms:

THEOREM 10.3. Let $E \subseteq \mathbb{Z}$ *be residue-unconstrained.*
Let $X_1, X_2, \ldots$ *be an iid sequence of* $\mathbb{Z}$*-valued* L^2 *random-variables.*
Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E$.
Let $\alpha \in \mathbb{R}$, $v \in [0; \infty]$. *Assume:* $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ *and* $\mathrm{Var}[X_n] = v$.

Then: $\quad 0 < v < \infty, \quad$ *and,* $\quad \forall t_1, t_2, \ldots \in \mathbb{Z},$
as $n \to \infty, \quad \sqrt{n} \cdot [\,(\Pr[X_1 + \cdots + X_n = t_n]) - (\Phi^{nv}_{n\alpha}(t_n))\,] \to 0.$

For a good exposition of this theorem and its proof,
search on "Terence Tao Local Limit Theorem".
Visit the website, and then expand "read the rest of this entry",
and then scroll down to "– 2. Local limit theorems –".

In Theorem 10.3, since $E \subseteq \mathbb{Z}$, we have, for each $n \in \mathbb{N}$,
$|X_n| \leqslant X_n^2$ a.s., so $\mathrm{E}[|X_n|] \leqslant \mathrm{E}[X_n^2]$,
so, since X_n is L^2, we get X_n is L^1,
and so $\mathrm{E}[X_n]$ and $\mathrm{Var}[X_n]$ are both defined.
Moreover, in Theorem 10.3, $\forall n \in \mathbb{N}$,
since $|\mathrm{E}[X_n]| \leqslant \mathrm{E}[|X_n|] \leqslant \mathrm{E}[X_n^2] < \infty$, we get: $\mathrm{E}[X_n]$ is finite.
In Theorem 10.3, the proof that $v > 0$ is relatively simple:
Since E is residue-unconstrained, we get: $\#E \geqslant 2$.
Then, $\quad \forall n \in \mathbb{N}, \quad \#\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} \geqslant 2$,
so X_n is not deterministic,
which implies that $\mathrm{Var}[X_n] > 0$,
and so $v > 0$.
In Theorem 10.3, the proof that $v < \infty$ is relatively simple:
$\forall n \in \mathbb{N}, \quad \mathrm{Var}[X_n] = \mathrm{E}[X_n^2] - (\mathrm{E}[X_n])^2 \leqslant E[X_n^2] < \infty,$
and so $v < \infty$.

Next is another version of the Discrete Local Limit Theorem;
this version is stated in measure-theoretic terms:

THEOREM 10.4. **Let** *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let $\mu \in \mathcal{P}_E$. *Assume:* $S_\mu = E$. *Assume:* $|\mu|_2 < \infty$.
Let $\alpha := M_\mu$, $v := V_\mu$. *Then:* $0 < v < \infty$, *and,* $\forall t_1, t_2, \ldots \in \mathbb{Z}$,
as $n \to \infty, \quad \sqrt{n} \cdot [\,(\mu^n\{f \in E^n \,|\, f_1 + \cdots + f_n = t_n\}) - (\Phi^{nv}_{n\alpha}(t_n))\,] \to 0.$

In Theorem 10.4, since $S_\mu = E \subseteq \mathbb{Z}$ we get: $|\mu|_1 \leqslant |\mu|_2^2$.
Since $|\mu|_1 \leqslant |\mu|_2^2 < \infty$, we get: M_μ and V_μ are both defined.
Moreover, since $|M_\mu| \leqslant |\mu|_1 \leqslant |\mu|_2^2 < \infty$, we get: M_μ is finite.
In Theorem 10.4, the proof that $v > 0$ is relatively simple:
Since E is residue-unconstrained, we get: $\#E \geqslant 2$.
Since $\#S_\mu = \#E \geqslant 2$, by Theorem 9.6, we get: $v > 0$.
In Theorem 10.4, the proof that $v < \infty$ is relatively simple:
$v = V_\mu = |\mu|_2^2 - M_\mu^2 \leqslant |\mu|_2^2 < \infty.$

Here is an application of Theorem 10.3:

THEOREM 10.5. Let *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let *$X_1, X_2, \ldots$ be an iid sequence of $\mathbb{Z}$-valued L^2 random-variables.*
Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$.
Let $\alpha \in \mathbb{R}$, $v \in [0; \infty]$. *Assume:* $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ *and* $\mathrm{Var}[X_n] = v$.
Then: $0 < v < \infty$. *Also,* $\forall t_1, t_2, \ldots \in \mathbb{Z}$,
if $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ *is bounded,*
then, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = t_n]) \to 1/\sqrt{2\pi v}$.

Proof. By Theorem 10.3, we get $0 < v < \infty$.
Given $t_1, t_2, \ldots \in \mathbb{Z}$, assume $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ is bounded,
want: as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = t_n]) \to 1/\sqrt{2\pi v}$.
By Theorem 10.3, **it suffices to show:**
as $n \to \infty$, $\sqrt{n} \cdot (\Phi^{nv}_{n\alpha}(t_n)) \to 1/\sqrt{2\pi v}$.
We have: $\forall n \in \mathbb{N}$, $\Phi^{nv}_{n\alpha}(t_n) = \dfrac{\exp(\,-(t_n - n\alpha)^2 \,/\, (2nv)\,)}{\sqrt{2\pi n v}}$.
Since $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ is bounded and since $0 < v < \infty$, we get:
as $n \to \infty$, $-(t_n - n\alpha)^2 \,/\, (2nv) \to 0$.
Then: as $n \to \infty$, $\exp(\,-(t_n - n\alpha)^2 \,/\, (2nv)\,) \to 1$.
Then: as $n \to \infty$, $\sqrt{n} \cdot (\Phi^{nv}_{n\alpha}(t_n)) \to 1/\sqrt{2\pi v}$. □

We record a measure-theoretic version of Theorem 10.5:

THEOREM 10.6. Let *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let $\mu \in \mathcal{P}_E$. *Assume:* $S_\mu = E$ *and* $|\mu|_2 < \infty$.
Let $\alpha := M_\mu$, $v := V_\mu$. *Then:* $0 < v < \infty$.
Also, $\forall t_1, t_2, \ldots \in \mathbb{Z}$,
if $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ *is bounded,*
then, as $n \to \infty$, $\sqrt{n} \cdot (\mu^n\{f \in E^n \,|\, f_1 + \cdots + f_n = t_n\}) \to 1/\sqrt{2\pi v}$.

We also record the $t_n = t_0 + n\alpha$ special case of the past two theorems:

THEOREM 10.7. Let *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let *$X_1, X_2, \ldots$ be an iid sequence of $\mathbb{Z}$-valued L^2 random-variables.*
Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$.
Let $t_0, \alpha \in \mathbb{Z}$, $v \in [0; \infty]$. *Assume:* $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ *and* $\mathrm{Var}[X_n] = v$.
Then: $0 < v < \infty$, *and,*
as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = t_0 + n\alpha]) \to 1/\sqrt{2\pi v}$.

THEOREM 10.8. **Let** *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let *$\mu \in \mathcal{P}_E$. Assume: $S_\mu = E$. Assume: $|\mu|_2 < \infty$.*
Let *$\alpha := M_\mu$, $v := V_\mu$. Assume: $\alpha \in \mathbb{Z}$.* **Let** *$t_0 \in \mathbb{Z}$.*
Then: $0 < v < \infty$, and,
as $n \to \infty$, $\sqrt{n} \cdot (\mu^n\{f \in E^n \,|\, f_1 + \cdots + f_n = t_0 + n\alpha\}) \to 1/\sqrt{2\pi v}$.

We also record the $t_0 = 0$ special case of the past two theorems:

THEOREM 10.9. **Let** *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let *$X_1, X_2, \ldots$ be an iid sequence of $\mathbb{Z}$-valued L^2 random-variables.*
Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$.
Let *$\alpha \in \mathbb{Z}$, $v \in [0; \infty]$. Assume: $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ and $\mathrm{Var}[X_n] = v$.*
Then: $0 < v < \infty$, and,
as $n \to \infty$, $\sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = n\alpha]) \to 1/\sqrt{2\pi v}$.

THEOREM 10.10. **Let** *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let *$\mu \in \mathcal{P}_E$. Assume: $S_\mu = E$. Assume: $|\mu|_2 < \infty$.*
Let *$\alpha := M_\mu$, $v := V_\mu$. Assume: $\alpha \in \mathbb{Z}$.*
Then: $0 < v < \infty$, and,
as $n \to \infty$, $\sqrt{n} \cdot (\mu^n\{f \in E^n \,|\, f_1 + \cdots + f_n = n\alpha\}) \to 1/\sqrt{2\pi v}$.

11. Average events have low information, particular case

Suppose, in secret, I flip a coin 1000 times,
then reveal to you that
the total number of heads was 1000,
and then ask you to guess the last flip.
The answer is that, since *all* the coin flips were heads,
the last flip must have been a head.
Similarly, if I had told you that
the total number of heads was 0,
then you would have known that the last flip was a tail.
By contrast, if I had told you that
the total number of heads was 500,
it seems intuitively clear that
you'd have had very little information about the last flip.
We wish to generalize and formalize that intuition,
and then provide rigorous proof of the resulting formal statement.
Our main theorem is Theorem 12.3, in the next section.
In this section, we go carefully through a special case:

Let $X_1, X_2 \ldots$ be $\mathbb{Z}$-valued iid random-variables s.t.,

$$\forall n \in \mathbb{N}, \quad \Pr[\,X_n = -1\,] = 1/2, \quad \Pr[\,X_n = 0\,] = 1/3, \quad \Pr[\,X_n = 3\,] = 1/6.$$

Then, $\forall n \in \mathbb{N}$, X_n is L^1 and X_n is L^2.
Also, $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = 0$ and $\mathrm{Var}[X_n] = 2$.
Also, $\forall n \in \mathbb{N}$, $-1 \leqslant X_n \leqslant 3$ a.s.
For all $n \in \mathbb{N}$, **let** $T_n := X_1 + \cdots + X_n$.
Then: $\forall n \in \mathbb{N}$, $-n \leqslant T_n \leqslant 3n$ a.s.
Then: $-1000 \leqslant T_{1000} \leqslant 3000$ a.s.
Also, $[\,T_{1000} = -1000\,] \Rightarrow [\,X_1 = \cdots = X_{1000} = -1\,]$,
and so $\Pr[X_{1000} = -1 \,|\, T_{1000} = -1000] = 1$.
Similarly, $\Pr[X_{1000} = 3 \,|\, T_{1000} = 3000] = 1$.
By contrast, the event $T_{1000} = 0$
would seem to give very little information about X_{1000}.
It therefore seems reasonable to expect that
$\Pr[X_{1000} = -1 \,|\, T_{1000} = 0] \approx 1/2$ and
$\Pr[X_{1000} = 0 \,|\, T_{1000} = 0] \approx 1/3$ and
$\Pr[X_{1000} = 3 \,|\, T_{1000} = 0] \approx 1/6$.
To make this precise, we will work "in the thermodynamic limit",
which means: we replace 1000 by a variable $n \in \mathbb{N}$, and let $n \to \infty$.
That is, more precisely, we expect that, as $n \to \infty$,
$\Pr[X_n = -1 \,|\, T_n = 0] \to 1/2$ and
$\Pr[X_n = 0 \,|\, T_n = 0] \to 1/3$ and
$\Pr[X_n = 3 \,|\, T_n = 0] \to 1/6$.
We will focus on proving the third of these limits;
proofs of the other two are similar.
By definition of conditional probability,

we wish to prove: As $n \to \infty$, $\dfrac{\Pr[(X_n = 3)\&(T_n = 0)]}{\Pr[T_n = 0]} \to 1/6$.

Claim: **Let** $n \in [2..\infty)$.
Then: $\Pr[(X_n = 3)\&(T_n = 0)] = (1/6) \cdot (\Pr[T_{n-1} = -3])$.

Proof of Claim: We have: $T_n = X_1 + \cdots + X_{n-1} + X_n$.
Since $\Pr[(X_n = 3)\&(T_n = 0)]$
$= \Pr[(X_n = 3)\&(X_1 + \cdots + X_{n-1} + X_n = 0)]$
$= \Pr[(X_n = 3)\&(X_1 + \cdots + X_{n-1} + 3 = 0)]$

$= \Pr[(X_n = 3)\&(X_1 + \cdots + X_{n-1} = -3)],$

it follows, from independence of $X_1, \ldots, X_n$, that

$\Pr[(X_n = 3)\&(T_n = 0)]$

$= (\Pr[X_n = 3]) \cdot (\Pr[X_1 + \cdots + X_{n-1} = -3]).$

So, since $\Pr[X_n = 3] = 1/6$ and $X_1 + \cdots + X_{n-1} = T_{n-1}$,

we get: $\Pr[(X_n = 3)\&(T_n = 0)] = (1/6) \cdot (\Pr[T_{n-1} = -3])$.

End of proof of Claim.

By the claim, **we wish to prove:**

$$\text{As } n \to \infty, \quad \frac{(1/6) \cdot (\Pr[T_{n-1} = -3])}{\Pr[T_n = 0]} \to 1/6.$$

We wish to prove: $$\text{As } n \to \infty, \quad \frac{\Pr[T_{n-1} = -3]}{\Pr[T_n = 0]} \to 1.$$

That is, **we wish to prove:**

As $n \to \infty$, $\Pr[T_{n-1} = -3]$ is asymptotic to $\Pr[T_n = 0]$.

So the question becomes:

How do we get a handle on the asymptotics, as $n \to \infty$, of both $\Pr[T_{n-1} = -3]$ and $\Pr[T_n = 0]$?

The Discrete Local Limit Theorem turns out to be just what we need.

Recall: $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = 0$ and $\mathrm{Var}[X_n] = 2$.

Let $\alpha := 0$ and $v := 2$. Then: ($\forall n \in \mathbb{N}$, $n\alpha = 0$) and ($2\pi v = 4\pi$).

Also, $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ and $\mathrm{Var}[X_n] = v$.

Let $E := \{-1, 0, 3\}$. Then E is residue-unconstrained.

Also, we have: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$.

By Theorem 10.9, as $n \to \infty$,

$$\sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = n\alpha]) \to 1/\sqrt{2\pi v},$$

Then: as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = 0]) \to 1/\sqrt{4\pi}$,

so, as $n \to \infty$, $\Pr[T_n = 0]$ is asymptotic to $1/\sqrt{4\pi n}$.

Want: as $n \to \infty$, $\Pr[T_{n-1} = -3]$ is asymptotic to $1/\sqrt{4\pi n}$.

Let $t_0 := -3$. Then, $\forall n \in \mathbb{N}$, $t_0 + n\alpha = -3$.

By Theorem 10.7, as $n \to \infty$,

$$\sqrt{n} \cdot (\Pr[X_1 + \cdots + X_n = t_0 + n\alpha]) \to 1/\sqrt{2\pi v}.$$

Recall: $\forall n \in \mathbb{N}$, $T_n = X_1 + \cdots + X_n$.

Then: as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = -3]) \to 1/\sqrt{4\pi}$.

Then, as $n \to \infty$, $\sqrt{n-1} \cdot (\Pr[T_{n-1} = -3]) \to 1/\sqrt{4\pi}$.

Then, as $n \to \infty$, $\Pr[T_{n-1} = -3]$ is asymptotic to $1/\sqrt{4\pi(n-1)}$,

which is asymptotic to $1/\sqrt{4\pi n}$.

12. Average events have low information, general result

We now seek to generalize our work in §11;
in the example at the end of this section, we show that
Theorem 12.3 reproduces the result of §11.

THEOREM 12.1. Let *$E \subseteq \mathbb{Z}$ be residue-unconstrained.*
Let *$X_1, X_2, \ldots$ be an iid sequence of $\mathbb{Z}$-valued L^2 random-variables.*
Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$. **Let** *$\alpha, P \in \mathbb{R}$.*
Assume: $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ and $\Pr[X_n = \varepsilon_0] = P$. **Let** *$\varepsilon_0 \in E$.*
Let *$t_1, t_2, \ldots \in \mathbb{Z}$. Assume: $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ is bounded.*
Then: as $n \to \infty$, $\Pr[\, X_n = \varepsilon_0 \,|\, X_1 + \cdots + X_n = t_n \,] \;\to\; P$.

I don't know whether "L^2" can be replaced by "L^1".

Part of the content of Theorem 12.1 is:
$\forall$sufficiently large $n \in \mathbb{N}$, $\Pr[X_1 + \cdots + X_n = t_n\,] \;>\; 0$
since, otherwise, $\Pr[\, X_n = \varepsilon_0 \,|\, X_1 + \cdots + X_n = t_n \,]$ would not be defined.

Proof. Since $X_1, X_2, \ldots$ are all $\mathbb{Z}$-valued and L^2,
and since $\forall t \in \mathbb{Z}$, $|t| \leqslant t^2$ we get: $X_1, X_2, \ldots$ are all L^1.
So, since $X_1, X_2, \ldots$ is an identically distributed sequence,
choose $v \in [0; \infty]$ s.t., $\forall n \in \mathbb{N}$, $\mathrm{Var}[X_n] = v$.
By Theorem 10.5, we have: $0 < v < \infty$ and
as $n \to \infty$, $\sqrt{n} \cdot (\, \Pr[X_1 + \cdots + X_n = t_n]\,) \;\to\; 1/\sqrt{2\pi v}$.
For all $n \in \mathbb{N}$, **let** $T_n := X_1 + \cdots + X_n$.
Then: as $n \to \infty$, $\sqrt{n} \cdot (\, \Pr[\; T_n \; = t_n]\,) \;\to\; 1/\sqrt{2\pi v}$.
Want: as $n \to \infty$, $\Pr[\, X_n = \varepsilon_0 \,|\, T_n = t_n \,] \;\to\; P$.
Let $D_1 := \{t_n - n\alpha \,|\, n \in \mathbb{N}\}$. By hypothesis, D_1 is bounded.
Let $D_2 := \{t_n - n\alpha \,|\, n \in [2..\infty)\}$. Then $D_2 \subseteq D_1$.
Let $D_3 := \{t_{n+1} - (n+1) \cdot \alpha \,|\, n \in \mathbb{N}\}$. Then $D_3 = D_2$.
For all $n \in \mathbb{N}$, **let** $\widetilde{t}_n := t_{n+1} - \varepsilon_0$.
Let $D_4 := \{\; \widetilde{t}_n \; - \; n\alpha \;|\, n \in \mathbb{N}\}$.
Since
$$\begin{aligned} D_4 - \alpha + \varepsilon &= \{\; \widetilde{t}_n - n\alpha - \alpha + \varepsilon \,|\, n \in \mathbb{N}\} \\ &= \{t_{n+1} - \varepsilon_0 - (n+1) \cdot \alpha + \varepsilon \,|\, n \in \mathbb{N}\} \\ &= \{t_{n+1} - (n+1) \cdot \alpha \,|\, n \in \mathbb{N}\} \\ &= D_3 = D_2 \subseteq D_1, \end{aligned}$$
and since D_1 is bounded,
we get $D_4 - \alpha + \varepsilon$ is bounded.
Then: $D_4 - \alpha + \varepsilon + (\alpha - \varepsilon)$ is bounded.

Then: $\qquad D_4$ is bounded.

Then, by Theorem 10.5, we have:

$$\text{as } n \to \infty, \qquad \sqrt{n} \cdot (\,\Pr[T_n = \widetilde{t}_n\,]\,) \to 1/\sqrt{2\pi v}.$$

Then, as $n \to \infty$, $\quad \sqrt{n-1} \cdot (\,\Pr[T_{n-1} = \widetilde{t}_{n-1}\,]\,) \to 1/\sqrt{2\pi v}$.

We have: $\forall n \in [2..\infty)$, $\quad \widetilde{t}_{n-1} = t_n - \varepsilon_0$.

Then, as $n \to \infty$, $\quad \sqrt{n-1} \cdot (\,\Pr[T_{n-1} = t_n - \varepsilon_0]\,) \to 1/\sqrt{2\pi v}$.

Recall: as $n \to \infty$, $\quad \sqrt{n} \cdot (\,\Pr[T_n = t_n]\,) \to 1/\sqrt{2\pi v}$.

Dividing the last two limits, we get:

$$\text{as } n \to \infty, \qquad \frac{\sqrt{n-1} \cdot (\,\Pr[T_{n-1} = t_n - \varepsilon_0]\,)}{\sqrt{n} \cdot (\,\Pr[T_n = t_n]\,)} \to 1.$$

Also, as $n \to \infty$, $\quad \dfrac{\sqrt{n}}{\sqrt{n-1}} \to 1$.

Multiplying the last two limits together, we get:

$$\text{as } n \to \infty, \qquad \frac{\Pr[T_{n-1} = t_n - \varepsilon_0]}{\Pr[T_n = t_n]} \to 1.$$

Since, $\forall n \in [2..\infty)$,

$$\begin{aligned}
\Pr[\,X_n = \varepsilon_0 \,|\, T_n = t_n\,] &= \frac{\Pr[(X_n = \varepsilon_0)\&(T_n = t_n)]}{\Pr[T_n = t_n]} \\
&= \frac{\Pr[(X_n = \varepsilon_0)\&(T_{n-1} + X_n = t_n)]}{\Pr[T_n = t_n]} \\
&= \frac{\Pr[(X_n = \varepsilon_0)\&(T_{n-1} + \varepsilon_0 = t_n)]}{\Pr[T_n = t_n]} \\
&= \frac{\Pr[(X_n = \varepsilon_0)\&(T_{n-1} = t_n - \varepsilon_0)]}{\Pr[T_n = t_n]} \\
&= \frac{(\Pr[X_n = \varepsilon_0]) \cdot (\Pr[T_{n-1} = t_n - \varepsilon_0])}{\Pr[T_n = t_n]} \\
&= P \cdot \frac{\Pr[T_{n-1} = t_n - \varepsilon_0]}{\Pr[T_n = t_n]},
\end{aligned}$$

and since, as $n \to \infty$, $\quad \dfrac{\Pr[T_{n-1} = t_n - \varepsilon_0]}{\Pr[T_n = t_n]} \to 1$,

we get: as $n \to \infty$,

$$\Pr[\,X_n = \varepsilon_0 \,|\, T_n = t_n\,] \to P. \qquad \square$$

Recall (§9): $\forall$countable set Θ,

$\mathcal{FM}_\Theta^\times$ is the set of nonzero finite measures on Θ

and $\mathcal{P}_\Theta$ is the set of probability measures on Θ.

Recall (§9): $\forall$nonempty countable set Θ, $\forall \mu \in \mathcal{FM}_\Theta^\times$,

$\mathcal{N}(\mu)$ is the normalization of μ.

Here is a measure-theoretic version of the preceding theorem:

THEOREM 12.2. **Let** $E \subseteq \mathbb{Z}$ *be residue-unconstrained.*
Let $\mu \in \mathcal{P}_E$. *Assume:* $S_\mu = E$. *Assume:* $|\mu|_2 < \infty$.
Let $\alpha := M_\mu$. **Let** $\varepsilon_0 \in E$, $P := \mu\{\varepsilon_0\}$.
Let $t_1, t_2, \ldots \in \mathbb{Z}$. *Assume:* $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ *is bounded.*
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in E^n \,|\, f_1 + \cdots + f_n = t_n\}$.
Then: *as* $n \to \infty$, $(\mathcal{N}(\mu^n|\Omega_n))\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to P$.

I don't know whether "$|\mu|_2 < \infty$" can be replaced by "$|\mu|_1 < \infty$".

Part of the content of Theorem 12.2 is:
$\forall$sufficiently large $n \in \mathbb{N}$, $\mu^n(\Omega_n) > 0$,
since, otherwise, $\mu^n|\Omega_n$ would be the zero measure on Ω_n,
and so $\mathcal{N}(\mu^n|\Omega_n)$ would not be defined.

We record the $t_n = n\alpha$ special case of the past two theorems:

THEOREM 12.3. **Let** $E \subseteq \mathbb{Z}$ *be residue-unconstrained.*
Let $X_1, X_2, \ldots$ *be an iid sequence of* $\mathbb{Z}$*-valued* L^2 *random-variables.*
Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$. **Let** $\alpha \in \mathbb{Z}$, $P \in \mathbb{R}$.
Let $\varepsilon_0 \in E$. *Assume:* $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$ *and* $\Pr[X_n = \varepsilon_0] = P$.
Then: *as* $n \to \infty$, $\Pr[\, X_n = \varepsilon_0 \,|\, X_1 + \cdots + X_n = n\alpha \,] \to P$.

THEOREM 12.4. **Let** $E \subseteq \mathbb{Z}$ *be residue-unconstrained.*
Let $\mu \in \mathcal{P}_E$. **Let** $\alpha := M_\mu$. *Assume:* $\alpha \in \mathbb{Z}$ *and* $S_\mu = E$ *and* $|\mu|_2 < \infty$.
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in E^n \,|\, f_1 + \cdots + f_n = n\alpha\}$.
Let $\varepsilon_0 \in E$. **Let** $P := \mu\{\varepsilon_0\}$.
Then: *as* $n \to \infty$, $(\mathcal{N}(\mu^n|\Omega_n))\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to P$.

Example: **Let** $E := \{-1, 0, 3\}$.
Then: $E \subseteq \mathbb{Z}$ and E is residue-unconstrained.
Let $X_1, X_2 \ldots$ be $\mathbb{Z}$-valued iid random-variables s.t.,
$\forall n \in \mathbb{N}$, $\Pr[\, X_n = -1\,] = 1/2$,
$\Pr[\, X_n = 0\,] = 1/3$,
$\Pr[\, X_n = 3\,] = 1/6$.
Then: $\forall n \in \mathbb{N}$, $\{\, t \in \mathbb{Z} \mid \Pr[\, X_n = t\,] > 0 \,\} = E$.
Let $\varepsilon_0 = 3$, $P := 1/6$.
Then: $\forall n \in \mathbb{N}$, $\Pr[\, X_n = \varepsilon_0] = P$.
We have: $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = 0$. **Let** $\alpha := 0$.
Then, $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$.
Then, by Theorem 12.3, we have:

as $n \to \infty$, $\Pr[\, X_n = \varepsilon_0 \,|\, X_1 + \cdots + X_n = n\alpha \,] \ \to \ P$.
Then: as $n \to \infty$, $\Pr[\, X_n = 3 \ |\, X_1 + \cdots + X_n = \ 0 \ ] \ \to \ 1/6$.
For all $n \in \mathbb{N}$, **let** $T_n := X_1 + \cdots + X_n$.
Then: as $n \to \infty$, $\Pr[\, X_n = 3 \ | \qquad T_n \qquad = \ 0 \ ] \ \to \ 1/6$.
Thus Theorem 12.3 reproduces the result of §11.

13. Solving the main problem

We finally have all we need to solve the main problem (end of §3).

$$\textbf{Let} \qquad (p, q, r) := \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}.$$

We compute $(p, q, r) \ \approx \ (\, 0.5225\,,\, 0.4194\,,\, 0.0581\,)$,
all accurate to four decimal places.

Again, let's say I am one of the professors applying to the GFA.
We will show: Under the GFA's *first* system (§3),
my probability of getting \$ 0 is p, approximately and
my probability of getting \$ 1 is q, approximately and
my probability of getting \$10 is r, approximately.

Recall: $\Omega \ = \ \left\{ \ \omega : [1..N] \to \{0, 1, 10\} \ \middle| \ \sum_{\ell=1}^{N} [\omega(\ell)] = N \ \right\}$.
Recall (§5): the notations i_ω, j_ω, k_ω.
Let $S := \sum_{\omega \in \Omega} p^{i_\omega} q^{j_\omega} r^{k_\omega}$.
By the work in §7, $p + q + r = 1$ and $q + 10r = 1$ and
$\forall \omega \in \Omega, \quad p^{i_\omega} q^{j_\omega} r^{k_\omega} \,/\, S \ = \ 1 \,/\, (\#\Omega)$.
Let $X_1, X_2, \ldots$ be $\mathbb{Z}$-valued iid random-variables s.t., $\forall n \in \mathbb{N}$,
$\Pr[\, X_n = \ \ 0\,] \ = \ p$,
$\Pr[\, X_n = \ \ 1\,] \ = \ q$,
$\Pr[\, X_n = 10\,] \ = \ r$.
Then $X_1, X_2, \ldots$ is a sequence of L^2 random-variables.
Also, $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = q + 10r$.
So, since $q + 10r = 1$, we get:
$\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = 1$.
We model the GFA's *second* system (§5) by: $\forall \ell \in [1..N]$,
Professor$\#\ell$ receives X_ℓ dollars.
For all $n \in \mathbb{N}$, **let** $T_n := X_1 + \cdots + X_n$.
We model the GFA's *third* system (§5) by: $\forall \ell \in [1..N]$,
Professor$\#\ell$ receives X_ℓ dollars, conditioned on $T_N = N$.
Since $\forall \omega \in \Omega$, $p^{i_\omega} q^{j_\omega} r^{k_\omega} \,/\, S \ = \ 1 \,/\, (\#\Omega)$,

it follows that: the third system is equivalent to the first.
For definiteness, let's assume that I am Professor#N.
Then, assuming N is large, **we wish to show:**

$$\Pr[\,X_N = \ \ 0\,|\,T_N = N\,] \ \approx \ p \qquad \text{and}$$
$$\Pr[\,X_N = \ \ 1\,|\,T_N = N\,] \ \approx \ q \qquad \text{and}$$
$$\Pr[\,X_N = 10\,|\,T_N = N\,] \ \approx \ r.$$

To be more precise, **we wish to show:** as $n \to \infty$,

$$\Pr[\,X_n \ = \ \ 0\,|\,T_n \ = \ n\,] \ \to \ p \qquad \text{and}$$
$$\Pr[\,X_n \ = \ \ 1\,|\,T_n \ = \ n\,] \ \to \ q \qquad \text{and}$$
$$\Pr[\,X_n \ = 10\,|\,T_n \ = \ n\,] \ \to \ r.$$

Let $E := \{0, 1, 10\}$. Then: E is residue-unconstrained.

Given $\varepsilon_0 \in E$, **let** $P := \begin{cases} p, & \text{if } \varepsilon_0 = 0 \\ q, & \text{if } \varepsilon_0 = 1 \\ r, & \text{if } \varepsilon_0 = 10, \end{cases}$

want: as $n \to \infty$, $\Pr[X_n = \varepsilon_0 | T_n = n] \to P$.
By definition of $X_1, X_2, \ldots$, we get: $\forall n \in \mathbb{N}$, $\Pr[X_n = \varepsilon_0] = P$.
Let $\alpha := 1$. Then: $\alpha \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$.
Also, $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z}\,|\,\Pr[X_n = t] > 0\} = E$.
Then, by Theorem 12.3, we have:

$$\text{as } n \to \infty, \qquad \Pr[X_n = \varepsilon_0 | X_1 + \cdots + X_n = n\alpha] \to P.$$

Then: $\qquad$ as $n \to \infty$, $\qquad \Pr[X_n = \varepsilon_0 | \qquad T_n \qquad = \ n \ ] \to P$.

14. Probability of two professors getting zero

Under the GFA's first system, since N is large, one would expect:
the award amounts of two different professors
are almost independent.
Then, for example, one would expect:
the probability that two professors both receive zero dollars
should be very close to the square of
the probability that one professor receives zero dollars.
We will formalize this statement and prove it, below.
For definiteness, we will assume that
the two professors are Professor #$(N-1)$ and Professor #N.

Let $\qquad (p, q, r) := \dfrac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$. $\qquad$ Then (§7): $p + q + r = 1$.
Let $X_1, X_2, \ldots$ be $\mathbb{Z}$-valued iid random-variables s.t., $\forall n \in \mathbb{N}$,
$\Pr[\,X_n = \ \ 0\,] \ = \ p,$

$\Pr[\,X_n = \;\;1\,] \;=\; q,$
$\Pr[\,X_n = 10\,] \;=\; r.$
Then $X_1, X_2, \ldots$ is a sequence of L^2 random-variables.
For all $n \in \mathbb{N}$, **let** $T_n := X_1 + \cdots + X_n$.
Assuming N is large, our goal is to prove:
$$\Pr\;[\;X_{N-1} = 0 = X_N \mid T_N = N\;] \approx p^2.$$
To be more precise, **we will prove:**
as $n \to \infty$, $\Pr\;[\;X_{n-1} \;=\; 0 \;=\; X_n \mid T_n = \;n\;\;] \to p^2.$

For all $n \in \mathbb{N}$, **define** $\psi_n : \mathbb{Z} \to \mathbb{R}$ by:
$$\forall t \in \mathbb{Z}, \quad \psi_n(t) = \Pr[T_n = t].$$
For all $n \in \mathbb{N}$, **let** $a_n := \psi_n(n+2)$, $z_n := \psi_n(n)$.
Since, $\forall n \in \mathbb{N}$, we have $\psi_n(n) = \Pr[T_n = n] = \Pr[X_1 + \cdots + X_n = n]$
$\geqslant \Pr[X_1 = \cdots = X_n = 1] = q^n > 0,$
we conclude: $\forall n \in \mathbb{N}$, $z_n > 0$.

Claim: **Let** $n \in [3..\infty)$. Then $\Pr[X_{n-1} = 0 = X_n \,|\, T_n = n] = p^2 \cdot \dfrac{a_{n-2}}{z_n}$.
Proof of Claim: We have $T_n = X_1 + \cdots + X_{n-2} + X_{n-1} + X_n$.
Since $\Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)]$
$= \Pr[(X_{n-1} = 0 = X_n) \& (X_1 + \cdots + X_{n-2} + X_{n-1} + X_n = n)]$
$= \Pr[(X_{n-1} = 0 = X_n) \& (X_1 + \cdots + X_{n-2} + \;\;0\;\; + \;\;0\;\; = n)]$
$= \Pr[(X_{n-1} = 0 = X_n) \& (X_1 + \cdots + X_{n-2} \qquad\qquad = n)]$,
it follows, from independence of $X_1, \ldots, X_n$, that
$\Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)]$
$= (\,\Pr[X_{n-1} = 0]\,) \cdot (\,\Pr[X_n = 0]\,) \cdot (\,\Pr[X_1 + \cdots + X_{n-2} = n]\,).$
So, since $\Pr[X_{n-1} = 0] = p = \Pr[X_n = 0]$
and since $X_1 + \cdots + X_{n-2} = T_{n-2}$,
we get: $\Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)] = p^2 \cdot (\Pr[T_{n-2} = n]).$
Then $\Pr[X_{n-1} = 0 = X_n \,|\, T_n = n] = \dfrac{\Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)]}{\Pr[T_n = n]}$
$$= \frac{p^2 \cdot (\Pr[T_{n-2} = n])}{\Pr[T_n = n]} = p^2 \cdot \frac{\psi_{n-2}(n)}{\psi_n(n)} = p^2 \cdot \frac{a_{n-2}}{z_n}.$$
End of proof of Claim.

Because of the Claim, **we want to show:** as $n \to \infty$, $p^2 \cdot \dfrac{a_{n-2}}{z_n} \to p^2$.
Want: as $n \to \infty$, $\dfrac{a_{n-2}}{z_n} \to 1.$
We compute: $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = q + 10r$.

Recall (§7): $q + 10r = 1$. Then: $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = 1$.
For all $n \in \mathbb{N}$, **let** $v := \mathrm{Var}[X_n]$.
Let $\alpha := 1$. Then: ($\alpha \in \mathbb{Z}$) and ($\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$).
Let $E := \{0, 1, 10\}$. Then, $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \,|\, \Pr[X_n = t] > 0\} = E$.
Also, E is residue-unconstrained.
By Theorem 10.9, we have: $0 < v < \infty$.
Let $\tau := 1/\sqrt{2\pi v}$. Then: $0 < \tau < \infty$.
By Theorem 10.9, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = n\alpha]) \to 1/\sqrt{2\pi v}$.
Then: as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = \; n \;]) \to \tau$.
Then: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(n)) \to \tau$.
Then: as $n \to \infty$, $\sqrt{n} \cdot z_n \to \tau$.
Let $t_0 := 2$. Then $t_0 \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$, $t_0 + n\alpha = n + 2$.
By Theorem 10.7, as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = t_0 + n\alpha]) \to 1/\sqrt{2\pi v}$.
Then: as $n \to \infty$, $\sqrt{n} \cdot (\Pr[T_n = \; n + 2 \;]) \to \tau$.
Then: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(n + 2)) \to \tau$.
Then: as $n \to \infty$, $\sqrt{n} \cdot a_n \to \tau$.
Then: as $n \to \infty$, $\sqrt{n - 2} \cdot a_{n-2} \to \tau$.
Recall: as $n \to \infty$, $\sqrt{n} \cdot z_n \to \tau$.
Dividing the last two limits, we get:

$$\text{as } n \to \infty, \quad \frac{\sqrt{n-2} \cdot a_{n-2}}{\sqrt{n} \cdot z_n} \to 1.$$

Also, $$\text{as } n \to \infty, \quad \frac{\sqrt{n}}{\sqrt{n-2}} \to 1.$$

Multiplying these last two limits, we get:

$$\text{as } n \to \infty, \quad \frac{a_{n-2}}{z_n} \to 1.$$

15. Fraction of professors getting a zero award

Let $$(p, q, r) := \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}.$$
We compute $(p, q, r) \approx (\,0.5225\,,\,0.4194\,,\,0.0581\,)$,
all accurate to four decimal places.
Let $X_1, X_2, \ldots$ be $\mathbb{Z}$-valued iid random-variables s.t., $\forall n \in \mathbb{N}$,

$\Pr[\,X_n = \;\;0\,] = p$,
$\Pr[\,X_n = \;\;1\,] = q$,
$\Pr[\,X_n = 10\,] = r$.

For all $n \in \mathbb{N}$, **let** $T_n := X_1 + \cdots + X_n$.
For all $n \in \mathbb{N}$, **let** I_n be the indicator variable of the event: $X_n = 0$.
For all $n \in \mathbb{N}$, **let** $J_n := (I_1 + \cdots + I_n)/n$.

Using the GFA's first (or third) awards system, the random-variable J_N conditioned on $T_N = N$ represents the fraction of professors receiving a \$0 award.
In this section, **we will prove the following:**

Claim: $\forall \delta > 0$, as $n \to \infty$, $\Pr[\, p - \delta < J_n < p + \delta \mid T_n = n] \to 1$.

Assume, for a moment, that this Claim is true.
Then: as $n \to \infty$, $\Pr[\, p - 0.02 < J_n < p + 0.02 \mid T_n = n] \to 1$.
From this, it follows that, if N is sufficiently large, then

$$\Pr[\, p - 0.02 < J_N < p + 0.02 \mid T_N = N\,] > 0.99,$$

so $$\Pr[\, p - 0.02 < J_N \mid T_N = N\,] > 0.99,$$

so $$\Pr[\, J_N > p - 0.02 \mid T_N = N\,] > 0.99.$$

Since $p \approx 0.5225$, accurate to four decimal places, we get

$$p - 0.02 > 0.5,$$

so $$[\, J_N > p - 0.02\,] \Rightarrow [\, J_n > 0.5\,],$$

so $$\Pr[\, J_N > p - 0.02 \mid T_N = N\,] \leqslant \Pr[\, J_N > 0.5 \mid T_N = N\,].$$

Therefore, if N is sufficiently large, then, since

$$\Pr[\, J_N > 0.5 \mid T_N = N\,] \geqslant \Pr[\, J_N > p - 0.02 \mid T_N = N\,] > 0.99,$$

we conclude: under the GFA's first system, with probability $> 99\%$, over 50% of the professors receive \$0.

Proof of Claim:
Given $\delta > 0$, **want:** as $n \to \infty$, $\Pr[\, p - \delta < J_n < p + \delta \mid T_n = n\,] \to 1$.
Let $E := \{0, 1, 10\}$. Then E is residue-unconstrained.
Also, $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E$.
Let $\alpha := 1$. Then: $\alpha \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$, $\mathrm{E}[X_n] = \alpha$.
For all $n \in \mathbb{N}$, **let** $\kappa_n := \mathrm{E}[\, I_n \mid T_n = n\,]$.
Then: $\forall n \in \mathbb{N}$, $\kappa_n = \Pr[\, X_n = 0 \mid T_n = n\,]$.
By Theorem 12.3, we get:

as $n \to \infty$, $\Pr[X_n = 0 \mid X_1 + \cdots + X_n = n\alpha] \to p$.
That is, as $n \to \infty$, $\Pr[X_n = 0 \mid T_n = n\,] \to p$.
Then: as $n \to \infty$, $\kappa_n \to p$.
So, $\exists n_0 \in \mathbb{N}$ s.t., $\forall n \in [n_0..\infty)$,
we have $p - (\delta/2) < \kappa_n < p + (\delta/2)$,
and so both $p - \delta < \kappa_n - (\delta/2)$ and $\kappa_n + (\delta/2) < p + \delta$,

and so $\quad [\,\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2)\,] \;\Rightarrow\; [\,p - \delta < J_n < p + \delta\,]$,

and so $\quad \Pr[\,\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2) \,|\, T_n = n\,]$

$\qquad \leqslant \quad \Pr[\;\; p - \delta \quad < J_n < \;\; p + \;\delta \quad |\, T_n = n\,]$.

It therefore suffices to show:

$$\text{as } n \to \infty, \qquad \Pr[\,\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2) \,|\, T_n = n\,] \;\to\; 1.$$

We have: $\quad \forall n \in \mathbb{N}$, $\;T_n$ is invariant under permutation of $X_1, \ldots, X_n$,

as is the joint-distribution of $X_1, \ldots, X_n$.

Then: $\;\forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathrm{E}\,[\;I_i \mid T_n = n\;] \;=\; \mathrm{E}\,[\;I_n \mid T_n = n\;]$.

Then: $\;\forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathrm{E}\,[\;I_i \mid T_n = n\;] \;=\; \kappa_n$.

Since, $\;\forall n \in \mathbb{N}, \quad J_n = (I_1 + \cdots + I_n)/n, \quad$ we get:

$\forall n \in \mathbb{N}, \quad \mathrm{E}\,[\;J_n \mid T_n = n\;] \;=\; (\;\sum_{i=1}^{n} \mathrm{E}\,[\;I_i \mid T_n = n\;]\;)\;/\;n$.

Then: $\;\forall n \in \mathbb{N}, \quad \mathrm{E}\,[\;J_n \mid T_n = n\;] \;=\; (\;\sum_{i=1}^{n} \kappa_n\;)\;/\;n$.

Then: $\;\forall n \in \mathbb{N}, \quad \mathrm{E}\,[\;J_n \mid T_n = n\;] \;=\; (\;n\kappa_n\;)\;/\;n$.

Then: $\;\forall n \in \mathbb{N}, \quad \mathrm{E}\,[\;J_n \mid T_n = n\;] \;=\; \kappa_n$.

For all $\;n \in \mathbb{N}$, **let** $v_n \;:=\; \mathrm{Var}\,[\,J_n \,|\, T_n = n\,]$.

Then, $\quad$ by Chebyshev's inequality, $\quad$ we have: $\quad \forall n \in \mathbb{N}$,

$$\Pr[\,\kappa_n - (\delta/2) < J_n < \kappa_n + (\delta/2) \,|\, T_n = n\,] \;\geqslant\; 1 - (v_n/(\delta/2)^2).$$

It therefore suffices to show: $\quad$ as $n \to \infty, \quad v_n \;\to\; 0$.

Recall: $\quad$ as $n \to \infty, \;\; \kappa_n \to p$.

Since $\quad \forall n \in \mathbb{N}$,

$$\begin{aligned} v_n &= \mathrm{Var}\,[\,J_n \,|\, T_n = n\,] \\ &= (\,\mathrm{E}\,[\,J_n^2 \,|\, T_n = n\,]\,) \;-\; (\,\mathrm{E}\,[\,J_n \,|\, T_n = n\,]\,)^2 \\ &= (\,\mathrm{E}\,[\,J_n^2 \,|\, T_n = n\,]\,) \;-\; \kappa_n^2. \end{aligned}$$

and since, $\quad$ as $n \to \infty, \qquad \kappa_n^2 \;\to\; p^2$,

we want: as $n \to \infty, \qquad \mathrm{E}\,[\,J_n^2 \,|\, T_n = n\,] \;\to\; p^2$.

For all $\;n \in [2..\infty)$, $\quad$ **let** $\;\lambda_n \;:=\; \mathrm{E}\;[\quad I_{n-1} \cdot I_n \quad \mid T_n = n\;]$.

Then: $\forall n \in [2..\infty), \qquad \lambda_n \;\;=\;\; \Pr\;[\;X_{n-1} = 0 = X_n \mid T_n = n\;]$.

So, by the result of §14, $\qquad$ we get: $\quad$ as $n \to \infty, \quad \lambda_n \to p^2$.

For all $n \in \mathbb{N}$, for all $i, j \in [1..n]$, **let** $c_{ijn} \;:=\; \mathrm{E}\;[\;I_i \cdot I_j \mid T_n = n\;]$.

Recall: $\;\forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathrm{E}\;[\;I_i \mid T_n = n\;] \;=\; \kappa_n$.

For all $i \in \mathbb{N}$, since I_i is an indicator variable, we get: $\quad I_i \in \{0, 1\}$ a.s.

Then: $\qquad \forall i \in \mathbb{N}, \qquad I_i^2 \;=\; I_i$ a.s.

Then: $\quad \forall n \in \mathbb{N}, \forall i \in [1..n], \quad \mathrm{E}\;[\;I_i^2 \mid T_n = n\;] \;=\; \mathrm{E}\;[\;I_i, \mid T_n = n\;]$.

Then: $\quad \forall n \in \mathbb{N}, \forall i \in [1..n], \qquad c_{iin} \;=\; \kappa_n$.

Recall: $\quad \forall n \in \mathbb{N}$, $\;T_n$ is invariant under permutation of $X_1, \ldots, X_n$,

as is the joint-distribution of $X_1, \ldots, X_n$.

Then, $\quad \forall n \in [2..\infty), \quad \forall i, j \in [1..n], \quad$ if $i \neq j$, $\quad$ then

$$\mathrm{E}\;[\;I_i \cdot I_j \mid T_n = n\;] \;\;=\;\; \mathrm{E}\;[\;I_{n-1} \cdot I_n \mid T_n = n\;],$$

so, $\qquad \mathrm{E}\;[\;I_i \cdot I_j \mid T_n = n\;] \;\;=\;\; \lambda_n$.

so, $\quad c_{ijn} = \lambda_n$.

Then: $\forall n \in \mathbb{N}, \forall i,j \in [1..n], \quad c_{ijn} = \begin{cases} \kappa_n, & \text{if } i = j \\ \lambda_n, & \text{if } i \neq j. \end{cases}$

Then: $\forall n \in \mathbb{N}, \quad \sum_{i=1}^n \sum_{j=1}^n c_{ijn} = n \cdot \kappa_n + (n^2 - n) \cdot \lambda_n$.

Recall: as $n \to \infty$, $\quad \kappa_n \to p \quad$ and $\quad \lambda_n \to p^2$.

Since $\quad \forall n \in \mathbb{N}, \quad J_n = (I_1 + \cdots + I_n)/n$,

we get: $\quad \forall n \in \mathbb{N}, \quad J_n^2 = \left(\sum_{i=1}^n \sum_{j=1}^n [\, I_i \cdot I_j \,] \right) / \, n^2$.

Then: $\quad \forall n \in \mathbb{N}, \quad \mathrm{E}\,[\, J_n^2 \mid T_n = n \,] = \left(\sum_{i=1}^n \sum_{j=1}^n c_{ijn} \right) / \, n^2$.

Then: $\quad \forall n \in \mathbb{N}, \quad \mathrm{E}\,[\, J_n^2 \mid T_n = n \,] = (1/n) \cdot \kappa_n + (1 - (1/n)) \cdot \lambda_n$.

Then: $\quad$ as $n \to \infty$, $\quad \mathrm{E}\,[\, J_n^2 \mid T_n = n \,] \to 0 \cdot p + 1 \cdot p^2$.

Then: $\quad$ as $n \to \infty$, $\quad \mathrm{E}\,[\, J_n^2 \mid T_n = n \,] \to p^2$.

End of proof of Claim.

16. Boltzmann distributions on nonempty finite sets

Recall (§9): $\forall$countable set Θ,

$\mathcal{M}_\Theta$ is the set of measures on Θ

and $\mathcal{FM}_\Theta^\times$ is the set of nonzero finite measures on Θ

and $\mathcal{P}_\Theta$ is the set of probability measures on Θ.

Recall (§9): $\forall$nonempty countable set Θ, $\quad \forall \mu \in \mathcal{FM}_\Theta^\times$,

$\mathcal{N}(\mu)$ is the normalization of μ.

DEFINITION 16.1. Let $\;E \subseteq \mathbb{R}$ *be nonempty and finite,* $\;\beta \in \mathbb{R}$. *The* $\boxed{\textbf{unnormalized-}\beta\textbf{-Boltzmann distribution } \textit{on } E}$ *is*

the measure $\boxed{\widehat{B}_\beta^E} \in \mathcal{FM}_E^\times$ **defined** *by:*

$$\forall \varepsilon \in E, \quad \widehat{B}_\beta^E\{\varepsilon\} = e^{-\beta \cdot \varepsilon}.$$

Also, the $\boxed{\beta\textbf{-Boltzmann distribution } \textit{on } E}$ *is*

$$\boxed{B_\beta^E} := \mathcal{N}(\widehat{B}_\beta^E) \in \mathcal{P}_E.$$

Then: $\forall \varepsilon \in E$, we have: $\quad B_\beta^E\{\varepsilon\} = (\widehat{B}_\beta^E\{\varepsilon\}) \, / \, (\widehat{B}_\beta^E(E))$.

Example: **Let** $E := \{0, 1, 10\}$ and **let** $\beta \in \mathbb{R}$.

Then: $\quad \widehat{B}_\beta^E\{0\} = 1, \quad \widehat{B}_\beta^E\{1\} = e^{-\beta}, \quad \widehat{B}_\beta^E\{10\} = e^{-10\beta}$.

Let $C := 1/(1 + e^{-\beta} + e^{-10\beta})$.

Then: $\quad B_\beta^E\{0\} = C, \quad B_\beta^E\{1\} = Ce^{-\beta}, \; B_\beta^E\{10\} = Ce^{-10\beta}$.

Example: **Let** $E := \{2, 4, 8, 9\}$ and **let** $\beta \in \mathbb{R}$.

Then: $\quad \widehat{B}_\beta^E\{2\} = e^{-2\beta}, \quad \widehat{B}_\beta^E\{4\} = e^{-4\beta},$

$$\widehat{B}^E_\beta\{8\} = e^{-8\beta}, \qquad \widehat{B}^E_\beta\{9\} = e^{-9\beta}.$$

Let $C := 1/(e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta})$.

Then: $\quad B^E_\beta\{2\} = Ce^{-2\beta}, \quad B^E_\beta\{4\} = Ce^{-4\beta},$

$$B^E_\beta\{8\} = Ce^{-8\beta}, \quad B^E_\beta\{9\} = Ce^{-9\beta}.$$

Recall (§9): For any countable set Θ, for any $\mu \in \mathcal{M}_\Theta$,

S_μ is the support of μ.

Note: $\forall$nonempty finite $E \subseteq \mathbb{R}$, $\forall \beta \in \mathbb{R}$, we have: $S_{\widehat{B}^E_\beta} = E = S_{B^E_\beta}$.

THEOREM 16.2. **Let** $E \subseteq \mathbb{R}$ *be nonempty and finite.*
Let $\varepsilon_0 \in E$, $\beta, \xi \in \mathbb{R}$. *Then:* $B^{E-\xi}_\beta\{\varepsilon_0 - \xi\} = B^E_\beta\{\varepsilon_0\}$.

Proof. We have:

$$\begin{aligned} B^{E-\xi}_\beta\{\varepsilon_0 - \xi\} &= \frac{e^{-\beta\cdot(\varepsilon_0-\xi)}}{\sum_{\varepsilon\in E}\left[e^{-\beta\cdot(\varepsilon-\xi)}\right]} \\ &= \frac{e^{-\beta\cdot\varepsilon_0}\cdot e^{\beta\cdot\xi}}{\sum_{\varepsilon\in E}\left[e^{-\beta\cdot\varepsilon}\cdot e^{\beta\cdot\xi}\right]} \\ &= \frac{e^{\beta\cdot\xi}\cdot e^{-\beta\cdot\varepsilon_0}}{e^{\beta\cdot\xi}\cdot\sum_{\varepsilon\in E}\left[e^{-\beta\cdot\varepsilon}\right]} \\ &= \frac{e^{-\beta\cdot\varepsilon_0}}{\sum_{\varepsilon\in E}\left[e^{-\beta\cdot\varepsilon}\right]} = B^E_\beta\{\varepsilon_0\}. \qquad \square \end{aligned}$$

Recall (§9): **Let** $\Theta \subseteq \mathbb{R}$ be countable, $\mu \in \mathcal{P}_\Theta$. Assume $\#S_\mu < \infty$.
Then $|\mu|_1 < \infty$ and M_μ is the mean of μ and V_μ is the variance of μ.

Let $E \subseteq \mathbb{R}$ be nonempty and finite. **Let** $\beta \in \mathbb{R}$. We **define**:

$$\boxed{\Gamma^E_\beta} := \sum_{\varepsilon\in E}\left[\varepsilon\cdot e^{\beta\cdot\varepsilon}\right],$$

$$\boxed{\Delta^E_\beta} := \sum_{\varepsilon\in E}\left[e^{\beta\cdot\varepsilon}\right],$$

$$\boxed{A^E_\beta} := \Gamma^E_\beta \,/\, \Delta^E_\beta.$$

Then: $\quad \Gamma^E_\beta = \sum_{\varepsilon\in E}\left[\varepsilon\cdot(\widehat{B}^E_\beta\{\varepsilon\})\right]$.

Also, $\quad \Delta^E_\beta = \sum_{\varepsilon\in E}\left[\widehat{B}^E_\beta\{\varepsilon\}\right]$, and so $\Delta^E_\beta = \widehat{B}^E_\beta(E)$.

Since
$$\frac{\Gamma^E_\beta}{\Delta^E_\beta} = \frac{\sum_{\varepsilon\in E}\left[\varepsilon\cdot(\widehat{B}^E_\beta\{\varepsilon\})\right]}{\widehat{B}^E_\beta(E)} = \sum_{\varepsilon\in E}\left[\varepsilon\cdot(B^E_\beta\{\varepsilon\})\right],$$

we conclude: $\quad A^E_\beta = M_{B^E_\beta}$.

Then: A^E_β is the average value of any E-valued random-variable whose distribution in E is B^E_β.

THEOREM 16.3. **Let** $E \subseteq \mathbb{R}$ *be nonempty and finite.* **Let** $\beta, \xi \in \mathbb{R}$.
Then: $A^{E-\xi}_\beta = A^E_\beta - \xi$.

Proof. **Want:** $M_{B_\beta^{E-\xi}} = M_{B_\beta^E} - \xi$.

Let $\lambda := B_\beta^{E-\xi}$, $\mu := B_\beta^E$. **Want:** $M_\lambda = M_\mu - \xi$.

We have: $\lambda \in \mathcal{P}_{E-\xi}$ and $\mu \in \mathcal{P}_E$.

By Theorem 16.2, we have: $\forall \varepsilon \in E,\ B_\beta^{E-\xi}\{\varepsilon - \xi\} = B_\beta^E\{\varepsilon\}$.

Then: $\forall \varepsilon \in E,\ \lambda\{\varepsilon - \xi\} = \mu\{\varepsilon\}$.

Since $\mu \in \mathcal{P}_E$, we get: $\mu(E) = 1$.

Then:
$$\begin{aligned} M_\lambda &= \textstyle\sum_{\varepsilon \in E} [\, (\varepsilon - \xi) \cdot (\lambda\{\varepsilon - \xi\})\,] \\ &= \textstyle\sum_{\varepsilon \in E} [\, (\varepsilon - \xi) \cdot (\mu\{\varepsilon\})\,] \\ &= \textstyle\sum_{\varepsilon \in E} [\, \varepsilon \cdot (\mu\{\varepsilon\}) - \xi \cdot (\mu\{\varepsilon\})\,] \\ &= (\textstyle\sum_{\varepsilon \in E} [\, \varepsilon \cdot (\mu\{\varepsilon\})\,]) - (\textstyle\sum_{\varepsilon \in E} [\, \xi \cdot (\mu\{\varepsilon\})\,]) \\ &= (\textstyle\sum_{\varepsilon \in E} [\, \varepsilon \cdot (\mu\{\varepsilon\})\,]) - \xi \cdot (\textstyle\sum_{\varepsilon \in E} [\, \mu\{\varepsilon\}\,]) \\ &= M_\mu - \xi \cdot (\mu(E)) \quad = \quad M_\mu - \xi \cdot 1 \quad = \quad M_\mu - \xi. \end{aligned}$$ □

THEOREM 16.4. **Let** $E \subseteq \mathbb{R}$ *be nonempty and finite. Then:*

as $\beta \to \infty$, $A_\beta^E \to \min E$

and *as* $\beta \to -\infty$, $A_\beta^E \to \max E$.

The proof is a matter of bookkeeping, best explained by example:

Let $E := \{2, 4, 8, 9\}$. Then $\min E = 2$ and $\max E = 9$.

Since, $\forall \beta \in \mathbb{R}$, $A_\beta^E = \dfrac{2e^{-2\beta} + 4e^{-4\beta} + 8e^{-8\beta} + 9e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta}}$,

we get as $\beta \to \infty$, $A_\beta^E \to 2/1$

and as $\beta \to -\infty$, $A_\beta^E \to 9/1$,

and so as $\beta \to \infty$, $A_\beta^E \to \min E$

and as $\beta \to -\infty$, $A_\beta^E \to \max E$.

For all nonempty, finite $E \subseteq \mathbb{R}$, **define** $\boxed{A_\bullet^E} : \mathbb{R} \to \mathbb{R}$ by:

$\forall \beta \in \mathbb{R}$, $A_\bullet^E(\beta) = A_\beta^E$.

Recall (§2): "C^ω" means "real-analytic".

THEOREM 16.5. **Let** $E \subseteq \mathbb{R}$. *Assume:* $2 \leqslant \#E < \infty$.

Then: $A_\bullet^E$ *is a strictly-decreasing* C^ω*-diffeomorphism*

from $\mathbb{R}$ *onto* $(\min E; \max E)$.

Proof. **Let** $\kappa := \#E$. **Choose** $\varepsilon_1, \ldots, \varepsilon_\kappa \in \mathbb{R}$ s.t. $E = \{\varepsilon_1, \ldots, \varepsilon_\kappa\}$.

Then: $2 \leqslant \kappa < \infty$ and $\varepsilon_1, \ldots, \varepsilon_\kappa$ are distinct.

Then: $\forall \beta \in \mathbb{R}$, $A_\bullet^E(\beta) = \dfrac{\sum_{i=1}^\kappa [\varepsilon_i \cdot e^{-\beta \cdot \varepsilon_i}]}{\sum_{j=1}^\kappa [e^{-\beta \cdot \varepsilon_j}]}$. Then $A_\bullet^E : \mathbb{R} \to \mathbb{R}$ is C^ω.

So, by Theorem 16.4 and the C^ω-Inverse Function Theorem and the Mean Value Theorem, **it suffices to show:** $(A_\bullet^E)' < 0$ on $\mathbb{R}$.

Given $\beta \in \mathbb{R}$, **want:** $(A_\bullet^E)'(\beta) < 0$.

Let $P := \sum_{i=1}^{\kappa} [\, \varepsilon_i \cdot e^{-\beta\cdot\varepsilon_i} \,]$, $P' := \sum_{i=1}^{\kappa} [\, (-\varepsilon_i^2) \cdot e^{-\beta\cdot\varepsilon_i} \,]$.

Let $Q := \sum_{j=1}^{\kappa} [\, e^{-\beta\cdot\varepsilon_j} \,]$, $Q' := \sum_{j=1}^{\kappa} [\, (-\varepsilon_j) \cdot e^{-\beta\cdot\varepsilon_j} \,]$.

Then $Q > 0$. Also, by the Quotient Rule, $(A_\bullet^E)'(\beta) = [QP' - PQ']/Q^2$.

Want: $QP' - PQ' < 0$.

We have: $QP' = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} [\, (-\varepsilon_i^2 \qquad) \cdot e^{-\beta\cdot(\varepsilon_i+\varepsilon_j)} \,]$.

We have: $PQ' = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} [\, (\qquad - \varepsilon_i\varepsilon_j) \cdot e^{-\beta\cdot(\varepsilon_i+\varepsilon_j)} \,]$.

Then: $QP' - PQ' = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} [\, (-\varepsilon_i^2 + \varepsilon_i\varepsilon_j) \cdot e^{-\beta\cdot(\varepsilon_i+\varepsilon_j)} \,]$.

Interchanging i and j, we get:

$$QP' - PQ' = \sum_{j=1}^{\kappa} \sum_{i=1}^{\kappa} [\, (-\varepsilon_j^2 + \varepsilon_j\varepsilon_i) \cdot e^{-\beta\cdot(\varepsilon_j+\varepsilon_i)} \,].$$

By commutativity of addition and multiplication,
adding the last two equations gives:

$$2 \cdot (QP' - PQ') = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} [\, (-\varepsilon_i^2 - \varepsilon_j^2 + 2\varepsilon_i\varepsilon_j) \cdot e^{-\beta\cdot(\varepsilon_i+\varepsilon_j)} \,].$$

Then: $2 \cdot (QP' - PQ') = \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} [\, -(\varepsilon_i - \varepsilon_j)^2 \cdot e^{-\beta\cdot(\varepsilon_i+\varepsilon_j)} \,]$.

Then: $2 \cdot (QP' - PQ') < 0$. Then: $QP' - PQ' < 0$. $\square$

DEFINITION 16.6. **Let** $E \subseteq \mathbb{R}$.
Assume: $2 \leqslant \#E < \infty$. **Let** $\alpha \in (\min E; \max E)$.
The $\boxed{\alpha\text{-}\textbf{Boltzmann-parameter } \textit{on } E}$ *is:* $\boxed{\mathrm{BP}_\alpha^E} := (A_\bullet^E)^{-1}(\alpha)$.

So the α-Boltzmann-parameter on E is the unique $\beta \in \mathbb{R}$ s.t. $A_\beta^E = \alpha$.

Example: Computations in §7 show:

$$\forall \beta \in \mathbb{R}, \quad \text{if } \beta = (\ln 9)/10, \text{ then } \frac{e^{-\beta} + 10e^{-10\beta}}{1 + e^{-\beta} + 10e^{-10\beta}} = 1.$$

Then, $\forall \beta \in \mathbb{R}$, if $\beta = (\ln 9)/10$, then $(A_\bullet^{\{0,1,10\}})(\beta) = 1$.

Then: $(A_\bullet^{\{0,1,10\}})((\ln 9)/10) = 1$.

Then: $(A_\bullet^{\{0,1,10\}})^{-1}(1) = (\ln 9)/10$.

Then: $\mathrm{BP}_1^{\{0,1,10\}} = (\ln 9)/10$.

Example: **Let** $E := \{2, 4, 8, 9\}$, $\alpha := 5$, $\beta := \mathrm{BP}_\alpha^E$.
To compute β, we need to solve $A_\bullet^E(\beta) = 5$ for β.
Since $A_\bullet^E$ is strictly-decreasing, there are iterative methods of solution,
and we get: $\beta \approx 0.0918$, accurate to four decimal places.
(Thanks to C. Prouty for these calculations. See §28.)

THEOREM 16.7. **Let** $E \subseteq \mathbb{R}$. *Assume:* $2 \leqslant \#E < \infty$.
Let $\alpha \in (\min E; \max E)$. **Let** $\xi \in \mathbb{R}$. *Then:* $\mathrm{BP}_{\alpha-\xi}^{E-\xi} = \mathrm{BP}_\alpha^E$.

Proof. **Let** $\beta := \mathrm{BP}^E_\alpha$. **Want:** $\mathrm{BP}^{E-\xi}_{\alpha-\xi} = \beta$.
Since $\beta = \mathrm{BP}^E_\alpha = (A^E_\bullet)^{-1}(\alpha)$, we get: $(A^E_\bullet)(\beta) = \alpha$.
By Theorem 16.3, $A^{E-\xi}_\beta = A^E_\beta - \xi$.
Since $(A^{E-\xi}_\bullet)(\beta) = A^{E-\xi}_\beta = A^E_\beta - \xi = ((A^E_\bullet)(\beta)) - \xi = \alpha - \xi$,
we get: $\beta = (A^{E-\xi}_\bullet)^{-1}(\alpha - \xi)$.
So, since $\mathrm{BP}^{E-\xi}_{\alpha-\xi} = (A^{E-\xi}_\bullet)^{-1}(\alpha - \xi)$, we get: $\mathrm{BP}^{E-\xi}_{\alpha-\xi} = \beta$. □

17. Residue-unconstrained finite award sets

In the next three theorems, we generalize our work in §13
from $\{0, 1, 10\}$ to arbitrary finite residue-unconstrained sets.
In the example at the end of this section,
we show that Theorem 17.3 below reproduces the result of §13.

Recall (§9): $\forall$countable set Θ,
$\mathcal{FM}_\Theta$ is the set of finite measures on Θ
and $\mathcal{FM}^\times_\Theta$ is the set of nonzero finite measures on Θ
and $\mathcal{P}_\Theta$ is the set of probability measures on Θ.
Recall (§9): $\forall$ nonempty finite set F, $\forall f \in F$, $\nu_F\{f\} = 1/(\#F)$.
Recall (Definition 9.2): $\forall$countable set Θ, $\forall \mu \in \mathcal{FM}_\Theta$, $\forall n \in \mathbb{N}$,
$\forall x \in \Theta^n$, $\mu^n\{x\} = (\mu\{x_1\}) \cdots (\mu\{x_n\})$.

THEOREM 17.1. Let *$E \subseteq \mathbb{Z}$ be finite and residue-unconstrained.*
Let *$\alpha \in (\min E; \max E)$.* **Let** *$\beta := \mathrm{BP}^E_\alpha$.*
Let *$t_1, t_2, \ldots \in \mathbb{Z}$. Assume: $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ is bounded.*
For all $n \in \mathbb{N}$, **let** *$\Omega_n := \{f \in E^n \,|\, f_1 + \cdots + f_n = t_n\}$.*
Let *$\varepsilon_0 \in E$. Then: as $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to B^E_\beta\{\varepsilon_0\}$.*

Recall (§9): $\nu_\varnothing(\varnothing) = -1$.
So, since $B^E_\beta\{\varepsilon_0\} > 0$, part of the content of this theorem is:
$\forall$sufficiently large $n \in \mathbb{N}$, $\Omega_n \neq \varnothing$;
see Claim 1 in the proof below.

Proof. Since E is residue-unconstrained, we get: $E \neq \varnothing$.
By hypothesis, $E \subseteq \mathbb{Z}$ and E is finite.
Then: $E \subseteq \mathbb{R}$ and E is nonempty and finite.
Let $\mu := B^E_\beta$. Then: $\mu \in \mathcal{P}_E$ and $S_\mu = E$.
So, since $\mu \in \mathcal{P}_E \subseteq \mathcal{FM}_E$, we get: $|\mu|_1 < \infty$ and $|\mu|_2 < \infty$.
Since $\beta = \mathrm{BP}^E_\alpha = (A^E_\bullet)^{-1}(\alpha)$, we get: $(A^E_\bullet)(\beta) = \alpha$.
So, since $(A^E_\bullet)(\beta) = A^E_\beta = M_{B^E_\beta} = M_\mu$, we get: $M_\mu = \alpha$.

For all $n \in \mathbb{N}$, **define** $\psi_n : \mathbb{Z} \to \mathbb{R}$ by:

$$\forall t \in \mathbb{Z}, \quad \psi_n(t) = \mu^n\{f \in E^n \mid f_1 + \cdots + f_n = t\}.$$

Then: $\forall n \in \mathbb{N}, \quad \psi_n(t_n) = \mu^n(\Omega_n)$.

Let $v := V_\mu$. By Theorem 10.6, we get: $0 < v < \infty$.

Let $\tau := 1/\sqrt{2\pi v}$. Then: $0 < \tau < \infty$.

By Theorem 10.6, we get:

$$\text{as } n \to \infty, \quad \sqrt{n} \cdot (\mu^n\{f \in E^n \mid f_1 + \cdots + f_n = t_n\}) \to 1/\sqrt{2\pi v}.$$

Then: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(t_n)) \to \tau$.

So, since $\tau > 0$, **choose** $n_0 \in \mathbb{N}$ s.t.: $\forall n \in [n_0..\infty)$, $\sqrt{n} \cdot (\psi_n(t_n)) > 0$.

Claim 1: **Let** $n \in [n_0..\infty)$. Then: $\mu^n(\Omega_n) > 0$.

Proof of Claim 1: Recall: $\psi_n(t_n) = \mu^n(\Omega_n)$. **Want:** $\psi_n(t_n) > 0$.

By the choice of n_0, we get: $\sqrt{n} \cdot (\psi_n(t_n)) > 0$. Then: $\psi_n(t_n) > 0$.

End of proof of Claim 1.

Recall: $\mu \in \mathcal{P}_E$.

Then: $\forall n \in \mathbb{N}$, $\mu^n \in \mathcal{P}_{E^n}$, so $\mu^n(\Omega_n) \leqslant 1$.

So, by Claim 1, $\forall n \in [n_0..\infty)$, $0 < \mu^n(\Omega_n) \leqslant 1$.

Also, we have: $\forall n \in \mathbb{N}$, $(\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n)$.

Then: $\forall n \in [n_0..\infty)$, $0 < (\mu^n|\Omega_n)(\Omega_n) \leqslant 1$.

Then: $\forall n \in [n_0..\infty)$, $\mu^n \,|\, \Omega_n \in \mathcal{FM}^\times_{\Omega_n}$.

Then: $\forall n \in [n_0..\infty)$, $\mathcal{N}(\mu^n \,|\, \Omega_n) \in \mathcal{P}_{\Omega_n}$.

Claim 2: **Let** $n \in [n_0..\infty)$. Then: $\mathcal{N}(\mu^n \,|\, \Omega_n) = \nu_{\Omega_n}$.

Proof of Claim 2: **Let** $\theta := \mathcal{N}(\mu^n|\Omega_n)$, $F := \Omega_n$. Then $\theta \in \mathcal{P}_F$.

Want: $\theta = \nu_F$. By Theorem 9.9, **given** $f, g \in F$, **want:** $\theta\{f\} = \theta\{g\}$.

By Claim 1, we have: $\mu^n(\Omega_n) > 0$.

Since $(\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n)$ and $\theta = \mathcal{N}(\mu^n|\Omega)$, we get: $\theta = \dfrac{\mu^n|\Omega_n}{\mu^n(\Omega_n)}$.

Want: $\dfrac{(\mu^n|\Omega_n)\{f\}}{\mu^n(\Omega_n)} = \dfrac{(\mu^n|\Omega_n)\{g\}}{\mu^n(\Omega_n)}$.

Want: $(\mu^n|\Omega_n)\{f\} = (\mu^n|\Omega_n)\{g\}$.

Since $f, g \in F = \Omega_n$, we get:

$$(\mu^n|\Omega_n)\{f\} = \mu^n\{f\} \quad \text{and} \quad (\mu^n|\Omega_n)\{g\} = \mu^n\{g\}.$$

Want: $\mu^n\{f\} = \mu^n\{g\}$.

Since $E \subseteq \mathbb{R}$ is nonempty and finite, we get: $\widehat{B}^E_\beta(E) > 0$.

Let $C := 1/(\widehat{B}^E_\beta(E))$. Then $\mathcal{N}(\widehat{B}^E_\beta) = C \cdot \widehat{B}^E_\beta$

By definition of $\widehat{B}^E_\beta$, we have: $\forall \varepsilon \in E$, $\widehat{B}^E_\beta\{\varepsilon\} = e^{-\beta \cdot \varepsilon}$.

So, since $\mu = B^E_\beta = \mathcal{N}(\widehat{B}^E_\beta) = C \cdot \widehat{B}^E_\beta$,
we get: $\forall \varepsilon \in E$, $\mu\{\varepsilon\} = Ce^{-\beta\cdot\varepsilon}$.
Since $f \in F = \Omega_n$, by definition of Ω_n, we get: $f_1 + \cdots + f_n = t_n$.
Since $g \in F = \Omega_n$, by definition of Ω_n, we get: $g_1 + \cdots + g_n = t_n$.
Since $f_1 + \cdots + f_n = t_n = g_1 + \cdots + g_n$,
we get: $C^n e^{-\beta\cdot(f_1+\cdots+f_n)} = C^n e^{-\beta\cdot(g_1+\cdots+g_n)}$.
Then: $(Ce^{-\beta\cdot f_1})\cdots(Ce^{-\beta\cdot f_n}) = (Ce^{-\beta\cdot g_1})\cdots(Ce^{-\beta\cdot g_n})$.
Then: $(\ \mu\{f_1\}\ )\cdots(\ \mu\{f_n\}\ ) = (\ \mu\{g_1\}\ )\cdots(\ \mu\{g_n\}\ )$.
Then: $\mu^n\{f\} = \mu^n\{g\}$.
End of proof of Claim 2.

By hypothesis, E is residue-unconstrained and $\varepsilon_0 \in E$ and $t_1, t_2, \ldots \in \mathbb{Z}$ and $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ is bounded.
Recall: $\mu \in \mathcal{P}_E$ and $S_\mu = E$ and $|\mu|_2 < \infty$ and $M_\mu = \alpha$.
Let $P := \mu\{\varepsilon_0\}$. Then, since $\mu = B^E_\beta$, we get: $P = B^E_\beta\{\varepsilon_0\}$.
We want: as $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to P$.
By Theorem 12.2, as $n \to \infty$, $(\mathcal{N}(\mu^n|\Omega_n))\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to P$.
So, by Claim 2, as $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to P$. $\square$

Recall (§2): $\forall t \in \mathbb{R}$, $\lfloor t \rfloor$ is the floor of t.
We record the $t_n = \lfloor n\alpha \rfloor$ version of the preceding theorem:

THEOREM 17.2. Let *$E \subseteq \mathbb{Z}$ be finite and residue-unconstrained.*
Let $\alpha \in (\min E; \max E)$. **Let** $\beta := \mathrm{BP}^E_\alpha$.
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in E^n \,|\, f_1 + \cdots + f_n = \lfloor n\alpha \rfloor\}$.
Let $\varepsilon_0 \in E$. *Then: as $n \to \infty$,* $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to B^E_\beta\{\varepsilon_0\}$.

We record the $\alpha \in \mathbb{Z}$ special case of the preceding theorem:

THEOREM 17.3. Let *$E \subseteq \mathbb{Z}$ be finite and residue-unconstrained.*
Let $\alpha \in (\min E; \max E)$. **Let** $\beta := \mathrm{BP}^E_\alpha$. *Assume $\alpha \in \mathbb{Z}$.*
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in E^n \,|\, f_1 + \cdots + f_n = n\alpha\}$.
Let $\varepsilon_0 \in E$. *Then: as $n \to \infty$,* $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \varepsilon_0\} \to B^E_\beta\{\varepsilon_0\}$.

Example: Suppose $E = \{0, 1, 10\}$ and $\alpha = 1$.
Then $\Omega_N = \{f \in E^N \,|\, f_1 + \cdots + f_N = N\}$,
so Ω_N represents the set of all GFA dispensations,
as described in §3.
The measure ν_{Ω_N} gives equal probability to each dispensation,
so ν_{Ω_N} represents the GFA's first system for awarding grants,
also described in §3.

Since $\beta = \mathrm{BP}_\alpha^E = \mathrm{BP}_1^{\{0,1,10\}}$, we calculate: $\beta = (\ln 9)/10$.
More calculation gives: $(B_\beta^E\{0\}, B_\beta^E\{1\}, B_\beta^E\{10\}) = \dfrac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$.
Since N is large, by Theorem 17.3, we get:

$$\nu_{\Omega_N}\{f \in \Omega_N \mid f_N = \varepsilon_0\} \;\approx\; B_\beta^E\{\varepsilon_0\}.$$

So, if I am the Nth professor, then, under the first system,
my probability of receiving ε_0 dollars
is approximately equal to $B_\beta^E\{\varepsilon_0\}$.
Thus Theorem 17.3 reproduces the result of §13.

18. Rational awards

In this section, we investigate what happens if
the set of awards is an arbitrary set of rational numbers.
Recall that, on our Earth, which is Earth-1218,
grants are \$0, \$1, \$10, with average grant \$1.

Example: In a parallel universe, on Earth-googol-plex,
there are N_0 professors applying for grants from its GFA.
By GFA rules, grant amounts are \$10, \$14.45, \$54,
and Congress allocates \$13.37 per professor.
Earth-googol-plex has its own GFA.
That GFA is using the "first system" for awarding grants,
in which every dispensation is equally likely.
Question: For any professor,
what is the approximate probability of receiving \$10? \$14.45? \$54?
To simplify this problem, we can imagine that
that GFA makes two rounds of awards.
In the first round, it simply dispenses \$10 to each professor.
In the second round, using the first system, it dispenses
additional grants of \$0, \$4.45, \$44, with average grant \$3.37.
We seek the approximate probability of the additional grant being
each of the numbers \$0, \$4.45, \$44.
To simplify this problem still more, we can
change monetary units so that the grant amounts are all integers:
Additional grants, in pennies, are 0, 445, 4400, with average grant 337,
and we seek the approximate probability of receiving 0, 445, 4400.
Unfortunately, $\{0, 445, 4400\} \subseteq 5\mathbb{Z} + 0$,
so $\{0, 445, 4400\}$ is not residue-unconstrained,

making it difficult to apply the Discrete Local Limit Theorem.
Since $\gcd\{0,445,4400\} = 5$, we can change monetary units again:
Additional grants, in nickels, are 0, 89, 880, with average grant 337/5,
and we seek the approximate probability of receiving 0, 89, 880.
Let $E := \{0,89,880\}$ and **let** $\alpha := 337/5$.
Since $0 \in E$ and $\gcd(E) = 1$, we get: E is residue-unconstrained.
The amount of money (in nickels) allocated by Congress is $N_0\alpha$,
to be dispensed among the N_0 professors.
Unfortunately, a census reveals that: N_0 *is not* divisible by 5.
Recall: $\alpha = 337/5$. Then $N_0\alpha \notin \mathbb{Z}$, while $0,89,880 \in \mathbb{Z}$.
It is therefore *impossible* to dispense the grant money.
The bureaucracy seizes up, there is pandemonium in the streets,
and the military steps in to impose order.
The superheroes of Earth-googol-plex are committed to democracy,
and so they reverse time and select a different time-line.
On this new time-line, E and α are unchanged, but
there is a new number, N_1, of professors,
and N_1 is blissfully divisible by 5. Then: $N_1\alpha \in \mathbb{Z}$.
Let $\varepsilon_0 \in E$ **be given**.
We want: the approximate probability of receiving ε_0 nickels.
Recall (§2): $\forall t \in \mathbb{R}$, $\lfloor t \rfloor$ is the floor of t.
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in E^n \mid f_1 + \cdots + f_n = \lfloor n\alpha \rfloor\}$.
Since $N_1\alpha \in \mathbb{Z}$, we get: $\Omega_{N_1} = \{f \in E^{N_1} \mid f_1 + \cdots + f_{N_1} = N_1\alpha\}$.
We want: an approximation to $\nu_{\Omega_{N_1}}\{f \in \Omega_{N_1} \mid f_{N_1} = \varepsilon_0\}$.
Recall: E is residue-unconstrained.
Let $\beta := \mathrm{BP}^E_\alpha$. By Theorem 17.2, we have:
as $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \mid f_n = \varepsilon_0\} \to B^E_\beta\{\varepsilon_0\}$.
So, assuming N_1 is large, we get

$$\nu_{\Omega_{N_1}}\{f \in \Omega_{N_1} \mid f_{N_1} = \varepsilon_0\} \approx B^E_\beta\{\varepsilon_0\}.$$

For each $\varepsilon_0 \in \{0,89,880\}$, **we want** to compute $B^E_\beta\{\varepsilon_0\}$.
We therefore want to compute $(\,B^E_\beta\{0\}\,,\,B^E_\beta\{89\}\,,\,B^E_\beta\{880\}\,)$.
Since $\beta = \mathrm{BP}^E_\alpha = \mathrm{BP}^{\{0,89,880\}}_{337/5}$, we see that:
to evaluate β, we must solve $A^{\{0,89,880\}}_\bullet(\beta) = 337/5$ for β.
Since, by Theorem 16.5, $A^{\{0,89,880\}}_\bullet$ is strictly-decreasing,
there are simple iterative methods to do this.
We calculate $\beta = 0.003144$, accurate to six decimal places.
We also calculate

$$(B^E_\beta\{0\} \, , \, B^E_\beta\{89\} \, , \, B^E_\beta\{880\} \,) \; = \; (\, 0.5498 \, , \, 0.4156 \, , \, 0.0345 \,),$$

all accurate to four decimal places.

(Thanks to C. Prouty for this calculation. See §28.)

Recall (§3): N is a large positive integer.
More generally: Imagine a parallel universe with N professors.
Let E_0 denote the set of grant-awards.
Assume $E_0 \subseteq \mathbb{Q}$ and $2 \leqslant \#E_0 < \infty$.
Let $\alpha_0 \in (\min E_0; \max E_0)$ denote the average award.
Since $\#E_0 \geqslant 2$, we get: $E_0 \neq \varnothing$. **Choose** $\varepsilon_0 \in E_0$. Then $\varepsilon_0 \in \mathbb{Q}$.
Let $E_1 := E_0 - \varepsilon_0$, $\alpha_1 := \alpha_0 - \varepsilon_0$. Then $\alpha_1 \in (\min E_1; \max E_1)$.
Also, $0 \in E_1$.
So, by giving out awards in two rounds (first ε_0, then the remainder),
we are reduced to a case where 0 is a possible grant-award.
Since $E_1 = E_0 - \varepsilon_0 \subseteq \mathbb{Q}$, **choose** $m \in \mathbb{N}$ s.t. $mE_1 \subseteq \mathbb{Z}$.
Let $E_2 := mE_1$, $\alpha_2 := m\alpha_1$. Then $\alpha_2 \in (\min E_2; \max E_2)$.
Also, $0 \in E_2 \subseteq \mathbb{Z}$.
So, by change of monetary unit,
we are reduced to a case where every grant-award is an integer
and where 0 is a possible grant-award.
Let $g := \gcd(E_2)$, $E := E_2/g$, $\alpha := \alpha_2/g$. Then $\alpha \in (\min E; \max E)$.
Also, $0 \in E \subseteq \mathbb{Z}$ and $\gcd(E) = 1$, so E is residue-unconstrained.
So, by change of monetary unit, we are reduced to a case where
the set of grant-awards is a residue-unconstrained set of integers.
If $N\alpha \notin \mathbb{Z}$, then, since every grant-award is an integer,
no dispensation is possible, leading to
your typical military dictatorship and superhero intervention.
If $N\alpha \in \mathbb{Z}$, then, using Theorem 17.2,
we can compute the approximate probability of each award.

19. Irrational awards

In this section, we briefly discuss what can happen if
NOT every grant award is a rational number.
Here, we only present an example to show that
the award probabilities may NOT follow a Boltzmann distribution.

Example: On Earth-aleph-1, the GFA gives
grants of 0 , $\sqrt{2}$, $\sqrt{3}$, $10 - \sqrt{2} - \sqrt{3}$ dollars,

with an average grant of 1 dollar,
giving equal probability to every possible dispensation.
Let K be the number of professors. Assume: K is divisible by 10.
Let $M := K/10$. Then $M \in \mathbb{N}$ and there are $10M$ professors.
Moreover, since the average grant is 1 dollar, we conclude:
there are $10M$ dollars to dispense among the $10M$ professors.

Claim: On Earth-aleph-1, every dispensation of awards has

$7M$	grants of	0	dollars,
M	grants of	$\sqrt{2}$	dollars,
M	grants of	$\sqrt{3}$	dollars,
M	grants of	$10 - \sqrt{2} - \sqrt{3}$	dollars.

Proof of Claim: **Given** a dispensation,

let w be	the number of	0	dollar grants	and
let x be	the number of	$\sqrt{2}$	dollar grants	and
let y be	the number of	$\sqrt{3}$	dollar grants	and
let z be	the number of	$10 - \sqrt{2} - \sqrt{3}$	dollar grants,	

want: $w = 7M$ and $x = y = z = M$.
Because the total money dispensed is $10M$ dollars, we get:
$$w \cdot 0 + x \cdot \sqrt{2} + y \cdot \sqrt{3} + z \cdot (10 - \sqrt{2} - \sqrt{3}) = 10M.$$
Then: $(10z - 10M) \cdot 1 + (x - z) \cdot \sqrt{2} + (y - z) \cdot \sqrt{3} = 0$.
So, since $1, \sqrt{2}, \sqrt{3}$ are linearly indpendent over $\mathbb{Q}$, we get:
$$10z - 10M = 0 \quad \text{and} \quad x - z = 0 \quad \text{and} \quad y - z = 0.$$
Then $z = M$ and $x = z$ and $y = z$. Then $x = y = z = M$.
It remains only to show: $w = 7M$.
Because there are $10M$ professors, we get: $w + x + y + z = 10M$.
Then: $w + M + M + M = 10M$. Then: $w = 7M$.
End of proof of Claim.

By the Claim, in each dispensation, there are
exactly M grants of $10 - \sqrt{2} - \sqrt{3}$ dollars.
Of the four grant amounts, the largest is $10 - \sqrt{2} - \sqrt{3}$.
So, if I am one of the $10M$ professors, then I would hope to be among
the lucky M receiving $10 - \sqrt{2} - \sqrt{3}$ dollars.
My probability of being so lucky is: $M/(10M)$, *i.e.*, 10%.
That is, we obtain a probaity of:
10% for $10 - \sqrt{2} - \sqrt{3}$ dollars.
Extending this reasoning, we obtain probabities of:

70%	for	0	dollars,
10%	for	$\sqrt{2}$	dollars,
10%	for	$\sqrt{3}$	dollars,
10%	for	$10-\sqrt{2}-\sqrt{3}$	dollars.

In a Boltzmann distribution, depending on whether $\beta = 0$ or $\beta \neq 0$,
either the probabilities are all equal
or the probabilities are all distinct from one another.
The numbers 70,10,10,10 are neither all equal nor all distinct.
Thus, the 70-10-10-10 distribution above is NOT Boltzmann.

20. Earth-minimum-Mahlo-cardinal and the BUA

Next, we wish to handle thermodynamic systems in which
many states may have a single energy-level.
One says that such an energy-level is "degenerate".
In this section, we develop a whimsical example.
In §21 and §22, we will develop a general theory.

Recall that $N \in \mathbb{N}$ is large.
In a parallel universe, on Earth-minimum-Mahlo-cardinal,
the BUA (Best University Anywhere) employs N professors.
Each professor has a number, from 1 to N.
Each professor wanders the campus,
carrying two bags: one red, one blue.
Each bag is closed from view, but has money in it or is empty.
The "state" of a professor is the pair $\sigma = (\sigma_1, \sigma_2)$ such that
σ_1 is the number of dollars in the professor's red bag,
σ_2 is the number of dollars in the professor's blue bag;
the professor's "wealth" is $\sigma_1 + \sigma_2$ dollars.
So, if I am one of the professors, and if my state is $(3, 2)$,
then I have: \$3 in my red bag and \$2 in my blue bag,
and my wealth is \$5.
By BUA rules, the amount of money in any bag is always
\$0 or \$1 or \$2 or \$3 or \$4,
and each professor's wealth is always $\leqslant$ \$7.
Therefore, the set of allowable states is
$(\,[0..4] \times [0..4]\,) \setminus \{\,(4,4)\,\}$.
Let $\Sigma := (\,[0..4] \times [0..4]\,) \setminus \{\,(4,4)\,\}$.
Since $\#(\,[0..4] \times [0..4]\,) = 5 \cdot 5 = 25$, we get: $\#\Sigma = 24$.

Define $\varepsilon : \Sigma \to [0..7]$ by: $\quad \forall \sigma \in \Sigma, \ \ \varepsilon(\sigma) = \sigma_1 + \sigma_2$.
For convenience of notation, $\quad \forall \sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
If I am one of the professors,
and if my state is $\sigma = (\sigma_1, \sigma_2) \in \Sigma$,
then I have: $\$\sigma_1$ in my red bag and $\$\sigma_2$ in my blue bag,
and my wealth is $\$\varepsilon_\sigma$.
Since $\varepsilon_{(3,2)} = 5 = \varepsilon_{(1,4)}$, we see that ε is not one-to-one,
and we have a so-called "degeneracy" at 5.
This function ε has many other degeneracies, as well.

Recall: The professors are numbered, from 1 to N.
At random moments,
random pairs of wandering professors cross paths, and interact.
Each interaction involves three steps:
a game and then
a verbal offer and then
a rejection or a money transfer.
The first step, the game, is played as follows:
one of the two professors flips a fair coin and
if heads, then the lower-numbered professor wins and
if tails, then the higher-numbered professor wins.
Next, without touching any money,
the losing professor verbally offers \$1 to the winning professor.
The losing professor then flips a fair coin, and
if heads, then the loser's red bag is opened and
if tails, then the loser's blue bag is opened.
If the loser's open bag is empty, then
then the winner gallantly rejects the \$1 offer and
the opened bag is closed, the interaction is over, and
the professors continue their wanderings.
On the other hand, if the loser's open bag is NOT empty, then,
both of the winner's bags are opened.
Recall that, by BUA rules, every professor's wealth must be $\leqslant \$7$.
If the winner's wealth is \$7,
then the winner rejects the \$1 offer and
the opened bags are closed, the interaction is over, and
the professors continue their wanderings.
On the other hand, if the winner's wealth is $\leqslant \$6$,

then the winner flips a fair coin, and
if heads, then the winner's red bag is closed and
if tails, then the winner's blue bag is closed.
At this point, the winner has one open bag, as does the loser.
Moreover, the loser's open bag is NOT empty.
Recall that no bag may have more than \$4.
If the winner's open bag has \$4,
then the winner rejects the \$1 offer and
the opened bags are closed, the interaction is over, and
the professors continue their wanderings.
On the other hand, if the winner's open bag has $\leqslant$ \$3,
then \$1 is transferred
from the losing professor's open bag
to the winning professor's open bag;
then the opened bags are closed, the interaction is over, and
the professors continue their wanderings.

Because of these interactions,
the wealth of an individual professor may change over time,
but the sum of the wealths of all of them is constant;
there is "conservation of (total) wealth".
An audit reveals that, at the BUA, that total wealth is always N.

Recall: $\Sigma = ([0..4] \times [0..4]) \backslash \{(4,4)\}$ is the set of states.
A "state-dispensation" is a function $[1..N] \to \Sigma$,
representing the states of all N professors.
So, if, at some point in time, the state-dispensation is $\omega : [1..N] \to \Sigma$,
then, for every $\ell \in [1..N]$, the state of Professor $\#\ell$ is $\omega(\ell)$,
and the wealth of Professor $\#\ell$ is $\varepsilon_{\omega(\ell)}$;
therefore, the total wealth of all the professors is $\sum_{\ell=1}^{N} \varepsilon_{\omega(\ell)}$.
As we mentioned, at the BUA, that total wealth is N.
Let $\Omega^* := \left\{ \omega : [1..N] \to \Sigma \;\middle|\; \sum_{\ell=1}^{N} \varepsilon_{\omega(\ell)} = N \right\}$.
Then Ω^* represents the set of all state-dispensations at the BUA.

The random interactions, described above,
induce a discrete Markov-chain on Ω^*.
This, in turn, induces a map $\Pi : \mathcal{P}_{\Omega^*} \to \mathcal{P}_{\Omega^*}$.

Let $T := \#\Omega^*$. **Fix** an ordering of Ω^*, *i.e.*, a bijection $[1..T] \leftrightarrow \Omega^*$.
The Markov-chain then has a $T \times T$ transition-matrix Φ,
with entries in $[0;1]$, whose column-sums are all $= 1$.
For every $\phi, \psi \in \Omega^*$, the probability of transitioning from ϕ to ψ
is equal to
the probability of transitioning from ψ to ϕ.
That is, the transition-matrix Φ is symmetric.
So, since the column-sums of Φ are all 1,
we get: the row-sums of Φ are all 1.
Let v be a $T \times 1$ column vector whose entries are all 1. Then $\Phi v = v$.
Let $w := v/T$. Then: all the entries of w are $1/T$ and $\Phi w = w$.
Recall that the probability-distribution $\nu_{\Omega^*} \in \mathcal{P}_{\Omega^*}$
assigns equal probability to each state-dispensation in Ω^*.
That is, $\forall \omega \in \Omega^*$, $\nu_{\Omega^*}\{\omega\} = 1/T$.
Since the entries of w are equal to these ν_{Ω^*}-probabilities,
and since $\Phi w = w$, we get: $\Pi(\nu_{\Omega^*}) = \nu_{\Omega^*}$.

We will say that two state-dispensations $\phi, \psi \in \Omega^*$ are "adjacent",
if there is an interaction that carries ϕ to ψ.
For any $\phi, \psi \in \Omega^*$,
$\exists$a finite sequence of interactions that carries ϕ to ψ.
That is: $\forall \phi, \psi \in \Omega^*$, $\exists m \in \mathbb{N}$, $\exists \omega_0, \ldots, \omega_m \in \Omega^*$
s.t. $\phi = \omega_0$ and $\omega_m = \psi$
and s.t. $\forall i \in [1..m]$, ω_{i-1} is adjacent to ω_i.
That is, any two state-dispensations
are connected by an adjacency-path.
That is, the Markov-chain on Ω^* is irreducible.
Recall that some interactions result in a rejection;
such interactions do not change the state-dispensation.
So, a state-dispensation is sometimes adjacent to itself.
That is, there are adjacency-cycles of length 1.
It follows that the Markov-chain is aperiodic.
So, since the Markov-chain is irreducible and since $\Pi(\nu_{\Omega^*}) = \nu_{\Omega^*}$,
by the Perron-Frobenius Theorem, we get:

$$\forall \mu \in \mathcal{P}_{\Omega^*}, \qquad \mu\,,\ \Pi(\mu)\,,\ \Pi(\Pi(\mu))\,,\ \Pi(\Pi(\Pi(\mu)))\,,\ \ldots \to \nu_{\Omega^*}.$$

That is, for any starting probability-distribution on Ω^*,
after enough random interactions,

the resulting probability-distribution on Ω^*
will be approximately equal to ν_{Ω^*},
to any desired level of accuracy.

Problem: Suppose I am Professor $\#N$ at the BUA.
Suppose that the probability-distribution μ of state-dispensations is approximately equal to ν_{Ω^*}.
For each $\sigma \in \Sigma$, compute my probability of being in state σ.
That is, $\forall \sigma \in \Sigma$, compute $\mu\{\omega \in \Omega^* \,|\, \omega(N) = \sigma\}$.
Since $\#\Sigma = 24$, there will be 24 answers.
Approximate answers are acceptable.

To make a precise mathematical problem,
we, in fact, assume that μ is *exactly* equal to ν_{Ω^*},
and we seek the exact "thermodynamic limit", meaning
we replace N with a variable $n \in \mathbb{N}$, and let $n \to \infty$.

In the next two sections, we will develop a theory
to solve problems like this one.
We need only adapt our earlier methods to allow for degeneracies.

Our main theorems are
Theorem 22.1 and Theorem 22.2 and Theorem 22.3,
and the solution to the above "precise mathematical problem"
appears in the example at the end of §22.

21. Boltzmann distributions on finite sets with degeneracy

In this section, we adapt our earlier work (§16)
on Boltzmann distributions to allow for degeneracies.

Recall (§9): $\forall$countable set Θ,
$\mathcal{FM}_\Theta^\times$ is the set of nonzero finite measures on Θ
and $\mathcal{P}_\Theta$ is the set of probability measures on Θ.
Recall (§9): $\forall$nonempty countable set Θ, $\forall \mu \in \mathcal{FM}_\Theta^\times$,
$\mathcal{N}(\mu)$ is the normalization of μ.

DEFINITION 21.1. Let Σ *be a nonempty finite set.*
Let $\varepsilon : \Sigma \to \mathbb{R}$. **Let** $\beta \in \mathbb{R}$.
Then $\boxed{\widehat{B}^\varepsilon_\beta} \in \mathcal{FM}^\times_\Sigma$ *is* **defined** *by:* $\forall \sigma \in \Sigma$, $\widehat{B}^\varepsilon_\beta\{\sigma\} = e^{-\beta\cdot(\varepsilon(\sigma))}$.
Also, we **define**: $\boxed{B^\varepsilon_\beta} := \mathcal{N}(\widehat{B}^\varepsilon_\beta) \in \mathcal{P}_\Sigma$.

Then: $\forall$nonempty finite set Σ, $\forall \varepsilon : \Sigma \to \mathbb{R}$, $\forall \beta \in \mathbb{R}$,
$\widehat{B}^\varepsilon_\beta(\Sigma) > 0$ and $\forall \sigma \in \Sigma$, $B^\varepsilon_\beta\{\sigma\} = (\widehat{B}^\varepsilon_\beta\{\sigma\}) / (\widehat{B}^\varepsilon_\beta(\Sigma))$
and $S_{\widehat{B}^\varepsilon_\beta} = \Sigma = S_{B^\varepsilon_\beta}$.

Example: **Let** $\Sigma := \{0, 1, 10\}$ and **let** $\beta \in \mathbb{R}$.
Define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$.
Then: $\widehat{B}^\varepsilon_\beta\{0\} = 1$, $\widehat{B}^\varepsilon_\beta\{1\} = e^{-\beta}$, $\widehat{B}^\varepsilon_\beta\{10\} = e^{-10\beta}$.
Let $C := 1/(1 + e^{-\beta} + e^{-10\beta})$.
Then: $B^\varepsilon_\beta\{0\} = C$, $B^\varepsilon_\beta\{1\} = Ce^{-\beta}$, $B^\varepsilon_\beta\{10\} = Ce^{-10\beta}$.

Example: **Let** $\Sigma := \{2, 4, 8, 9\}$ and **let** $\beta \in \mathbb{R}$.
Define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$.
Then: $\widehat{B}^\varepsilon_\beta\{2\} = e^{-2\beta}$, $\widehat{B}^\varepsilon_\beta\{4\} = e^{-4\beta}$,
$\widehat{B}^\varepsilon_\beta\{8\} = e^{-8\beta}$, $\widehat{B}^\varepsilon_\beta\{9\} = e^{-9\beta}$.
Let $C := 1/(e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta})$.
Then: $B^\varepsilon_\beta\{2\} = Ce^{-2\beta}$, $B^\varepsilon_\beta\{4\} = Ce^{-4\beta}$,
$B^\varepsilon_\beta\{8\} = Ce^{-8\beta}$, $B^\varepsilon_\beta\{9\} = Ce^{-9\beta}$.

Example: **Let** $\Sigma := \{1, 2, 3, 4\}$ and **let** $\beta \in \mathbb{R}$.
Define $\varepsilon : \Sigma \to \mathbb{R}$ by:
$\varepsilon(1) = 2$, $\varepsilon(2) = 4$, $\varepsilon(3) = 8$, $\varepsilon(4) = 9$.
Then: $\widehat{B}^\varepsilon_\beta\{1\} = e^{-2\beta}$, $\widehat{B}^\varepsilon_\beta\{2\} = e^{-4\beta}$,
$\widehat{B}^\varepsilon_\beta\{3\} = e^{-8\beta}$, $\widehat{B}^\varepsilon_\beta\{4\} = e^{-9\beta}$.
Let $C := 1/(e^{-2\beta} + e^{-4\beta} + e^{-8\beta} + e^{-9\beta})$.
Then: $B^\varepsilon_\beta\{1\} = Ce^{-2\beta}$, $B^\varepsilon_\beta\{2\} = Ce^{-4\beta}$,
$B^\varepsilon_\beta\{3\} = Ce^{-8\beta}$, $B^\varepsilon_\beta\{4\} = Ce^{-9\beta}$.

In the preceding three examples, ε is one-to-one.
That is, ε has no degeneracies.
In the next, ε has one degeneracy, at energy-level 9.

Example: **Let** $\Sigma := \{1, 2, 3, 4\}$ and **define** $\varepsilon : \Sigma \to \mathbb{R}$ by:

$\varepsilon(1) = 2, \quad \varepsilon(2) = 4, \quad \varepsilon(3) = 9, \quad \varepsilon(4) = 9.$
Then: $\widehat{B}^\varepsilon_\beta\{1\} = e^{-2\beta}, \quad \widehat{B}^\varepsilon_\beta\{2\} = e^{-4\beta},$
$\widehat{B}^\varepsilon_\beta\{3\} = e^{-9\beta}, \quad \widehat{B}^\varepsilon_\beta\{4\} = e^{-9\beta}.$
Let $C := 1/(e^{-2\beta} + e^{-4\beta} + 2e^{-9\beta})$.
Then: $B^\varepsilon_\beta\{1\} = Ce^{-2\beta}, \quad B^\varepsilon_\beta\{2\} = Ce^{-4\beta},$
$B^\varepsilon_\beta\{3\} = Ce^{-9\beta}, \quad B^\varepsilon_\beta\{4\} = Ce^{-9\beta}.$

In the next example, ε has many degeneracies.

Example: **Let** $\Sigma := (\,[0..4] \times [0..4]\,) \setminus \{\,(4,4)\,\}$.
Let $\beta \in \mathbb{R}$ and **define** $\varepsilon : \Sigma \to \mathbb{R}$ by: $\quad \forall \sigma \in \Sigma, \ \ \varepsilon(\sigma) = \sigma_1 + \sigma_2.$
Then: $\widehat{B}^\varepsilon_\beta\{(3,2)\} = e^{-5\beta}, \ \ \widehat{B}^\varepsilon_\beta\{(1,4)\} = e^{-5\beta}, \ \ \widehat{B}^\varepsilon_\beta\{(0,0)\} = 1.$
Generally, $\quad \forall \sigma \in \Sigma, \ \ \widehat{B}^\varepsilon_\beta\{\sigma\} = e^{-(\sigma_1+\sigma_2)\cdot\beta}.$
Let $C := 1/(\sum_{\sigma\in\Sigma}[e^{-(\sigma_1+\sigma_2)\cdot\beta}])$.
Then: $B^\varepsilon_\beta\{(3,2)\} = Ce^{-5\beta}, \ \ B^\varepsilon_\beta\{(1,4)\} = Ce^{-5\beta}, \ \ B^\varepsilon_\beta\{(0,0)\} = C.$
Generally, $\quad \forall \sigma \in \Sigma, \ \ B^\varepsilon_\beta\{\sigma\} = Ce^{-(\sigma_1+\sigma_2)\cdot\beta}.$

THEOREM 21.2. Let Σ *be a nonempty finite set.*
Let $\varepsilon : \Sigma \to \mathbb{R}, \ \ \xi, \beta \in \mathbb{R}.$ *Then:* $B^\varepsilon_\beta = B^{\varepsilon-\xi}_\beta.$

Proof. For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Since, $\forall \sigma \in \Sigma, \ \ \widehat{B}^\varepsilon_\beta\{\sigma\} = e^{-\beta\cdot\varepsilon_\sigma} = e^{-\beta\cdot\xi} \cdot e^{-\beta\cdot(\varepsilon_\sigma-\xi)} = e^{-\beta\cdot\xi} \cdot (\widehat{B}^{\varepsilon-\xi}_\beta\{\sigma\}),$
we get: $\quad \widehat{B}^\varepsilon_\beta = e^{-\beta\cdot\xi} \cdot \widehat{B}^{\varepsilon-\xi}_\beta.$
Since $e^{-\beta\xi} > 0$, we get: $\mathcal{N}(e^{-\beta\cdot\xi} \cdot \widehat{B}^{\varepsilon-\xi}_\beta) = \mathcal{N}(\widehat{B}^{\varepsilon-\xi}_\beta)$.
Then: $B^\varepsilon_\beta = \mathcal{N}(\widehat{B}^\varepsilon_\beta) = \mathcal{N}(e^{-\beta\cdot\xi} \cdot \widehat{B}^{\varepsilon-\xi}_\beta) = \mathcal{N}(\widehat{B}^{\varepsilon-\xi}_\beta) = B^{\varepsilon-\xi}_\beta.$ $\square$

DEFINITION 21.3. Let Σ *be a nonempty finite set,* $\varepsilon : \Sigma \to \mathbb{R}$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
For all $\beta \in \mathbb{R}$, **let** $\boxed{\Gamma^\varepsilon_\beta} := \sum_{\sigma\in\Sigma}[\varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma}],$
$\boxed{\Delta^\varepsilon_\beta} := \sum_{\sigma\in\Sigma}[e^{-\beta\cdot\varepsilon_\sigma}],$
$\boxed{A^\varepsilon_\beta} := \Gamma^\varepsilon_\beta \,/\, \Delta^\varepsilon_\beta.$

Let Σ be a nonempty finite set, $\varepsilon : \Sigma \to \mathbb{R}$.
Since $\quad \Gamma^\varepsilon_\beta = \sum_{\sigma\in\Sigma}[\varepsilon_\sigma \cdot (\widehat{B}^\varepsilon_\beta\{\sigma\})],$
we get: $\quad \Gamma^\varepsilon_\beta$ is the integral of ε wrt $\widehat{B}^\varepsilon_\beta$.
Since $\quad \Delta^\varepsilon_\beta = \sum_{\sigma\in\Sigma}[\widehat{B}^\varepsilon_\beta\{\sigma\}],$
we get: $\quad \Delta^\varepsilon_\beta = \widehat{B}^\varepsilon_\beta(\Sigma).$

$$\text{Since}\qquad \frac{\Gamma^\varepsilon_\beta}{\Delta^\varepsilon_\beta} \;=\; \frac{\sum_{\sigma\in\Sigma}\left[\varepsilon_\sigma\cdot(\widehat{B}^\varepsilon_\beta\{\sigma\})\right]}{\widehat{B}^\varepsilon_\beta(\Sigma)},$$

we get: $\quad A^\varepsilon_\beta \;=\; \sum_{\sigma\in\Sigma}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right]$.

Then: $\quad A^\varepsilon_\beta$ is the average value of ε wrt B^ε_β.

Recall (§2) the notations $\mathbb{I}_f$, f^*A. Recall (§9) the notation $\varepsilon_*\mu$. Recall (Definition 9.5) the notation M_μ.

THEOREM 21.4. Let Σ *be a nonempty finite set.*
Let $\varepsilon:\Sigma\to\mathbb{R}$, $\beta\in\mathbb{R}$. *Then:* $M_{\varepsilon_* B^\varepsilon_\beta} = A^\varepsilon_\beta$.

Proof. For all $\sigma\in\Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Because Σ is the disjoint union, over $t\in\mathbb{I}_\varepsilon$, of $\varepsilon^*\{t\}$,
we get: $\sum_{t\in\mathbb{I}_\varepsilon}\sum_{\sigma\in\varepsilon^*\{t\}}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right] = \sum_{\sigma\in\Sigma}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right]$.
Also, $\quad A^\varepsilon_\beta = \sum_{\sigma\in\Sigma}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right]$.
Then: $\quad \sum_{t\in\mathbb{I}_\varepsilon}\sum_{\sigma\in\varepsilon^*\{t\}}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right] = A^\varepsilon_\beta$.
So, since $\quad \sum_{t\in\mathbb{I}_\varepsilon}\left[t\cdot((\varepsilon_* B^\varepsilon_\beta)\{t\})\right] = M_{\varepsilon_* B^\varepsilon_\beta}$,
we want: $\quad \sum_{t\in\mathbb{I}_\varepsilon}\left[t\cdot((\varepsilon_* B^\varepsilon_\beta)\{t\})\right] = \sum_{t\in\mathbb{I}_\varepsilon}\sum_{\sigma\in\varepsilon^*\{t\}}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right]$.
Want: $\forall t\in\mathbb{I}_\varepsilon$, $\quad t\cdot((\varepsilon_* B^\varepsilon_\beta)\{t\}) \;=\; \sum_{\sigma\in\varepsilon^*\{t\}}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right]$.
Given $t\in\mathbb{I}_\varepsilon$, **want:** $t\cdot((\varepsilon_* B^\varepsilon_\beta)\{t\}) \;=\; \sum_{\sigma\in\varepsilon^*\{t\}}\left[\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\right]$.
For all $\sigma\in\varepsilon^*\{t\}$, since $\varepsilon_\sigma = \varepsilon(\sigma)\in\{t\}$, we get: $\varepsilon_\sigma = t$.
Want: $t\cdot((\varepsilon_* B^\varepsilon_\beta)\{t\}) \;=\; \sum_{\sigma\in\varepsilon^*\{t\}}\left[\,t\,\cdot(B^\varepsilon_\beta\{\sigma\})\,\right]$.
Because $\varepsilon^*\{t\}$ is the disjoint union, over $\sigma\in\varepsilon^*\{t\}$, of $\{\sigma\}$,
we get: $\quad B^\varepsilon_\beta(\varepsilon^*\{t\}) \;=\; \sum_{\sigma\in\varepsilon^*\{t\}}\left[\quad B^\varepsilon_\beta\{\sigma\}\quad\right]$.
Also, $\quad (\varepsilon_* B^\varepsilon_\beta)\{t\} \;=\; B^\varepsilon_\beta(\varepsilon^*\{t\})$.
Then: $t\cdot((\varepsilon_* B^\varepsilon_\beta)\{t\}) = t\cdot(B^\varepsilon_\beta(\varepsilon^*\{t\})) \;=\; \sum_{\sigma\in\varepsilon^*\{t\}}\left[\,t\;\cdot(B^\varepsilon_\beta\{\sigma\})\right]$. $\square$

THEOREM 21.5. Let Σ *be a nonempty finite set.*
Let $\varepsilon:\Sigma\to\mathbb{R}$, $\beta,\xi\in\mathbb{R}$. *Then:* $A^{\varepsilon-\xi}_\beta = A^\varepsilon_\beta - \xi$.

Proof. We have: $\quad B^\varepsilon_\beta(\Sigma) \;=\; \sum_{\sigma\in\Sigma}\left[\,B^\varepsilon_\beta\{\sigma\}\,\right]$.
Since $B^\varepsilon_\beta\in\mathcal{P}_\Sigma$, we get: $\quad B^\varepsilon_\beta(\Sigma) \;=\; 1$.
By Theorem 21.2, we have: $\quad B^\varepsilon_\beta = B^{\varepsilon-\xi}_\beta$.
For all $\sigma\in\Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.

$$\begin{aligned}
\text{Then:}\quad A^{\varepsilon-\xi}_\beta &= \textstyle\sum_{\sigma\in\Sigma}\left[\,(\varepsilon_\sigma-\xi)\cdot(B^{\varepsilon-\xi}_\beta\{\sigma\})\,\right]\\
&= \textstyle\sum_{\sigma\in\Sigma}\left[\,(\varepsilon_\sigma-\xi)\cdot(B^{\varepsilon}_\beta\{\sigma\})\,\right]\\
&= \left(\textstyle\sum_{\sigma\in\Sigma}\left[\,\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\,\right]\right) - \left(\textstyle\sum_{\sigma\in\Sigma}\left[\,\xi\cdot(B^\varepsilon_\beta\{\sigma\})\,\right]\right)\\
&= \left(\textstyle\sum_{\sigma\in\Sigma}\left[\,\varepsilon_\sigma\cdot(B^\varepsilon_\beta\{\sigma\})\,\right]\right) - \xi\cdot\left(\textstyle\sum_{\sigma\in\Sigma}\left[\,B^\varepsilon_\beta\{\sigma\}\,\right]\right)\\
&= A^\varepsilon_\beta - \xi\cdot(B^\varepsilon_\beta(\Sigma)) \;=\; A^\varepsilon_\beta - \xi\cdot 1 \;=\; A^\varepsilon_\beta - \xi. \qquad\square
\end{aligned}$$

THEOREM 21.6. Let Σ *be a nonempty finite set,* $\varepsilon : \Sigma \to \mathbb{R}$.

Then: *as* $\beta \to \infty$, $A^\varepsilon_\beta \to \min \mathbb{I}_\varepsilon$

and *as* $\beta \to -\infty$, $A^\varepsilon_\beta \to \max \mathbb{I}_\varepsilon$.

The proof is a matter of bookkeeping, best explained by example:
Let $\Sigma := \{1,2,3,4\}$ and **define** $\varepsilon : \Sigma \to \mathbb{R}$ by:
$\varepsilon(1) = 2$, $\varepsilon(2) = 4$, $\varepsilon(3) = 9$, $\varepsilon(4) = 9$.
Then $\mathbb{I}_\varepsilon = \{2,4,9\}$, so $\min \mathbb{I}_\varepsilon = 2$ and $\max \mathbb{I}_\varepsilon = 9$.

Since $\forall\, \beta \in \mathbb{R}$,
$$A^\varepsilon_\beta = \frac{2e^{-2\beta} + 4e^{-4\beta} + 9e^{-9\beta} + 9e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + e^{-9\beta} + e^{-9\beta}}, \quad = \frac{2e^{-2\beta} + 4e^{-4\beta} + 18e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + 2e^{-9\beta}},$$
we get as $\beta \to \infty$, $A^\varepsilon_\beta \to 2/1$
and as $\beta \to -\infty$, $A^\varepsilon_\beta \to 18/2$,
and so as $\beta \to \infty$, $A^\varepsilon_\beta \to \min \mathbb{I}_\varepsilon$
and as $\beta \to -\infty$, $A^\varepsilon_\beta \to \max \mathbb{I}_\varepsilon$.

For any nonempty finite set Σ, for any $\varepsilon : \Sigma \to \mathbb{R}$,
define $\boxed{A^\varepsilon_\bullet} : \mathbb{R} \to \mathbb{R}$ by: $\forall \beta \in \mathbb{R}$, $A^\varepsilon_\bullet(\beta) = A^\varepsilon_\beta$.

Recall (§2): "C^ω" means "real-analytic".

THEOREM 21.7. Let Σ *be a finite set.*
Let $\varepsilon : \Sigma \to \mathbb{R}$. *Assume:* $\#\mathbb{I}_\varepsilon \geqslant 2$.
Then: $A^\varepsilon_\bullet$ *is a strictly-decreasing* C^ω*-diffeomorphism from* $\mathbb{R}$ *onto* $(\min \mathbb{I}_\varepsilon; \max \mathbb{I}_\varepsilon)$.

Proof. For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
We have: $\forall \beta \in \mathbb{R}$, $A^\varepsilon_\bullet(\beta) = \dfrac{\sum_{\sigma\in\Sigma} [\varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma}]}{\sum_{\tau\in\Sigma} [e^{-\beta\cdot\varepsilon_\tau}]}$. Then $A^\varepsilon_\bullet : \mathbb{R} \to \mathbb{R}$ is C^ω.
So, by Theorem 21.6 and the C^ω-Inverse Function Theorem and the Mean Value Theorem, **it suffices to show:** $(A^\varepsilon_\bullet)' < 0$ on $\mathbb{R}$.
Given $\beta \in \mathbb{R}$, **want:** $(A^\varepsilon_\bullet)'(\beta) < 0$.
Let $P := \sum_{\sigma\in\Sigma} [\, \varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma} \,]$, $P' := \sum_{\sigma\in\Sigma} [\, (-\varepsilon_\sigma^2) \cdot e^{-\beta\cdot\varepsilon_\sigma} \,]$.
Let $Q := \sum_{\tau\in\Sigma} [\, e^{-\beta\cdot\varepsilon_\tau} \,]$, $Q' := \sum_{\tau\in\Sigma} [\, (-\varepsilon_\tau) \cdot e^{-\beta\cdot\varepsilon_\tau} \,]$.
Then $Q > 0$. Also, by the Quotient Rule, $(A^\varepsilon_\bullet)'(\beta) = [QP' - PQ']/Q^2$.
Want: $QP' - PQ' < 0$.
We have: $QP' = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (-\varepsilon_\sigma^2 \qquad) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.
We have: $PQ' = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (\qquad - \varepsilon_\sigma\varepsilon_\tau) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.
Then: $QP' - PQ' = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (-\varepsilon_\sigma^2 + \varepsilon_\sigma\varepsilon_\tau) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.

Interchanging σ and τ, we get:

$$QP' - PQ' = \sum_{\tau\in\Sigma} \sum_{\sigma\in\Sigma} [\,(-\varepsilon_\tau^2 + \varepsilon_\tau\varepsilon_\sigma) \cdot e^{-\beta\cdot(\varepsilon_\tau+\varepsilon_\sigma)}\,].$$

By commutativity of addition and multiplication,
adding the last two equations gives:

$$2\cdot(QP' - PQ') = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\,(-\varepsilon_\sigma^2 - \varepsilon_\tau^2 + 2\varepsilon_\sigma\varepsilon_\tau) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)}\,].$$

Then: $2\cdot(QP' - PQ') = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\quad -(\varepsilon_\sigma - \varepsilon_\tau)^2 \quad \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)}\,]$.

Then: $2\cdot(QP' - PQ') < 0$. Then: $QP' - PQ' < 0$. $\square$

DEFINITION 21.8. Let Σ *be a finite set.* **Let** $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\#\mathbb{I}_\varepsilon \geqslant 2$. **Let** $\alpha \in (\min \mathbb{I}_\varepsilon; \max \mathbb{I}_\varepsilon)$.
The $\boxed{\alpha\textbf{-Boltzmann-parameter } \textit{on } \varepsilon}$ *is:* $\boxed{\mathrm{BP}_\alpha^\varepsilon} := (A_\bullet^\varepsilon)^{-1}(\alpha)$.

So the α-Boltzmann-parameter on ε is the unique $\beta \in \mathbb{R}$ s.t. $A_\beta^\varepsilon = \alpha$.

Example: **Let** $\Sigma := \{0, 1, 10\}$ and **define** $\varepsilon : \Sigma \to \mathbb{R}$ by:
$\forall \sigma \in \Sigma,\ \ \varepsilon(\sigma) = \sigma$.
Computation shows: $A_{(\ln 9)/10}^\varepsilon = 1$. Then: $\mathrm{BP}_1^\varepsilon = (\ln 9)/10$.

Example: **Let** $\Sigma := \{2, 4, 8, 9\}$ and **define** $\varepsilon : \Sigma \to \mathbb{R}$ by:
$\forall \sigma \in \Sigma,\ \ \varepsilon(\sigma) = \sigma$.
To evaluate $\mathrm{BP}_5^\varepsilon$, we must solve $A_\bullet^\varepsilon(\beta) = 5$ for β,
and, since, by Theorem 21.7, $A_\bullet^\varepsilon$ is strictly-decreasing,
there are simple iterative methods to do this.
We compute: $\mathrm{BP}_5^\varepsilon \approx 0.0918$, accurate to four decimal places.
(Thanks to C. Prouty for this calculation. See §28.)
Next, **let** $\overline{\Sigma} := \{1, 2, 3, 4\}$ and **define** $\overline{\varepsilon} : \overline{\Sigma} \to \mathbb{R}$ by:
$\overline{\varepsilon}(1) = 2,\quad \overline{\varepsilon}(2) = 4,\quad \overline{\varepsilon}(3) = 8,\quad \overline{\varepsilon}(4) = 9$.
Then $A_\bullet^{\overline{\varepsilon}} = A_\bullet^\varepsilon$, so $\mathrm{BP}_5^{\overline{\varepsilon}} = \mathrm{BP}_5^\varepsilon$.
Then $\mathrm{BP}_5^{\overline{\varepsilon}} \approx 0.0918$, accurate to four decimal places.

Example: **Let** $\Sigma := \{1, 2, 3, 4\}$ and **define** $\varepsilon : \Sigma \to \mathbb{R}$ by:
$\varepsilon(1) = 2,\quad \varepsilon(2) = 4,\quad \varepsilon(3) = 9,\quad \varepsilon(4) = 9$.
To evaluate $\mathrm{BP}_5^\varepsilon$, we must solve $A_\bullet^\varepsilon(\beta) = 5$ for β,
and, since, by Theorem 21.7, $A_\bullet^\varepsilon$ is strictly-decreasing,
there are simple iterative methods to do this.
We compute: $\mathrm{BP}_5^\varepsilon \approx 0.1060$, accurate to four decimal places.
(Thanks to C. Prouty for this calculation. See §28.)

Example: **Let** $\Sigma := (\,[0..4] \times [0..4]\,) \setminus \{\,(4, 4)\,\}$.

Define $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma_1 + \sigma_2$.
To evaluate $\mathrm{BP}^\varepsilon_1$, we must solve $A^\varepsilon_\bullet(\beta) = 1$ for β,
and, since, by Theorem 21.7, $A^\varepsilon_\bullet$ is strictly-decreasing,
there are simple iterative methods to do this.
We compute: $\mathrm{BP}^\varepsilon_1 \approx 1.0670$, accurate to four decimal places.
(Thanks to C. Prouty for this calculation. See §28.)

THEOREM 21.9. Let Σ *be a finite set.*
Let $\varepsilon : \Sigma \to \mathbb{R}$. *Assume:* $\#\mathbb{I}_\varepsilon \geqslant 2$.
Let $\alpha \in (\min \mathbb{I}_\varepsilon ; \max \mathbb{I}_\varepsilon)$. **Let** $\xi \in \mathbb{R}$. *Then:* $\mathrm{BP}^{\varepsilon-\xi}_{\alpha-\xi} = \mathrm{BP}^\varepsilon_\alpha$.

Proof. **Let** $\beta := \mathrm{BP}^\varepsilon_\alpha$. **Want:** $\mathrm{BP}^{\varepsilon-\xi}_{\alpha-\xi} = \beta$.
Since $\beta = \mathrm{BP}^\varepsilon_\alpha = (A^\varepsilon_\bullet)^{-1}(\alpha)$, we get: $(A^\varepsilon_\bullet)(\beta) = \alpha$.
By Theorem 21.5, $A^{\varepsilon-\xi}_\beta = A^\varepsilon_\beta - \xi$.
Since $(A^{\varepsilon-\xi}_\bullet)(\beta) = A^{\varepsilon-\xi}_\beta = A^\varepsilon_\beta - \xi = ((A^\varepsilon_\bullet)(\beta)) - \xi = \alpha - \xi$,
we get: $\beta = (A^{\varepsilon-\xi}_\bullet)^{-1}(\alpha - \xi)$.
So, since $\mathrm{BP}^{\varepsilon-\xi}_{\alpha-\xi} = (A^{\varepsilon-\xi}_\bullet)^{-1}(\alpha - \xi)$, we get: $\mathrm{BP}^{\varepsilon-\xi}_{\alpha-\xi} = \beta$. □

22. Degenerate energy levels

Recall (§2): the notations $\mathbb{I}_f$ and f^*A.
Recall (§9): the notation ν_F.

THEOREM 22.1. Let Σ *be a finite set.*
Let $\varepsilon : \Sigma \to \mathbb{Z}$. *Assume* $\mathbb{I}_\varepsilon$ *is residue-unconstrained.*
Let $\alpha \in (\min \mathbb{I}_\varepsilon ; \max \mathbb{I}_\varepsilon)$. **Let** $\beta := \mathrm{BP}^\varepsilon_\alpha$.
Let $t_1, t_2, \ldots \in \mathbb{Z}$. *Assume:* $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ *is bounded.*
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in \Sigma^n \,|\, (\varepsilon(f_1)) + \cdots + (\varepsilon(f_n)) = t_n\}$.
Let $\sigma_0 \in \Sigma$. *Then: as* $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \sigma_0\} \to B^\varepsilon_\beta\{\sigma_0\}$.

Recall (§9): $\nu_\varnothing(\varnothing) = -1$.
So, since $B^\varepsilon_\beta\{\sigma_0\} > 0$, part of the content of Theorem 22.1 is:
$\forall$sufficiently large $n \in \mathbb{N}$, $\Omega_n \neq \varnothing$;
see Claim 1 in the proof below.

Proof. Since $\mathbb{I}_\varepsilon$ is residue-unconstrained, we get: $\mathbb{I}_\varepsilon \neq \varnothing$.
So, since $\varepsilon : \Sigma \to \mathbb{Z}$, we conclude: $\Sigma \neq \varnothing$.
By hypothesis, Σ is finite. Then: Σ is a nonempty finite set.
Since $\beta = \mathrm{BP}^\varepsilon_\alpha = (A^\varepsilon_\bullet)^{-1}(\alpha)$, we get: $A^\varepsilon_\bullet(\beta) = \alpha$.
By Theorem 21.4, we have: $M_{\varepsilon_* B^\varepsilon_\beta} = A^\varepsilon_\beta$.
So, since $A^\varepsilon_\beta = A^\varepsilon_\bullet(\beta) = \alpha$, we get: $M_{\varepsilon_* B^\varepsilon_\beta} = \alpha$.

Let $\mu := B_\beta^\varepsilon$. Then: $\mu \in \mathcal{P}_\Sigma$ and $M_{\varepsilon_*\mu} = \alpha$.
Let $E := \mathbb{I}_\varepsilon$, $\widetilde{\mu} := \varepsilon_*\mu$. Then: $\widetilde{\mu} \in \mathcal{P}_E$ and $M_{\widetilde{\mu}} = \alpha$.
By hypothesis, E is residue-unconstrained.
Since $\varepsilon : \Sigma \to \mathbb{Z}$, we get: $E \subseteq \mathbb{Z}$.
Since Σ is finite, we get: E is finite.
So, since $\widetilde{\mu} \in \mathcal{P}_E \subseteq \mathcal{FM}_E$, we get: $|\widetilde{\mu}|_1 < \infty$ and $|\widetilde{\mu}|_2 < \infty$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Then: $\forall n \in \mathbb{N}$, $\Omega_n = \{f \in \Sigma^n \,|\, \varepsilon_{f_1} + \cdots + \varepsilon_{f_n} = t_n\}$.
For all $n \in \mathbb{N}$, **define** $\varepsilon^n : \Sigma^n \to E^n$ by:

$$\forall f_1, \ldots, f_n \in \Sigma, \quad \varepsilon^n(f_1, \ldots, f_n) = (\varepsilon_{f_1}, \ldots, \varepsilon_{f_n}).$$

Then, since $\varepsilon_*\mu = \widetilde{\mu}$, we get: $\forall n \in \mathbb{N}$, $(\varepsilon^n)_*(\mu^n) = \widetilde{\mu}^n$.
For all $n \in \mathbb{N}$, **let** $\widetilde{\Omega}_n := \{\widetilde{f} \in E^n \,|\, \widetilde{f}_1 + \cdots + \widetilde{f}_n = t_n\}$;
then $(\varepsilon^n)^*\widetilde{\Omega}_n = \Omega_n$.
Then: $\forall n \in \mathbb{N}$, $\mu^n((\varepsilon^n)^*\widetilde{\Omega}_n) = \mu^n(\Omega_n)$.
Then: $\forall n \in \mathbb{N}$, $((\varepsilon^n)_*\mu^n)(\widetilde{\Omega}_n) = \mu^n(\Omega_n)$.
Then: $\forall n \in \mathbb{N}$, $\widetilde{\mu}^n(\widetilde{\Omega}_n) = \mu^n(\Omega_n)$.
For all $n \in \mathbb{N}$, **define** $\psi_n : \mathbb{Z} \to \mathbb{R}$ by:

$$\forall t \in \mathbb{Z}, \quad \psi_n(t) = \widetilde{\mu}^n\{\widetilde{f} \in E^n \,|\, \widetilde{f}_1 + \cdots + \widetilde{f}_n = t\}.$$

Then: $\forall n \in \mathbb{N}$, $\psi_n(t_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n)$.
Since E is finite and residue-unconstrained, we get: $2 \leqslant \#E < \infty$.
Since $\varepsilon : \Sigma \to \mathbb{Z}$, we get: $S_{B_\beta^\varepsilon} = \Sigma$.
So, since $\mu = B_\beta^\varepsilon$, we get: $S_\mu = \Sigma$.
So, since $\varepsilon : \Sigma \to \mathbb{Z}$, we get: $S_{\varepsilon_*\mu} = \mathbb{I}_\varepsilon$.
So, since $\varepsilon_*\mu = \widetilde{\mu}$ and $\mathbb{I}_\varepsilon = E$, we get: $S_{\widetilde{\mu}} = E$.
Since E is finite, we get: E is countable.
Let $v := V_{\widetilde{\mu}}$. By Theorem 10.6, we get: $0 < v < \infty$.
Let $\tau := 1/\sqrt{2\pi v}$. Then: $0 < \tau < \infty$.
By Theorem 10.6, we get:

$$\text{as } n \to \infty, \quad \sqrt{n} \cdot (\mu^n\{\widetilde{f} \in E^n \,|\, \widetilde{f}_1 + \cdots + \widetilde{f}_n = t_n\}) \to 1/\sqrt{2\pi v}.$$

Then: as $n \to \infty$, $\sqrt{n} \cdot (\quad \psi_n(t_n) \quad) \to \tau$.
So, since $\tau > 0$, **choose** $n_0 \in [2..\infty)$ such that:

$$\forall n \in [n_0..\infty), \quad \sqrt{n} \cdot (\psi_n(t_n)) > 0.$$

Claim 1: **Let** $n \in [n_0..\infty)$. Then: $\mu^n(\Omega_n) > 0$.
Proof of Claim 1: Recall: $\widetilde{\mu}^n(\widetilde{\Omega}_n) = \mu^n(\Omega_n)$ and $\psi_n(t_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n)$.
By the choice of n_0, we get: $\sqrt{n} \cdot (\psi_n(t_n)) > 0$. Then: $\psi_n(t_n) > 0$.
Then: $\mu^n(\Omega_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n) = \psi_n(t_n) > 0$.
End of proof of Claim 1.

Recall: $\Sigma \neq \varnothing$ and $\varepsilon : \Sigma \to \mathbb{Z}$. Then $\widehat{B}^{\varepsilon}_{\beta}(\Sigma) > 0$.
Let $C := 1/(\widehat{B}^{\varepsilon}_{\beta}(\Sigma))$. Then $\mathcal{N}(\widehat{B}^{\varepsilon}_{\beta}) = C \cdot \widehat{B}^{\varepsilon}_{\beta}$
By definition of $\widehat{B}^{\varepsilon}_{\beta}$, we have: $\forall \sigma \in \Sigma$, $\widehat{B}^{\varepsilon}_{\beta}\{\sigma\} = e^{-\beta \cdot \varepsilon_\sigma}$.
So, since $\mu = B^{\varepsilon}_{\beta} = \mathcal{N}(\widehat{B}^{\varepsilon}_{\beta}) = C \cdot \widehat{B}^{\varepsilon}_{\beta}$,
we get: $\forall \sigma \in \Sigma$, $\mu\{\sigma\} = Ce^{-\beta \cdot \varepsilon_\sigma}$.
Since $\mu \in \mathcal{P}_\Sigma$, we get: $\forall n \in \mathbb{N}$, $\mu^n \in \mathcal{P}_{\Sigma^n}$, so $\mu^n(\Omega_n) \leqslant 1$.
So, by Claim 1, $\forall n \in [n_0..\infty)$, $0 < \mu^n(\Omega_n) \leqslant 1$.
Also, we have: $\forall n \in \mathbb{N}$, $(\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n)$.
Then: $\forall n \in [n_0..\infty)$, $0 < (\mu^n|\Omega_n)(\Omega_n) \leqslant 1$.
Then: $\forall n \in [n_0..\infty)$, $\mu^n \,|\, \Omega_n \in \mathcal{FM}^{\times}_{\Omega_n}$.
Then: $\forall n \in [n_0..\infty)$, $\mathcal{N}(\mu^n \,|\, \Omega_n) \in \mathcal{P}_{\Omega_n}$.
Also, $\forall n \in \mathbb{N}, \forall S \subseteq \Omega_n$, $(\mu^n|\Omega_n)(\,S\,) = \mu^n(\,S\,)$.
Then: $\forall n \in \mathbb{N}$, $(\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n)$.
For all $n \in \mathbb{N}$, **let** $z_n := \mu^n(\Omega_n)$.
Then: $\forall n \in [n_0..\infty)$, $(\mu^n|\Omega_n)(\Omega_n) = z_n$ and $0 < z_n \leqslant 1$.
For all $n \in [n_0..\infty)$, **let** $\lambda_n := \mathcal{N}(\mu^n|\Omega_n)$.
Then: $\forall n \in [n_0..\infty)$, $\lambda_n = (\mu^n|\Omega_n)/z_n$.
Then: $\forall n \in [n_0..\infty)$, $\forall S \subseteq \Omega_n$, $\lambda_n(S) = (\mu^n(S))/z_n$.

Claim 2: **Let** $n \in [n_0..\infty)$. Then: $\lambda_n = \nu_{\Omega_n}$.
Proof of Claim 2: **Let** $F := \Omega_n$. **Want:** $\lambda_n = \nu_F$.
Since $\lambda_n = \mathcal{N}(\mu^n|\Omega_n) = \mathcal{N}(\mu^n|F)$, we get: $\lambda_n \in \mathcal{P}_F$.
By Theorem 9.9, **given** $f, g \in F$, **want:** $\lambda_n\{f\} = \lambda_n\{g\}$.
Want: $(\mu^n\{f\})/z_n = (\mu^n\{g\})/z_n$. **Want:** $\mu^n\{f\} = \mu^n\{g\}$.
For all $i \in [1..n]$, **let** $\widetilde{f}_i := \varepsilon_{f_i}$ and $\widetilde{g}_i := \varepsilon_{g_i}$.
Recall: $\forall \sigma \in \Sigma$, $\mu\{\sigma\} = Ce^{-\beta \cdot \varepsilon_\sigma}$.
Then: $\forall i \in [1..n]$, $\mu\{f_i\} = Ce^{-\beta \cdot \widetilde{f}_i}$ and $\mu\{g_i\} = Ce^{-\beta \cdot \widetilde{g}_i}$.
Since $f \in F = \Omega_n$, we get: $\varepsilon_{f_1} + \cdots + \varepsilon_{f_n} = t_n$.
Since $g \in F = \Omega_n$, we get: $\varepsilon_{g_1} + \cdots + \varepsilon_{g_n} = t_n$.
Since $\widetilde{f}_1 + \cdots + \widetilde{f}_n = \varepsilon_{f_1} + \cdots + \varepsilon_{f_n} = t_n$
$= \varepsilon_{g_1} + \cdots + \varepsilon_{g_n} = \widetilde{g}_1 + \cdots + \widetilde{g}_n$,
we get: $C^n e^{-\beta \cdot (\widetilde{f}_1 + \cdots + \widetilde{f}_n)} = C^n e^{-\beta \cdot (\widetilde{g}_1 + \cdots + \widetilde{g}_n)}$.
Then: $(Ce^{-\beta \cdot \widetilde{f}_1}) \cdots (Ce^{-\beta \cdot \widetilde{f}_n}) = (Ce^{-\beta \cdot \widetilde{g}_1}) \cdots (Ce^{-\beta \cdot \widetilde{g}_n})$.
Then: $(\ \mu\{f_1\}\ ) \cdots (\ \mu\{f_n\}\ ) = (\ \mu\{g_1\}\ ) \cdots (\ \mu\{g_n\}\ )$.
Then: $\mu^n\{f\} = \mu^n\{g\}$.
End of proof of Claim 2.

Claim 3: **Let** $\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$. Then: $\mu\{\sigma\} = \mu\{\sigma_0\}$.
Proof of Claim 3: Since $\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$, we get: $\varepsilon(\sigma) \in \{\varepsilon_{\sigma_0}\}$.
Since $\varepsilon_\sigma = \varepsilon(\sigma) \in \{\varepsilon_{\sigma_0}\}$, we get: $\varepsilon_\sigma = \varepsilon_{\sigma_0}$.
Then: $\mu\{\sigma\} = Ce^{-\beta\cdot\varepsilon_\sigma} = Ce^{-\beta\cdot\varepsilon_{\sigma_0}} = \mu\{\sigma_0\}$.
End of proof of Claim 3.

Since $\varepsilon(\sigma_0) = \varepsilon_{\sigma_0} \in \{\varepsilon_{\sigma_0}\}$, we get: $\sigma_0 \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$.
Then $\varepsilon^*\{\varepsilon_{\sigma_0}\} \neq \varnothing$, so $\#(\varepsilon^*\{\varepsilon_{\sigma_0}\}) \geqslant 1$.
Let $k := \#(\varepsilon^*\{\varepsilon_{\sigma_0}\})$. Then: $k \geqslant 1$.

Claim 4: $\mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\})$.
Proof of Claim 4: Since $\varepsilon^*\{\varepsilon_{\sigma_0}\}$ is equal to
the disjoint union, over $\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$, of $\{\,\sigma\,\}$,
we get: $\mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = \sum_{\sigma\in\varepsilon^*\{\varepsilon_{\sigma_0}\}} [\mu\{\,\sigma\,\}]$,
So, by Claim 3, we get: $\mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = \sum_{\sigma\in\varepsilon^*\{\varepsilon_{\sigma_0}\}} [\mu\{\sigma_0\}]$.
So, since $k = \#(\varepsilon^*\{\varepsilon_{\sigma_0}\})$, we get: $\mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\})$.
End of proof of Claim 4.

Claim 5: **Let** $n \in [2..\infty)$. **Let** $\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$.
Then: $\mu^n\{f \in \Omega_n \,|\, f_n = \sigma\} = \mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\}$.
Proof of Claim 5:
Let $X := \{f \in \Sigma^{n-1} \,|\, \varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_\sigma\}$.
Recall: $\Omega_n = \{f \in \Sigma^n \,|\, \varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} + \varepsilon_{f_n} = t_n\}$.
Since
$$\begin{aligned}
&\{f \in \Omega_n \,|\, f_n = \sigma\,\}\\
&= \{f \in \Sigma^n \,|\, [\varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} + \varepsilon_{f_n} = t_n] \,\&\, [f_n = \sigma]\}\\
&= \{f \in \Sigma^n \,|\, [\varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} + \varepsilon_\sigma = t_n] \,\&\, [f_n = \sigma]\}\\
&= \{f \in \Sigma^n \,|\, [\varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_\sigma\,] \,\&\, [f_n = \sigma]\},
\end{aligned}$$
it follows that, under the standard bijection $\Sigma^n \leftrightarrow \Sigma^{n-1} \times \Sigma$, we have:
$$\{f \in \Omega_n \,|\, f_n = \sigma\} \subseteq \Sigma^n$$
corresponds to
$$X \times \{\sigma\} \subseteq \Sigma^{n-1} \times \Sigma.$$
Then: $\mu^n\{f \in \Omega_n \,|\, f_n = \sigma\,\} = (\mu^{n-1}(X)) \cdot (\mu\{\sigma\})$.
Want: $\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\} = (\mu^{n-1}(X)) \cdot (\mu\{\sigma\})$.
By Claim 3, we have: $\mu\{\sigma\} = \mu\{\sigma_0\}$.
Want: $\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\} = (\mu^{n-1}(X)) \cdot (\mu\{\sigma_0\})$.
Since $\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$, we get: $\varepsilon(\sigma) \in \{\varepsilon_{\sigma_0}\}$.
Since $\varepsilon_\sigma = \varepsilon(\sigma) \in \{\varepsilon_{\sigma_0}\}$, we get: $\varepsilon_\sigma = \varepsilon_{\sigma_0}$.
Then $X = \{f \in \Sigma^{n-1} \,|\, \varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_{\sigma_0}\}$.

Since $\{f \in \Omega_n \quad | \quad f_n = \sigma_0 \ \}$
$= \{f \in \Sigma^n \quad | \, [\varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} + \varepsilon_{f_n} = t_n] \,\&\, [f_n = \sigma_0]\}$
$= \{f \in \Sigma^n \quad | \, [\varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} + \varepsilon_{\sigma_0} = t_n] \,\&\, [f_n = \sigma_0]\}$
$= \{f \in \Sigma^n \quad | \, [\varepsilon_{f_1} + \cdots + \varepsilon_{f_{n-1}} = t_n - \varepsilon_{\sigma_0}] \,\&\, [f_n = \sigma_0]\}$,
it follows that, under the standard bijection $\Sigma^n \leftrightarrow \Sigma^{n-1} \times \Sigma$, we have:
$\{f \in \Omega_n \,|\, f_n = \sigma_0\} \quad \subseteq \quad \Sigma^n$
corresponds to $\quad X \ \times \ \{\sigma_0\} \quad \subseteq \quad \Sigma^{n-1} \times \Sigma$.
Then: $\quad \mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\} = (\mu^{n-1}(X)) \cdot (\mu\{\sigma_0\})$.
End of proof of Claim 5.

Claim 6: **Let** $n \in [2..\infty)$.
Then: $\quad \widetilde{\mu}^n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} \ = \ k \cdot (\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\})$.
Proof of Claim 6: Recall: $(\varepsilon^n)^*\widetilde{\Omega}_n = \Omega_n$.
Then $\quad (\varepsilon^n)^*\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} \ = \ \{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\}$,
and so $\ \mu^n((\varepsilon^n)^*\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\}) = \mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\}$.
Then: $\quad ((\varepsilon^n)_*(\mu^n))\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} \ = \mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\}$.
Recall: $\quad (\varepsilon^n)_*(\mu^n) = \widetilde{\mu}^n$.
Then: $\quad \widetilde{\mu}^n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} \ = \mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\}$.
Want: $\mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\} \ = \ k \cdot (\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\})$.
Since $\quad \{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\}$
is the disjoint union, over $\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}$, of
$\{f \in \Omega_n \,|\, f_n = \sigma\}$,
we get: $\mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\} \ = \ \sum_{\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}} [\mu^n\{f \in \Omega_n \,|\, f_n = \sigma \ \}]$.
Then, by Claim 5, we conclude:
$\mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\} \ = \ \sum_{\sigma \in \varepsilon^*\{\varepsilon_{\sigma_0}\}} [\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\}]$.
So, since $k = \#(\varepsilon^*\{\varepsilon_{\sigma_0}\})$, we get:
$\mu^n\{f \in \Omega_n \,|\, f_n \in \varepsilon^*\{\varepsilon_{\sigma_0}\}\} \ = \quad k \quad \cdot \quad (\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\})$.
End of proof of Claim 6.

Recall: $\forall n \in \mathbb{N}, \qquad \mu^n(\Omega_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n)$.
Recall: $\forall n \in [n_0..\infty), \qquad 0 < \mu^n(\Omega_n) \leqslant 1$.
Then: $\forall n \in [n_0..\infty), \qquad 0 < \widetilde{\mu}^n(\widetilde{\Omega}_n) \leqslant 1$.
Also, $\forall n \in \mathbb{N}, \forall S \subseteq \widetilde{\Omega}_n, \qquad (\widetilde{\mu}^n|\widetilde{\Omega}_n)(\ S\ ) \ = \ \widetilde{\mu}^n(\ S\ )$.
Then: $\forall n \in \mathbb{N}, \qquad (\widetilde{\mu}^n|\widetilde{\Omega}_n)(\ \widetilde{\Omega}_n) \ = \ \widetilde{\mu}^n(\ \widetilde{\Omega}_n)$.
By dividing the last two equations, we get:
$\forall n \in [n_0..\infty), \forall S \subseteq \widetilde{\Omega}_n, \ (\mathcal{N}(\widetilde{\mu}^n|\widetilde{\Omega}_n))(S) = (\widetilde{\mu}^n(S))/(\widetilde{\mu}^n(\widetilde{\Omega}_n))$.
For all $\quad n \in [n_0..\infty), \quad$ **let** $\widetilde{\lambda}_n := \mathcal{N}(\widetilde{\mu}^n|\widetilde{\Omega}_n)$.

Then: $\quad \forall n \in [n_0..\infty),\ \forall S \subseteq \widetilde{\Omega}_n, \qquad \widetilde{\lambda}_n(S) = (\widetilde{\mu}^n(S))/(\widetilde{\mu}^n(\widetilde{\Omega}_n))$.
So, since $\forall n \in \mathbb{N}, \quad z_n = \mu^n(\Omega_n) = \widetilde{\mu}^n(\widetilde{\Omega}_n)$, we get:
$\forall n \in [n_0..\infty),\ \forall S \subseteq \widetilde{\Omega}_n, \qquad \widetilde{\lambda}_n(S) = (\widetilde{\mu}^n(S))/z_n$.
Recall: $\quad \forall n \in [n_0..\infty), \qquad \lambda_n = \mathcal{N}(\mu^n|\Omega_n)$.
Recall: $\quad \forall n \in [n_0..\infty),\ \forall S \subseteq \Omega_n, \qquad \lambda_n(S) = (\mu^n(S))/z_n$.

Claim 7: **Let** $\quad n \in [n_0..\infty)$.
Then: $\quad \widetilde{\lambda}_n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} = k \cdot (\lambda_n\{f \in \Omega_n \,|\, f_n = \sigma_0\})$.
Proof of Claim 7: By choice of n_0, we have: $\quad n_0 \in [2..\infty)$.
Then $[n_0..\infty) \subseteq [2..\infty)$, so, since $n \in [n_0..\infty)$, we get: $\quad n \in [2..\infty)$.
Then, by Claim 6, $\quad \widetilde{\mu}^n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} = k \cdot (\mu^n\{f \in \Omega_n \,|\, f_n = \sigma_0\})$.
Dividing this last equation by z_n yields
$$\widetilde{\lambda}_n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} = k \cdot (\lambda_n\{f \in \Omega_n \,|\, f_n = \sigma_0\}).$$
End of proof of Claim 7.

Let $\quad P := \mu\{\sigma_0\}$ and $\widetilde{P} := \widetilde{\mu}\{\varepsilon_{\sigma_0}\}$. $\qquad$ Recall: $k \geqslant 1$.
By Claim 4, we have: $\qquad \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\})$.
Recall: $\quad \widetilde{\mu} = \varepsilon_*\mu$.
Since $\widetilde{P} = \widetilde{\mu}\{\varepsilon_{\sigma_0}\} = (\varepsilon_*\mu)\{\varepsilon_{\sigma_0}\} = \mu(\varepsilon^*\{\varepsilon_{\sigma_0}\}) = k \cdot (\mu\{\sigma_0\}) = k \cdot P$,
we get: $\quad \widetilde{P}/k = P$.
Recall: $\quad M_{\widetilde{\mu}} = \alpha$ and $\widetilde{\mu} \in \mathcal{P}_E$ and $S_{\widetilde{\mu}} = E$.
Recall: $\quad E$ is residue-unconstrained and $|\widetilde{\mu}|_2 < \infty$.
Since $\varepsilon_{\sigma_0} = \varepsilon(\sigma_0) \in \mathbb{I}_\varepsilon = E$, we get: $\quad \varepsilon_{\sigma_0} \in E$.
Let $\widetilde{\varepsilon}_0 := \varepsilon_{\sigma_0}$. $\qquad$ Then: $\quad \widetilde{\varepsilon}_0 \in E$ and $\widetilde{P} = \widetilde{\mu}\{\widetilde{\varepsilon}_0\}$.
Recall: $\quad \forall n \in \mathbb{N},\ \widetilde{\Omega}_n := \{\widetilde{f} \in E^n \,|\, \widetilde{f}_1 + \cdots + \widetilde{f}_n = t_n\}$.
By hypothesis, $\quad t_1, t_2, \ldots \in \mathbb{Z}$ and $\{t_n - n\alpha \,|\, n \in \mathbb{N}\}$ is bounded.
By Theorem 12.2, as $n \to \infty$, $\mathcal{N}(\widetilde{\mu}^n|\widetilde{\Omega}_n)\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \widetilde{\varepsilon}_0\} \to \widetilde{P}$.
Recall: $\quad \forall n \in [n_0..\infty), \quad \widetilde{\lambda}_n = \mathcal{N}(\widetilde{\mu}^n|\widetilde{\Omega}_n)$.
Then: $\quad$ as $n \to \infty$, $\quad \widetilde{\lambda}_n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \widetilde{\varepsilon}_0\} \to \widetilde{P}$.
Then: $\quad$ as $n \to \infty$, $\quad \widetilde{\lambda}_n\{\widetilde{f} \in \widetilde{\Omega}_n \,|\, \widetilde{f}_n = \varepsilon_{\sigma_0}\} \to \widetilde{P}$.
So, by Claim 7, as $n \to \infty$, $k \cdot (\lambda_n\{f \in \Omega_n \,|\, f_n = \sigma_0\}) \to \widetilde{P}$.
Then: $\quad$ as $n \to \infty$, $\quad \lambda_n\{f \in \Omega_n \,|\, f_n = \sigma_0\} \to \widetilde{P}/k$.
So, by Claim 2, as $n \to \infty$, $\quad \nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \sigma_0\} \to \widetilde{P}/k$.
Recall: $\qquad \mu = B_\beta^\varepsilon$.
Then, since $\widetilde{P}/k = P = \mu\{\sigma_0\} = B_\beta^\varepsilon\{\sigma_0\}$, we get:
$$\text{as } n \to \infty, \qquad \nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \sigma_0\} \to B_\beta^\varepsilon\{\sigma_0\}. \qquad \square$$

The possibility of degeneracy at $\widetilde{\varepsilon}_0$ (*i.e.*, the possibility that $k \neq 1$) causes a number of complications in the preceding proof.
Here is another approach to proving Theorem 22.1:
By density of the set of injective functions $\Sigma \to \mathbb{R}$ in the topological space of all functions $\Sigma \to \mathbb{R}$, we reduce to the case where ε is injective.
Then the proof can follow the proof of Theorem 17.1, avoiding the degeneracy complications in the preceding proof.

Recall (§2): $\forall t \in \mathbb{R}$, $\lfloor t \rfloor$ is the floor of t.
Next, we record the $t_n = \lfloor n\alpha \rfloor$ version of the preceding theorem:

THEOREM 22.2. **Let** Σ *be a finite set.*
Let $\varepsilon : \Sigma \to \mathbb{Z}$. *Assume* $\mathbb{I}_\varepsilon$ *is residue-unconstrained.*
Let $\alpha \in (\min \mathbb{I}_\varepsilon; \max \mathbb{I}_\varepsilon)$. **Let** $\beta := \mathrm{BP}^\varepsilon_\alpha$.
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in \Sigma^n \,|\, (\varepsilon(f_1)) + \cdots + (\varepsilon(f_n)) = \lfloor n\alpha \rfloor\}$.
Let $\sigma_0 \in \Sigma$. *Then: as* $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \sigma_0\} \to B^\varepsilon_\beta\{\sigma_0\}$.

We record the $\alpha \in \mathbb{Z}$ special case of the preceding theorem:

THEOREM 22.3. **Let** Σ *be a finite set.*
Let $\varepsilon : \Sigma \to \mathbb{Z}$. *Assume* $\mathbb{I}_\varepsilon$ *is residue-unconstrained.*
Let $\alpha \in (\min \mathbb{I}_\varepsilon; \max \mathbb{I}_\varepsilon)$. *Assume* $\alpha \in \mathbb{Z}$. **Let** $\beta := \mathrm{BP}^\varepsilon_\alpha$.
For all $n \in \mathbb{N}$, **let** $\Omega_n := \{f \in \Sigma^n \,|\, (\varepsilon(f_1)) + \cdots + (\varepsilon(f_n)) = n\alpha\}$.
Let $\sigma_0 \in \Sigma$. *Then: as* $n \to \infty$, $\nu_{\Omega_n}\{f \in \Omega_n \,|\, f_n = \sigma_0\} \to B^\varepsilon_\beta\{\sigma_0\}$.

Example: Suppose $\Sigma = \{0, 1, 10\}$ and $\alpha = 1$.
Suppose, also, $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$.
Then Ω_N represents the set of all GFA dispensations to the N professors.
Since ν_{Ω_N} gives equal probability to each dispensation, ν_{Ω_N} represents the GFA's first system for awarding grants.
Since $\beta = \mathrm{BP}^\varepsilon_\alpha = \mathrm{BP}^\varepsilon_1$, we calculate: $\beta = (\ln 9)/10$.
More calculation gives: $(B^\varepsilon_\beta\{0\}, B^\varepsilon_\beta\{1\}, B^\varepsilon_\beta\{10\}) = \dfrac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$.
Since N is large, by Theorem 22.3, we get:

$$\nu_{\Omega_N}\{f \in \Omega_N \,|\, f_N = \sigma_0\} \approx B^\varepsilon_\beta\{\sigma_0\}.$$

So, if I am the Nth professor, then, under the first system, my probability of receiving σ_0 dollars is approximately equal to $B^\varepsilon_\beta\{\sigma_0\}$.
Thus Theorem 22.3 reproduces the result of §13.

Example: Suppose $\Sigma = (\,[0..4] \times [0..4]\,) \setminus \{\,(4,4)\,\}$.
Suppose, also, $\alpha = 1$ and $\forall \sigma \in \Sigma,\ \varepsilon(\sigma) = \sigma_1 + \sigma_2$.
Then Ω_N represents
the set of all state-distributions at the BUA. (See §20.)
Since $\beta = \mathrm{BP}^\varepsilon_\alpha = \mathrm{BP}^\varepsilon_1$, we calculate:
$\beta \approx 1.0670$, accurate to four decimal places.
Let $M \in \mathbb{R}^{5\times 5}$ be the matrix defined by: $M_{55} = 0$ and
$\forall (i,j) \in (\,[1..5] \times [1..5]\,) \setminus \{\,(5,5)\,\}$, $M_{ij} = B^\varepsilon_\beta\{(i-1, j-1)\}$.

$$\text{Then}\quad M \approx \begin{bmatrix} 0.4345 & 0.1495 & 0.0514 & 0.0177 & 0.0061 \\ 0.1495 & 0.0514 & 0.0177 & 0.0061 & 0.0021 \\ 0.0514 & 0.0177 & 0.0061 & 0.0021 & 0.0007 \\ 0.0177 & 0.0061 & 0.0021 & 0.0007 & 0.0002 \\ 0.0061 & 0.0021 & 0.0007 & 0.0002 & 0 \end{bmatrix}$$

all accurate to four decimal places.
(Thanks to C. Prouty for these calculations. See §28.)
According to Theorem 22.3, this answers
the "precise mathematical problem" formulated near the end of §20.
Since $B^\varepsilon_\beta\{(0,0)\} = M_{11} \approx 0.4345$, it is possible (cf. §15) to prove:
If N is sufficiently large, then, more than 99% of the time,
over 43% of the BUA professors have \$0 wealth.

23. ∞-PROPERNESS AND $(-\infty)$-PROPERNESS

Recall (§2): the notations $\mathbb{I}_f$ and f^*A.

DEFINITION 23.1. Let Σ *be a set.* **Let** $\varepsilon : \Sigma \to \mathbb{R}$.
By ε *is* $\boxed{\infty\textbf{-proper}}$, *we mean:* $\forall t \in \mathbb{R},\ \#\{\sigma \in \Sigma \,|\, \varepsilon(\sigma) \leqslant t\} < \infty$.

That is, $\forall t \in \mathbb{R},\ \#(\ \varepsilon^*(-\infty; t]\ ) < \infty$.

Note that, for any *finite* set Σ, for any $\varepsilon : \Sigma \to \mathbb{R}$,
we have: ε is ∞-proper.

THEOREM 23.2. Let Σ *be a set.*
If $\exists \varepsilon : \Sigma \to \mathbb{R}$ *s.t.* ε *is* ∞*-proper, then* Σ *is countable.*

The next result asserts that, for a nonempty set Σ,
if $\varepsilon : \Sigma \to \mathbb{R}$ is ∞-proper,
then its image, $\mathbb{I}_\varepsilon$, has a minimal element, *i.e.*, $\min \mathbb{I}_\varepsilon$ exists.

THEOREM 23.3. **Let** Σ *be a set.* **Let** $\varepsilon : \Sigma \to \mathbb{R}$ *be* ∞*-proper.*
Assume: $\Sigma \neq \varnothing$. *Then:* $\exists t_0 \in \mathbb{I}_\varepsilon$ *s.t.,* $\forall t \in \mathbb{I}_\varepsilon$, $t \geqslant t_0$.

THEOREM 23.4. **Let** Σ *be a set.* **Let** $\varepsilon : \Sigma \to \mathbb{R}$ *be* ∞*-proper.*
Then: $\mathbb{I}_\varepsilon$ *is bounded below* *and* $\forall t \in \mathbb{I}_\varepsilon$, $\varepsilon^*\{t\}$ *is finite.*

The preceding three theorems are basic; we omit proofs.
When ε is $\mathbb{Z}$-valued, the converse of Theorem 23.4 is also true:

THEOREM 23.5. **Let** Σ *be a set.* **Let** $\varepsilon : \Sigma \to \mathbb{Z}$.
Then: [ε *is* ∞*-proper*]
$\Leftrightarrow$ [($\mathbb{I}_\varepsilon$ *is bounded below*) & ($\forall t \in \mathbb{I}_\varepsilon$, $\varepsilon^*\{t\}$ *is finite*)].

The preceding is basic; we omit proof.
The following two results are corollaries of Theorem 23.5:

THEOREM 23.6. **Let** Σ *be a set.* **Let** $\varepsilon : \Sigma \to \mathbb{Z}$ *be injective.*
Then: [ε ∞*-proper*] $\Leftrightarrow$ [$\mathbb{I}_\varepsilon$ *is bounded below*].

THEOREM 23.7. **Let** $\Sigma \subseteq \mathbb{Z}$.
Define $\varepsilon : \Sigma \to \mathbb{R}$ *by:* $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$.
Then: [ε ∞*-proper*] $\Leftrightarrow$ [Σ *is bounded below*].

DEFINITION 23.8. **Let** Σ *be a set.* **Let** $\varepsilon : \Sigma \to \mathbb{R}$.
By ε *is* $\boxed{(-\infty)\textbf{-proper}}$, *we mean:* $\forall t \in \mathbb{R}$, $\#\{\sigma \in \Sigma \,|\, \varepsilon(\sigma) \geqslant t\} < \infty$.

Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$.
Then: (ε is $(-\infty)$-proper) $\Leftrightarrow$ ($-\varepsilon$ is ∞-proper).

THEOREM 23.9. **Let** Σ *be a finite set.*
Then: $\forall \varepsilon : \Sigma \to \mathbb{R}$, ε *is both* ∞*-proper and* $(-\infty)$*-proper.*

THEOREM 23.10. **Let** Σ *be a set.*
Assume: $\exists \varepsilon : \Sigma \to \mathbb{R}$ *s.t.* ε *is both* ∞*-proper and* $(-\infty)$*-proper.*
Then: Σ *is finite.*

The preceding two theorems are basic; we omit proofs.

24. Boltzmann distributions on countable sets

In the next few sections,
we generalize our earlier work on Boltzmann distributions (§21)
to allow for a countably infinite set of states.

Recall (§8): the notation $\sum^{\text{SP}}$.

DEFINITION 24.1. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
Then: $\boxed{\Delta_\beta^\varepsilon} := \sum_{\sigma\in\Sigma}^{\mathrm{SP}} [\, e^{-\beta\cdot(\varepsilon(\sigma))} \,] \in [0;\infty]$.

We have: $\forall$nonempty set Σ, $\forall \varepsilon : \Sigma \to \mathbb{R}$, $\forall \beta \in \mathbb{R}$, $\Delta_\beta^\varepsilon > 0$.

DEFINITION 24.2. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Then the $\boxed{\textbf{Delta-finite-set}}$ **of** ε *is:* $\boxed{\mathrm{DF}_\varepsilon} := \{\beta \in \mathbb{R} \,|\, \Delta_\beta^\varepsilon < \infty\}$.

We have: $\forall$finite set Σ, $\forall \varepsilon : \Sigma \to \mathbb{R}$, $\forall \beta \in \mathbb{R}$, $\Delta_\beta^\varepsilon < \infty$.
Then: $\forall$finite set Σ, $\forall \varepsilon : \Sigma \to \mathbb{R}$, $\mathrm{DF}_\varepsilon = \mathbb{R}$.

Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$.
Since $\forall \beta \in \mathbb{R}$, $\Delta_{-\beta}^{-\varepsilon} = \Delta_\beta^\varepsilon$, we get: $\mathrm{DF}_{-\varepsilon} = -\mathrm{DF}_\varepsilon$.

Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\xi \in \mathbb{R}$.
Since $\forall \beta \in \mathbb{R}$, $\Delta_\beta^{\varepsilon+\xi} = e^{-\beta\cdot\xi} \cdot \Delta_\beta^\varepsilon$, we get: $\mathrm{DF}_{\varepsilon+\xi} = \mathrm{DF}_\varepsilon$.

Recall (§9) the notations: $\mathcal{M}_\Theta$, $\mathcal{FM}_\Theta^\times$, $\mathcal{P}_\Theta$, $\mathcal{N}(\mu)$.

DEFINITION 24.3. Let Σ *be a countable set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
Then $\boxed{\widehat{B}_\beta^\varepsilon} \in \mathcal{M}_\Sigma$ *is defined by:* $\forall \sigma \in \Sigma$, $\widehat{B}_\beta^\varepsilon\{\sigma\} = e^{-\beta\cdot(\varepsilon(\sigma))}$.

Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
Since $\Delta_\beta^\varepsilon = \sum_{\sigma\in\Sigma} [\widehat{B}_\beta^\varepsilon\{\sigma\}]$, we get: $\Delta_\beta^\varepsilon = \widehat{B}_\beta^\varepsilon(\Sigma)$.

For any countable set Σ, for any $\varepsilon : \Sigma \to \mathbb{R}$, for any $\beta \in \mathbb{R}$,

$$(\,\Sigma \neq \varnothing \text{ and } \beta \in \mathrm{DF}_\varepsilon\,) \Leftrightarrow$$

$$(\,0 < \Delta_\beta^\varepsilon < \infty\,) \Leftrightarrow (\,0 < \widehat{B}_\beta^\varepsilon(\Sigma) < \infty\,) \Leftrightarrow (\,\widehat{B}_\beta^\varepsilon \in \mathcal{FM}_\Sigma^\times\,).$$

DEFINITION 24.4. Let Σ *be a countable set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
Assume: $0 < \Delta_\beta^\varepsilon < \infty$. *Then:* $\boxed{B_\beta^\varepsilon} := \mathcal{N}(\widehat{B}_\beta^\varepsilon) \in \mathcal{P}_\Sigma$.

Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$.
If $\mathrm{DF}_\varepsilon = \varnothing$, then, $\forall \beta \in \mathbb{R}$, since $\widehat{B}_\beta^\varepsilon(\Sigma) = \Delta_\beta^\varepsilon = \infty$,
we see that $\widehat{B}_\beta^\varepsilon$ cannot be normalized, *i.e.*, there is no B_β^ε.
So, if $\mathrm{DF}_\varepsilon = \varnothing$, then we have no Boltzmann distributions to study.
So, going forward, we often focus on cases where $\mathrm{DF}_\varepsilon \neq \varnothing$.
If $\Sigma = \varnothing$, ε is the empty function, and there is nothing to say.
If Σ is nonempty and finite,
we already developed a satisfactory Boltzmann theory, in §21.
So, going forward, we often focus on cases where Σ is infinite.

Recall (§2): the notations $\mathbb{I}_f$ and f^*A.

Let Σ be an infinite set, $\varepsilon : \Sigma \to \mathbb{R}$. Then: $\varepsilon^*\mathbb{R} = \Sigma$,
We have: $(-\infty;0] \bigcup [0;\infty) = \mathbb{R}$.
Since $(\ \varepsilon^*(-\infty;0]\ ) \bigcup (\ \varepsilon^*[0;\infty)\ ) = \varepsilon^*\mathbb{R} = \Sigma$,
we get: either $\varepsilon^*(-\infty;0]$ is infinite or $\varepsilon^*[0;\infty)$ is infinite.
Assuming Σ is countable,
the Boltzmann theory splits into these two cases;
replacing ε with $-\varepsilon$ interchanges the two cases,
so the theory in one case parallels the theory in the other.
Also, by Theorem 24.7 below, if $\mathrm{DF}_\varepsilon \neq \varnothing$,
then only one of the two cases can happen.

THEOREM 24.5. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*[0;\infty)$ *is infinite.* *Then:* $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$.

Proof. **Given** $\beta \in \mathrm{DF}_\varepsilon$, **want:** $\beta \in (0;\infty)$.
Since $\mathrm{DF}_\varepsilon \subseteq \mathbb{R}$, we get: $\beta \in \mathbb{R}$.
Want: $\beta > 0$. Assume: $\beta \leqslant 0$. **Want:** Contradiction.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
For all $\sigma \in \varepsilon^*[0;\infty)$, since $\varepsilon_\sigma = \varepsilon(\sigma) \in [0;\infty)$, we get: $\varepsilon_\sigma \geqslant 0$.
So, since $\beta \leqslant 0$, we get: $\forall \sigma \in \varepsilon^*[0;\infty)$, $-\beta \cdot \varepsilon_\sigma \geqslant 0$.
Then: $\forall \sigma \in \varepsilon^*[0;\infty)$, $e^{-\beta \cdot \varepsilon_\sigma} \geqslant 1$.
So, since $\varepsilon^*[0;\infty)$ is infinite, we get: $\sum^{\mathrm{SP}}_{\sigma \in \varepsilon^*[0;\infty)} [e^{-\beta \cdot \varepsilon_\sigma}] = \infty$.
Since $\Delta^\varepsilon_\beta = \sum^{\mathrm{SP}}_{\sigma \in \Sigma} [e^{-\beta \cdot \varepsilon_\sigma}] \geqslant \sum^{\mathrm{SP}}_{\sigma \in \varepsilon^*[0;\infty)]} [e^{-\beta \cdot \varepsilon_\sigma}] = \infty$,
we get: $\beta \notin \mathrm{DF}_\varepsilon$. Contradiction. ☐

THEOREM 24.6. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*(-\infty;0]$ *is infinite.* *Then:* $\mathrm{DF}_\varepsilon \subseteq (-\infty;0)$.

Proof. Since $(-\varepsilon)^*[0;\infty) = \varepsilon^*(-\infty;0]$, we get: $(-\varepsilon)^*[0;\infty)$ is infinite.
Then, by Theorem 24.5, we get: $\mathrm{DF}_{-\varepsilon} \subseteq (0;\infty)$.
Then: $\mathrm{DF}_\varepsilon = -\mathrm{DF}_{-\varepsilon} \subseteq -(0;\infty) = (-\infty;0)$. ☐

THEOREM 24.7. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*(-\infty;0]$ *and* $\varepsilon^*[0;\infty)$ *are both infinite.* *Then:* $\mathrm{DF}_\varepsilon = \varnothing$.

Proof. By Theorem 24.5, we get: $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$.
By Theorem 24.6, we get: $\mathrm{DF}_\varepsilon \subseteq (-\infty;0)$.
Since $\mathrm{DF}_\varepsilon \subseteq (-\infty;0) \bigcap (0;\infty) = \varnothing$, we get: $\mathrm{DF}_\varepsilon = \varnothing$. ☐

THEOREM 24.8. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) \neq \varnothing$. *Then:* ε *is* ∞*-proper.*

Proof. **Given** $t \in \mathbb{R}$, **let** $\Sigma_0 := \{\sigma \in \Sigma \,|\, \varepsilon(\sigma) \leqslant t\}$, **want:** $\#\Sigma_0 < \infty$.
Since $\mathrm{DF}_\varepsilon \bigcap [0;\infty) \neq \varnothing$, **choose** $\beta \in \mathrm{DF}_\varepsilon \bigcap [0;\infty)$.
Then $\beta \in \mathrm{DF}_\varepsilon$ and $\beta \in [0;\infty)$.
Since $\beta \in \mathrm{DF}_\varepsilon$, we get: $\Delta^\varepsilon_\beta < \infty$. Then: $e^{\beta\cdot t} \cdot \Delta^\varepsilon_\beta < \infty$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$. Then: $\Delta^\varepsilon_\beta = \sum^{\mathrm{SP}}_{\sigma\in\Sigma} [e^{-\beta\cdot\varepsilon_\sigma}]$.
By definition of Σ_0, we have: $\forall \sigma \in \Sigma_0$, $\varepsilon(\sigma) \leqslant t$.
Since $\beta \in [0;\infty)$ and since $\forall \sigma \in \Sigma_0$, $t \geqslant \varepsilon(\sigma) = \varepsilon_\sigma$,
we get: $\forall \sigma \in \Sigma_0$, $-\beta \cdot t \leqslant -\beta \cdot \varepsilon_\sigma$.
Then: $\forall \sigma \in \Sigma_0$, $e^{-\beta\cdot t} \leqslant e^{-\beta\cdot\varepsilon_\sigma}$.
Then: $\#\Sigma_0 = \sum^{\mathrm{SP}}_{\sigma\in\Sigma_0} [1] = e^{\beta\cdot t} \cdot \sum^{\mathrm{SP}}_{\sigma\in\Sigma_0} [e^{-\beta\cdot t}] \leqslant e^{\beta\cdot t} \cdot \sum^{\mathrm{SP}}_{\sigma\in\Sigma_0} [e^{-\beta\cdot\varepsilon_\sigma}]$
$\leqslant e^{\beta\cdot t} \cdot \sum^{\mathrm{SP}}_{\sigma\in\Sigma} [e^{-\beta\cdot\varepsilon_\sigma}] = e^{\beta\cdot t} \cdot \Delta^\varepsilon_\beta < \infty$. ☐

THEOREM 24.9. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \bigcap (-\infty;0] \neq \varnothing$. *Then:* ε *is* $(-\infty)$*-proper.*

Proof. Since $-(\mathrm{DF}_\varepsilon \bigcap (-\infty;0]) \neq \varnothing$,
we get: $\mathrm{DF}_{-\varepsilon} \bigcap [0;\infty) \neq \varnothing$.
Then, by Theorem 24.8, $-\varepsilon$ is ∞-proper, and so ε is $(-\infty)$-proper. ☐

THEOREM 24.10. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \neq \varnothing$. *Then:* Σ *is countable.*

Proof. Since $(\mathrm{DF}_\varepsilon \bigcap (-\infty;0]) \bigcup (\mathrm{DF}_\varepsilon \bigcap [0;\infty)) = \mathrm{DF}_\varepsilon \neq \varnothing$,
it follows that: either $\mathrm{DF}_\varepsilon \bigcap (-\infty;0] \neq \varnothing$ or $\mathrm{DF}_\varepsilon \bigcap [0;\infty) \neq \varnothing$.
Then, by Theorem 24.9 or Theorem 24.8,
we get: either ε is $(-\infty)$-proper or ε is ∞-proper.
Then: either $-\varepsilon$ is ∞-proper or ε is ∞-proper.
In either case, by Theorem 23.2, we get: Σ is countable. ☐

THEOREM 24.11. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \bigcap (-\infty;0] \neq \varnothing \neq \mathrm{DF}_\varepsilon \bigcap [0;\infty)$. *Then:* Σ *is finite.*

Proof. By Theorem 24.8, we get: ε is ∞-proper.
By Theorem 24.9, we get: ε is $(-\infty)$-proper.
Then, by Theorem 23.10, we get: Σ is finite. ☐

THEOREM 24.12. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*[0;\infty)$ *is infinite and* $\mathrm{DF}_\varepsilon \neq \varnothing$. *Then:* ε *is* ∞*-proper.*

Proof. By Theorem 24.5, we have: $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$.
Since $\mathrm{DF}_\varepsilon \subseteq (0;\infty) \subseteq [0;\infty)$, we get: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = \mathrm{DF}_\varepsilon$.
Since $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = \mathrm{DF}_\varepsilon \neq \varnothing$, by Theorem 24.8,
we get: ε is ∞-proper. □

THEOREM 24.13. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*(-\infty;0]$ *is infinite and* $\mathrm{DF}_\varepsilon \neq \varnothing$. *Then:* ε *is* $(-\infty)$*-proper.*

Proof. Since $(-\varepsilon)^*[0;\infty) = \varepsilon^*(-\infty;0]$, we get: $(-\varepsilon)^*[0;\infty)$ is infinite.
Since $\mathrm{DF}_{-\varepsilon} = -\mathrm{DF}_\varepsilon$, we get: $\mathrm{DF}_{-\varepsilon} \neq \varnothing$.
Then, by Theorem 24.12, $-\varepsilon$ is ∞-proper, so ε is $(-\infty)$-proper. □

DEFINITION 24.14. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Then, $\forall \rho \in [0;\infty)$, *the* $\boxed{\rho\textbf{-absolute-exponent } (\beta,\varepsilon)\textbf{-sum}}$ *is:*

$$\boxed{\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta} \;:=\; \sum\nolimits^{\mathrm{SP}}_{\sigma\in\Sigma} \left[\, |\varepsilon_\sigma|^\rho \cdot |e^{-\beta\cdot\varepsilon_\sigma}| \,\right] \;\in\; [0;\infty].$$

Also, $\forall \rho \in [0..\infty)$, *if* $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta < \infty$,
then the $\boxed{\rho\textbf{-exponent } (\beta,\varepsilon)\textbf{-sum}}$ *is:*

$$\boxed{\mathrm{X}^\rho \mathrm{S}^\varepsilon_\beta} \;:=\; \sum\nolimits_{\sigma\in\Sigma} \left[\, \varepsilon^\rho_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma} \,\right] \;\in\; \mathbb{R}.$$

Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$, $\rho \in [0..\infty)$.
If $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta < \infty$, then, by subadditivity of absolute value,
we get: $|\mathrm{X}^\rho \mathrm{S}^\varepsilon_\beta| \leqslant \overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta$.

Let Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
Recall our convention (§2): $0^0 = 1$. Then: $\overline{\mathrm{X}}^0 \mathrm{S}^\varepsilon_\beta = \Delta^\varepsilon_\beta$.
Also, if $\overline{\mathrm{X}}^0 \mathrm{S}^\varepsilon_\beta < \infty$, then $X^0 S^\varepsilon_\beta = \Delta^\varepsilon_\beta$.

THEOREM 24.15. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \neq \varnothing$ *and* $\mathbb{I}_\varepsilon$ *is bounded below.* **Let** $\rho \geqslant 0$ *be real.*
Let $\beta \in \mathrm{DF}_\varepsilon$ *and* **let** $\gamma > \beta$ *be real.* *Then:* $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma < \infty$.

We cannot replace "$\gamma > \beta$" with "$\gamma \geqslant \beta$"; see Theorem 24.17 below.

Proof. Since $\mathbb{I}_\varepsilon$ is bounded below, **choose** $t_0 \in \mathbb{R}$ s.t., $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) \geqslant t_0$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$. Then: $\forall \sigma \in \Sigma$, $\varepsilon_\sigma \geqslant t_0$.
Let $\delta := \gamma - \beta$. Then $\delta > 0$, so, as $t \to \infty$, $|t|^\rho \cdot e^{-\delta\cdot t} \to 0$.
So, since $t \mapsto |t|^\rho \cdot e^{-\delta\cdot t} : [t_0;\infty) \to \mathbb{R}$ is continuous,
by the Extreme Value Theorem, **choose** $M \in \mathbb{R}$ s.t.,
$\forall$real $t \geqslant t_0$, $|\,t\,|^\rho \cdot e^{-\delta\cdot t} \leqslant M$.

Recall: $\forall \sigma \in \Sigma$, we have: $\varepsilon_\sigma \geqslant t_0$.
Then: $\forall \sigma \in \Sigma$, $|\varepsilon_\sigma|^\rho \cdot e^{-\delta \cdot \varepsilon_\sigma} \leqslant M$.
By definition of $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma$, we get: $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma = \sum^{\mathrm{SP}}_{\sigma \in \Sigma} [\ |\varepsilon_\sigma|^\rho \cdot e^{-\gamma \cdot \varepsilon_\sigma}\ ]$.
So, since $-\gamma = -\delta - \beta$, we get: $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma = \sum^{\mathrm{SP}}_{\sigma \in \Sigma} [\,(|\varepsilon_\sigma|^\rho \cdot e^{-\delta \cdot \varepsilon_\sigma}) \cdot (e^{-\beta \cdot \varepsilon_\sigma})\,]$.
Since $\beta \in \mathrm{DF}_\varepsilon$, we get: $\Delta^\varepsilon_\beta < \infty$. Then: $M \cdot \Delta^\varepsilon_\beta < \infty$.

$$
\begin{aligned}
\text{Then:}\quad \overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma &= \textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma} [\,(|\varepsilon_\sigma|^\rho \cdot e^{-\delta \cdot \varepsilon_\sigma}) \cdot (e^{-\beta \cdot \varepsilon_\sigma})\,] \\
&\leqslant \textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma} [\qquad M \qquad \cdot (e^{-\beta \cdot \varepsilon_\sigma})\,] \\
&= M \cdot (\textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma} [\,e^{-\beta \cdot \varepsilon_\sigma}\,]) = M \cdot \Delta^\varepsilon_\beta < \infty. \qquad \square
\end{aligned}
$$

THEOREM 24.16. Let *Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta, \rho \in \mathbb{R}$.*
Assume: $\rho \geqslant 0$, ε is ∞-proper, $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta < \infty$. Then: $\beta \in \mathrm{DF}_\varepsilon$.

The proof below shows that we can weaken the hypothesis
"ε is ∞-proper" to "$\{\sigma \in \Sigma \,|\, \varepsilon(\sigma) \leqslant 1\}$ is finite".
However, it cannot be dropped altogether; see Theorem 24.18 below.

Proof. **Want:** $\Delta^\varepsilon_\beta < \infty$.
Let $F := \{\sigma \in \Sigma \,|\, \varepsilon(\sigma) \leqslant 1\}$. Since ε is ∞-proper, we get: F is finite.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$. Then: $F = \{\sigma \in \Sigma \,|\, \varepsilon_\sigma \leqslant 1\}$.
Since F is finite, we get: $\sum^{\mathrm{SP}}_{\sigma \in F} [e^{-\beta \cdot \varepsilon_\sigma}] < \infty$.
So, since $\Delta^\varepsilon_\beta = (\sum^{\mathrm{SP}}_{\sigma \in F} [e^{-\beta \cdot \varepsilon_\sigma}]) + (\sum^{\mathrm{SP}}_{\sigma \in \Sigma \backslash F} [e^{-\beta \cdot \varepsilon_\sigma}])$,
it suffices to show: $\sum^{\mathrm{SP}}_{\sigma \in \Sigma \backslash F} [e^{-\beta \cdot \varepsilon_\sigma}] < \infty$.
Since $F = \{\sigma \in \Sigma \,|\, \varepsilon_\sigma \leqslant 1\}$,
we get: $\forall \sigma \in \Sigma \backslash F$, $\varepsilon_\sigma > 1$.
Then: $\forall \sigma \in \Sigma \backslash F$, since $\varepsilon_\sigma > 1 > 0$,
we get: $\varepsilon_\sigma = |\varepsilon_\sigma|$.
Since $\forall \sigma \in \Sigma \backslash F$, $1 < \varepsilon_\sigma = |\varepsilon_\sigma|$,
we get: $\forall \sigma \in \Sigma \backslash F$, $1^\rho \leqslant |\varepsilon_\sigma|^\rho$.
Then: $\forall \sigma \in \Sigma \backslash F$, $1^\rho \cdot e^{-\beta \cdot \varepsilon_\sigma} \leqslant |\varepsilon_\sigma|^\rho \cdot e^{-\beta \cdot \varepsilon_\sigma}$.

$$
\begin{aligned}
\text{Then:}\quad \textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma \backslash F} [e^{-\beta \cdot \varepsilon_\sigma}] &= \textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma \backslash F} [1^\rho \cdot e^{-\beta \cdot \varepsilon_\sigma}] \leqslant \textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma \backslash F} [|\varepsilon_\sigma|^\rho \cdot e^{-\beta \cdot \varepsilon_\sigma}] \\
&\leqslant \textstyle\sum^{\mathrm{SP}}_{\sigma \in \Sigma} [|\varepsilon_\sigma|^\rho \cdot e^{-\beta \cdot \varepsilon_\sigma}] = \overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta < \infty. \qquad \square
\end{aligned}
$$

THEOREM 24.17. Let $\Sigma := [3..\infty)$.
Define $\varepsilon : \Sigma \to \mathbb{R}$ *by:* $\forall k \in \Sigma$, $\varepsilon(k) = (\ln k) + 2 \cdot (\ln(\ln k))$.
Let $\beta := 1$, $\gamma := 1$, $\rho := 1$. *Then:* $\beta \in \mathrm{DF}_\varepsilon$ *and* $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma = \infty$.

Proof. For all $k \in \Sigma$, **let** $\varepsilon_k := \varepsilon(k)$.
Then: $\forall k \in [3..\infty)$, $\varepsilon_k = (\ln k) + 2 \cdot (\ln(\ln k))$.
Since $\Delta^\varepsilon_\beta = \Delta^\varepsilon_1 = \sum^{\mathrm{SP}}_{k \in \Sigma} [e^{-\varepsilon_k}] = \sum^{\infty}_{k=3} [e^{-\varepsilon_k}]$

$$= \sum_{k=3}^{\infty} \left[\frac{1}{e^{\varepsilon_k}} \right] = \sum_{k=3}^{\infty} \left[\frac{1}{e^{(\ln k)+2(\ln(\ln k))}} \right] = \sum_{k=3}^{\infty} \left[\frac{1}{k \cdot (\ln k)^2} \right] < \infty,$$

we get: $\beta \in \mathrm{DF}_\varepsilon$. **It remains only to show:** $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma = \infty$.

We have: $\forall k \in [3..\infty)$, $k > e$, so $\ln k > 1$, so $\ln(\ln k) > 0$.

For all $k \in [3..\infty)$, since $\varepsilon_k = (\ln k) + 2 \cdot (\ln(\ln k)) > 1 + 2 \cdot 0 = 1 > 0$,

we get: $|\varepsilon_k| = \varepsilon_k$.

Since

$$\begin{aligned}
\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma = \overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_1 &= \textstyle\sum_{k \in \Sigma}^{\mathrm{SP}} \left[|\varepsilon_k| \cdot e^{-\varepsilon_k} \right] \\
&= \textstyle\sum_{k=3}^{\infty} \left[|\varepsilon_k| \cdot e^{-\varepsilon_k} \right] \\
&= \textstyle\sum_{k=3}^{\infty} \left[\ \varepsilon_k \ \cdot e^{-\varepsilon_k} \right] \\
&= \sum_{k=3}^{\infty} \left[\frac{\varepsilon_k}{e^{\varepsilon_k}} \right] = \sum_{k=3}^{\infty} \left[\frac{(\ln k) + 2 \cdot (\ln(\ln k))}{e^{(\ln k)+2(\ln(\ln k))}} \right] \\
&= \sum_{k=3}^{\infty} \left[\frac{(\ln k) + 2 \cdot (\ln(\ln k))}{k \cdot (\ln k)^2} \right] \\
&\geqslant \sum_{k=3}^{\infty} \left[\frac{\ln k}{k \cdot (\ln k)^2} \right] \\
&= \sum_{k=3}^{\infty} \left[\frac{1}{k \cdot (\ln k)} \right] = \infty,
\end{aligned}$$

we get: $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma = \infty$. □

THEOREM 24.18. Let $\Sigma := \mathbb{N}$.
Define $\varepsilon : \Sigma \to \mathbb{R}$ *by:* $\forall k \in \Sigma$, $\varepsilon(k) = 1/k^2$.
Let $\beta := 1$, $\rho := 1$. *Then:* $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta < \infty$ *and* $\beta \notin \mathrm{DF}_\varepsilon$.

Proof. For all $k \in \Sigma$, **let** $\varepsilon_k := \varepsilon(k)$. Then: $\forall k \in \Sigma$, $\varepsilon_k = 1/k^2$.

We have: $\forall k \in \mathbb{N}$, both $|\varepsilon_k| = 1/k^2$ and $-\varepsilon_k = -1/k^2$.

Since

$$\begin{aligned}
\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\beta = \overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_1 &= \textstyle\sum_{k \in \Sigma}^{\mathrm{SP}} \left[\ |\varepsilon_k| \ \cdot e^{-\varepsilon_k} \ \right] \\
&= \textstyle\sum_{k=1}^{\infty} \left[\ |\varepsilon_k| \ \cdot e^{-\varepsilon_k} \ \right] \\
&= \textstyle\sum_{k=1}^{\infty} \left[(1/k^2) \cdot e^{-1/k^2} \right] \\
&\leqslant \textstyle\sum_{k=1}^{\infty} \left[(1/k^2) \cdot \ 1 \ \right] \\
&= \textstyle\sum_{k=1}^{\infty} \left[\ 1/k^2 \ \right] < \infty,
\end{aligned}$$

it remains only to show: $\beta \notin \mathrm{DF}_\varepsilon$ **Want:** $\Delta^\varepsilon_\beta = \infty$.

We have: as $k \to \infty$, $e^{-1/k^2} \to 1$. Then: $\sum_{k=1}^{\infty} \left[e^{-1/k^2} \right] = \infty$.

Then: $\Delta^\varepsilon_\beta = \Delta^\varepsilon_1 = \sum_{k \in \Sigma}^{\mathrm{SP}} \left[e^{-\varepsilon_k} \right] = \sum_{k=1}^{\infty} \left[e^{-\varepsilon_k} \right] = \sum_{k=1}^{\infty} \left[e^{-1/k^2} \right] = \infty$. □

THEOREM 24.19. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \neq \varnothing$ *and* $\mathbb{I}_\varepsilon$ *is bounded below.* **Let** $\rho \geqslant 0$ *be real.*
Let $\beta_0 := \inf \mathrm{DF}_\varepsilon$ *and* **let** $\gamma \in (\beta_0; \infty)$. *Then:* $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma < \infty$.

Proof. Since $\gamma > \beta_0 = \inf \mathrm{DF}_\varepsilon$, **choose** $\beta \in \mathrm{DF}_\varepsilon$ s.t. $\gamma > \beta$.
Then, by Theorem 24.15, we get: $\overline{\mathrm{X}}^\rho \mathrm{S}^\varepsilon_\gamma < \infty$. □

THEOREM 24.20. Let *Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$.*
Assume: $\mathrm{DF}_\varepsilon \neq \varnothing$ and $\mathbb{I}_\varepsilon$ is bounded below.
Let $\beta_0 := \inf \mathrm{DF}_\varepsilon$ *and* **let** $\gamma \in (\beta_0; \infty)$. *Then: $\gamma \in \mathrm{DF}_\varepsilon$.*

Proof. By Theorem 24.19, we have: $\overline{\mathrm{X}}^0 \mathrm{S}^\varepsilon_\gamma < \infty$.
Since $\Delta^\varepsilon_\gamma = \overline{\mathrm{X}}^0 \mathrm{S}^\varepsilon_\gamma < \infty$, we get: $\gamma \in \mathrm{DF}_\varepsilon$. □

THEOREM 24.21. Let *Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$.*
Assume: $\varepsilon^*[0; \infty)$ is infinite and $\mathrm{DF}_\varepsilon \neq \varnothing$. **Let** $\beta_0 := \inf \mathrm{DF}_\varepsilon$.
Then: $0 \leqslant \beta_0 < \infty$ and $(\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$.

Proof. By Theorem 24.5, $\mathrm{DF}_\varepsilon \subseteq (0; \infty)$. Then: $\inf \mathrm{DF}_\varepsilon \geqslant \inf(0; \infty)$.
Since $\mathrm{DF}_\varepsilon \neq \varnothing$, we get: $\inf \mathrm{DF}_\varepsilon < \infty$.
Since $\beta_0 = \inf \mathrm{DF}_\varepsilon \geqslant \inf(0; \infty) = 0$ and since $\beta_0 = \inf \mathrm{DF}_\varepsilon < \infty$,
we get: $0 \leqslant \beta_0 < \infty$.
It remains to show: $(\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$.
Given $\gamma \in (\beta_0; \infty)$, **want:** $\gamma \in \mathrm{DF}_\varepsilon$.
By Theorem 24.12, we have: ε is ∞-proper.
Then, by Theorem 23.4, we have: $\mathbb{I}_\varepsilon$ is bounded below.
Then, by Theorem 24.20, we have: $\gamma \in \mathrm{DF}_\varepsilon$. □

THEOREM 24.22. Let *Σ be a set, $\varepsilon : \Sigma \to \mathbb{R}$.*
Assume: $\varepsilon^*[0; \infty)$ is infinite and $\mathrm{DF}_\varepsilon \neq \varnothing$. **Let** $\beta_0 := \inf \mathrm{DF}_\varepsilon$.
Then either ($\mathrm{DF}_\varepsilon = [\beta_0; \infty)$ and $0 < \beta_0 < \infty$)
or ($\mathrm{DF}_\varepsilon = (\beta_0; \infty)$ and $0 \leqslant \beta_0 < \infty$).

Proof. By Theorem 24.21, we get: $0 \leqslant \beta_0 < \infty$ and $(\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$.
Since $\beta_0 = \inf \mathrm{DF}_\varepsilon$, we get: $\mathrm{DF}_\varepsilon \subseteq [\beta_0; \infty)$.
By Theorem 24.5, we get: $\mathrm{DF}_\varepsilon \subseteq (0; \infty)$.

Case 1: $\beta_0 \in \mathrm{DF}_\varepsilon$. **Want:** $\mathrm{DF}_\varepsilon = [\beta_0; \infty)$ and $0 < \beta_0 < \infty$.
Recall: $(\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$ and $\mathrm{DF}_\varepsilon \subseteq [\beta_0; \infty)$ and $\mathrm{DF}_\varepsilon \subseteq (0; \infty)$.
Since $\beta_0 \in \mathrm{DF}_\varepsilon$ and $(\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$,
we get: $\{\beta_0\} \bigcup (\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$.
Since $[\beta_0; \infty) = \{\beta_0\} \bigcup (\beta_0; \infty) \subseteq \mathrm{DF}_\varepsilon$ and since $\mathrm{DF}_\varepsilon \subseteq [\beta_0; \infty)$,
we get: $\mathrm{DF}_\varepsilon = [\beta_0; \infty)$.
It remains only to show: $0 < \beta_0 < \infty$.
Recall: $0 \leqslant \beta_0 < \infty$. Then: $\beta_0 < \infty$.

It remains only to show: $0 < \beta_0$.
Since $\beta_0 \in [\beta_0;\infty) = \mathrm{DF}_\varepsilon \subseteq (0;\infty)$, we get: $0 < \beta_0$.
End of Case 1.

Case 2: $\beta_0 \notin \mathrm{DF}_\varepsilon$. **Want:** $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$ and $0 \leqslant \beta_0 < \infty$.
Recall: $0 \leqslant \beta_0 < \infty$.
It remains only to show: $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$.
Recall: $\mathrm{DF}_\varepsilon \subseteq [\beta_0;\infty)$,
Since $\beta_0 \notin \mathrm{DF}_\varepsilon$ and $\mathrm{DF}_\varepsilon \subseteq [\beta_0;\infty)$,
we get: $\mathrm{DF}_\varepsilon \subseteq [\beta_0;\infty)\backslash\{\beta_0\}$. Recall: $(\beta_0;\infty) \subseteq \mathrm{DF}_\varepsilon$.
Since $\mathrm{DF}_\varepsilon \subseteq [\beta_0;\infty)\backslash\{\beta_0\} = (\beta_0;\infty)$ and $(\beta_0;\infty) \subseteq \mathrm{DF}_\varepsilon$,
we get: $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$.
End of Case 2. □

THEOREM 24.23. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*(-\infty;0]$ *is infinite and* $\mathrm{DF}_\varepsilon \neq \varnothing$. **Let** $\beta_0 := -\sup \mathrm{DF}_\varepsilon$.
Then one of the following holds:
Either ($\mathrm{DF}_\varepsilon = (-\infty;-\beta_0]$ *and* $0 < \beta_0 < \infty$)
or ($\mathrm{DF}_\varepsilon = (-\infty;-\beta_0)$ *and* $0 \leqslant \beta_0 < \infty$).

Proof. Since $(-\varepsilon)^*[0;\infty)$ is infinite and $\mathrm{DF}_{-\varepsilon} \neq \varnothing$ and $\beta_0 = \inf \mathrm{DF}_{-\varepsilon}$,
by Theorem 24.22, we get:
either ($\mathrm{DF}_{-\varepsilon} = [\beta_0;\infty)$ and $0 < \beta_0 < \infty$)
or ($\mathrm{DF}_{-\varepsilon} = (\beta_0;\infty)$ and $0 \leqslant \beta_0 < \infty$).
Then: either ($\mathrm{DF}_\varepsilon = (-\infty;-\beta_0]$ and $0 < \beta_0 < \infty$)
or ($\mathrm{DF}_\varepsilon = (-\infty;-\beta_0)$ and $0 \leqslant \beta_0 < \infty$). □

THEOREM 24.24. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$. *Assume:* $\mathrm{DF}_\varepsilon \neq \varnothing$.
Then one of the following holds:
(i) $\mathrm{DF}_\varepsilon = \mathbb{R}$
or *(ii)* $\exists$*real* $\beta_0 \geqslant 0$ *s.t.* $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$
or *(ii')* $\exists$*real* $\beta_0 > 0$ *s.t.* $\mathrm{DF}_\varepsilon = [\beta_0;\infty)$
or *(iii)* $\exists$*real* $\beta_0 \geqslant 0$ *s.t.* $\mathrm{DF}_\varepsilon = (-\infty;-\beta_0)$
or *(iii')* $\exists$*real* $\beta_0 > 0$ *s.t.* $\mathrm{DF}_\varepsilon = (-\infty;-\beta_0]$.

Below, in each of
Theorem 24.28, Theorem 24.30, Theorem 24.32,
we give examples of ∞-proper $\varepsilon : \Sigma \to \mathbb{Z}$ such that
$\mathrm{DF}_\varepsilon = \varnothing$, $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$, $\mathrm{DF}_\varepsilon = [\beta_0;\infty)$, respectively;
it follows that $-\varepsilon$ is $(-\infty)$-proper and that
$\mathrm{DF}_{-\varepsilon} = \varnothing$, $\mathrm{DF}_{-\varepsilon} = (-\infty;-\beta_0)$, $\mathrm{DF}_{-\varepsilon} = (-\infty;-\beta_0]$, respectively.

Proof. Since $\varepsilon : \Sigma \to \mathbb{R}$, we get: $\quad \varepsilon^*\mathbb{R} = \Sigma$.
Since $(-\infty; 0] \bigcup [0; \infty) = \mathbb{R}$, we get: $\varepsilon^*(-\infty; 0] \bigcup \varepsilon^*[0; \infty) = \varepsilon^*\mathbb{R}$.
In case $\#\Sigma < \infty$, we get: (i) holds. We therefore assume $\#\Sigma = \infty$.
Want: (ii) or (ii') or (iii) or (iii') holds.
Because $\varepsilon^*(-\infty; 0] \bigcup \varepsilon^*[0; \infty) = \varepsilon^*\mathbb{R} = \Sigma$,
and because Σ is infinite, we get:
either $\varepsilon^*(-\infty; 0]$ is infinite or $\varepsilon^*[0; \infty)$ is infinite.
Then, by Theorem 24.23 or Theorem 24.22, we get:
either (iii) or (iii') holds or (ii) or (ii') holds.
Then: (ii) or (ii') or (iii) or (iii') holds. $\square$

THEOREM 24.25. Let $n_1, n_2, \ldots \in [0..\infty)$.
Let $\Sigma := \{(k, j) \in \mathbb{N} \times \mathbb{N} \,|\, j \leqslant n_k\}$.
Define $\varepsilon : \Sigma \to [0..\infty)$ *by:* $\quad \forall (k, j) \in \Sigma, \quad \varepsilon(k, j) = k - 1$.
Then: $\qquad \forall k \in \mathbb{N}, \qquad \#(\,\varepsilon^*[k-1; k)\,) = n_k$.

Proof. **Given** $k \in \mathbb{N}$, **want:** $\#(\,\varepsilon^*[k-1; k)\,) = n_k$.
Since
$$\begin{aligned}
\varepsilon^*[k-1; k) &= \{(\ell, j) \in \Sigma \,|\, \varepsilon(\ell, j) \in [k-1; k)\} \\
&= \{(\ell, j) \in \Sigma \,|\, \ell - 1 \in [k-1; k)\} \\
&= \{(\ell, j) \in \Sigma \,|\, \ell - 1 = k - 1\} \\
&= \{(\ell, j) \in \Sigma \,|\, \ell = k\} \\
&= \{(\ell, j) \in \mathbb{N} \times \mathbb{N} \,|\, \ell = k \,,\; j \leqslant n_\ell\} \\
&= \{(\ell, j) \in \mathbb{N} \times \mathbb{N} \,|\, \ell = k \,,\; j \leqslant n_k\} \\
&= \{\,(k, 1)\,,\, \ldots\,,\, (k, n_k)\,\},
\end{aligned}$$
we get: $\#(\,\varepsilon^*[k-1; k)\,) = n_k$. $\square$

THEOREM 24.26. Let Σ *be a set,* $\varepsilon : \Sigma \to [0; \infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \#(\,\varepsilon^*[k-1; k)\,)$.
Let $\beta \in [0; \infty)$. *Then:* $\quad (\, \beta \in \mathrm{DF}_\varepsilon \,) \quad \Leftrightarrow \quad (\, \sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty \,)$.

Proof. For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.

Proof of $\Rightarrow$*:* Assume: $\beta \in \mathrm{DF}_\varepsilon$. **Want:** $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$.
Since $\beta \in \mathrm{DF}_\varepsilon$, we get: $\quad \Delta_\beta^\varepsilon < \infty$.
Because Σ is the disjoint union, over $k = 1$ to ∞, of $\varepsilon^*[k-1; k)$,
we get: $\qquad \sum_{\sigma \in \Sigma}^{\mathrm{SP}} [e^{-\beta \cdot \varepsilon_\sigma}] = \sum_{k=1}^{\infty} \sum_{\sigma \in \varepsilon^*[k-1;k)}^{\mathrm{SP}} [e^{-\beta \cdot \varepsilon_\sigma}]$.
For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1; k)$, since $\varepsilon_\sigma = \varepsilon(\sigma) \in [k-1; k)$,
we have: $\qquad k > \qquad \varepsilon_\sigma$.
Since $\beta \in [0; \infty)$, we get: $\quad -\beta \leqslant 0$.
For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1; k)$, we have: $-\beta \cdot k \leqslant -\beta \cdot \varepsilon_\sigma$.

For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, we have: $e^{-\beta \cdot k} \leqslant e^{-\beta \cdot \varepsilon_\sigma}$.
Then: $\forall k \in \mathbb{N}$, $\sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot k}] \leqslant \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_\sigma}]$.
Also, $\forall k \in \mathbb{N}$, $\sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot k}] = n_k e^{-\beta \cdot k}$.
Then: $\forall k \in \mathbb{N}$, $n_k e^{-\beta \cdot k} \leqslant \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_\sigma}]$.
Then:
$$\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] \leqslant \sum_{k=1}^{\infty} \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot \varepsilon_\sigma}] = \sum^{\text{SP}}_{\sigma \in \Sigma} [e^{-\beta \cdot \varepsilon_\sigma}] = \Delta^\varepsilon_\beta < \infty.$$
End of proof of $\Rightarrow$.

Proof of $\Leftarrow$*:* Assume: $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$. **Want:** $\beta \in \text{DF}_\varepsilon$.
Because Σ is the disjoint union, over $k = 1$ to ∞, of $\varepsilon^*[k-1;k)$,
we get: $\sum^{\text{SP}}_{\sigma \in \Sigma} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}] = \sum_{k=1}^{\infty} \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}]$.
For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, since $\varepsilon_\sigma = \varepsilon(\sigma) \in [k-1;k)$,
we have: $\varepsilon_\sigma \geqslant k - 1$.
For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, we have: $\varepsilon_\sigma + 1 \geqslant k$.
Since $\beta \in [0;\infty)$, we get: $-\beta \leqslant 0$.
For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, we have: $-\beta \cdot (\varepsilon_\sigma + 1) \leqslant -\beta \cdot k$.
For all $k \in \mathbb{N}$, for all $\sigma \in \varepsilon^*[k-1;k)$, we have: $e^{-\beta \cdot (\varepsilon_\sigma + 1)} \leqslant e^{-\beta \cdot k}$.
Then: $\forall k \in \mathbb{N}$, $\sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}] \leqslant \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot k}]$.
Also, $\forall k \in \mathbb{N}$, $n_k e^{-\beta \cdot k} = \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot k}]$.
Then: $\forall k \in \mathbb{N}$, $\sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}] \leqslant n_k e^{-\beta \cdot k}$.
Then: $\sum_{k=1}^{\infty} \sum^{\text{SP}}_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}] \leqslant \sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}]$.
By assumption, $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$. Then $e^\beta \cdot \sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$.
Since
$$\begin{aligned} \Delta^\varepsilon_\beta &= \sum^{\text{SP}}_{\sigma \in \Sigma} [e^{-\beta \cdot \varepsilon_\sigma}] \\ &= \sum^{\text{SP}}_{\sigma \in \Sigma} [e^\beta \cdot e^{-\beta \cdot (\varepsilon_\sigma + 1)}] \\ &= e^\beta \cdot \sum^{\text{SP}}_{\sigma \in \Sigma} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}] \\ &= e^\beta \cdot \sum_{k=1}^{\infty} \sum_{\sigma \in \varepsilon^*[k-1;k)} [e^{-\beta \cdot (\varepsilon_\sigma + 1)}] \\ &\leqslant e^\beta \cdot \sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty, \end{aligned}$$
we get: $\beta \in \text{DF}_\varepsilon$.
End of proof of $\Leftarrow$. □

THEOREM 24.27. **Let** *Σ be a set,* $\varepsilon : \Sigma \to [0;\infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \#(\,\varepsilon^*[k-1;k)\,)$.
Assume: $\forall k \in \mathbb{N}$, $n_k \geqslant e^{k^2}$. *Then:* $\text{DF}_\varepsilon = \varnothing$.

Proof. Since $\forall k \in \mathbb{N}$, $n_k \geqslant e^{k^2} > 1$, we get: $\sum_{k=1}^{\infty} n_k = \infty$.
Since $\#(\varepsilon^*[0;\infty)) = \sum_{k=1}^{\infty} [\#(\varepsilon^*[k-1;k))] = \sum_{k=1}^{\infty} n_k = \infty$,
it follows, from Theorem 24.5, that: $\text{DF}_\varepsilon \subseteq (0;\infty)$.
It therefore suffices to show: $\forall \beta \in (0;\infty)$, $\beta \notin \text{DF}_\varepsilon$.
Given $\beta \in (0;\infty)$, **want:** $\beta \notin \text{DF}_\varepsilon$.

Since, as $k \to \infty$, $e^{k^2-\beta\cdot k} \to \infty$, we get: $\sum_{k=1}^{\infty}[e^{k^2-\beta\cdot k}] = \infty$.
Since $\sum_{k=1}^{\infty}[n_k e^{-\beta\cdot k}] \geqslant \sum_{k=1}^{\infty}[e^{k^2}e^{-\beta\cdot k}] = \sum_{k=1}^{\infty}[e^{k^2-\beta\cdot k}] = \infty$,
and since $\beta \in (0;\infty) \subseteq [0;\infty)$,
by Theorem 24.26, we get: $\beta \notin \mathrm{DF}_\varepsilon$. □

Recall (§2): $\forall t \in \mathbb{R}$, $\lfloor t \rfloor$ denotes the floor of t.

THEOREM 24.28. *For all* $k \in \mathbb{N}$, **let** $n_k := \lfloor e^{k^2} + 1 \rfloor$.
Let $\Sigma := \{(k,j) \in \mathbb{N} \times \mathbb{N} \,|\, j \leqslant n_k\}$.
Define $\varepsilon : \Sigma \to [0..\infty)$ *by:* $\forall (k,j) \in \Sigma$, $\varepsilon(k,j) = k-1$.
Then: $\mathrm{DF}_\varepsilon = \varnothing$.

Proof. We have: $\forall k \in \mathbb{N}$, $n_k \geqslant e^{k^2}$.
By Theorem 24.25, we get: $\forall k \in \mathbb{N}$, $\#(\,\varepsilon^*[k-1;k)\,) = n_k$.
Then, by Theorem 24.27, we get: $\mathrm{DF}_\varepsilon = \varnothing$. □

THEOREM 24.29. Let Σ *be a set,* $\varepsilon : \Sigma \to [0;\infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \#(\,\varepsilon^*[k-1;k)\,)$. **Let** $\beta_0 \in [0;\infty)$.
Assume: *as* $k \to \infty$, $n_k e^{-\beta_0\cdot k} \to 1$. *Then:* $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$.

Proof. Since as $k \to \infty$, $n_k e^{-\beta_0\cdot k} \to 1$, we get:
$\#\{k \in \mathbb{N} \,|\, n_k e^{-\beta_0\cdot k} = 0\} < \infty$.
Then: $\#\{k \in \mathbb{N} \,|\, n_k = 0\} < \infty$.
Then $\#\{k \in \mathbb{N} \,|\, n_k \geqslant 1\} = \infty$, and so $\sum_{k=1}^{\infty} n_k = \infty$.
Since $\#(\varepsilon^*[0;\infty)) = \sum_{k=1}^{\infty}[\#(\varepsilon^*[k-1;k))] = \sum_{k=1}^{\infty} n_k = \infty$,
it follows, from Theorem 24.5, that: $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$.
Since $\mathrm{DF}_\varepsilon \subseteq (0;\infty) \subseteq [0;\infty)$, we get: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = \mathrm{DF}_\varepsilon$.
Since $\beta_0 \in [0;\infty)$, we get: $(\beta_0;\infty) \subseteq (0;\infty)$.
Since $(\beta_0;\infty) \subseteq (0;\infty) \subseteq [0;\infty)$, we get: $(\beta_0;\infty) \bigcap [0;\infty) = (\beta_0;\infty)$.
We have: $\forall \beta \in \mathbb{R}$, $\forall k \in \mathbb{N}$, $[n_k e^{-\beta\cdot k}]\,/\,[e^{-(\beta-\beta_0)\cdot k}] = n_k e^{-\beta_0\cdot k}$.
By hypothesis, as $k \to \infty$, $n_k e^{-\beta_0\cdot k} \to 1$.
Then: $\forall \beta \in \mathbb{R}$, as $k \to \infty$, $[n_k e^{-\beta\cdot k}]\,/\,[e^{-(\beta-\beta_0)\cdot k}] \to 1$.
Then: $\forall \beta \in \mathbb{R}$, $(\sum_{k=1}^{\infty}[n_k e^{-\beta\cdot k}] < \infty) \Leftrightarrow (\sum_{k=1}^{\infty}[e^{-(\beta-\beta_0)\cdot k}] < \infty)$.
Also, $\forall \beta \in \mathbb{R}$, $(\,\beta \in (\beta_0;\infty)\,) \Leftrightarrow (\sum_{k=1}^{\infty}[e^{-(\beta-\beta_0)\cdot k}] < \infty)$.
Then: $\forall \beta \in \mathbb{R}$, $(\sum_{k=1}^{\infty}[n_k e^{-\beta\cdot k}] < \infty) \Leftrightarrow (\,\beta \in (\beta_0;\infty)\,)$.
Then, by Theorem 24.26,
$\forall \beta \in [0;\infty)$, $(\beta \in \mathrm{DF}_\varepsilon) \Leftrightarrow (\,\beta \in (\beta_0;\infty)\,)$.
Then: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = (\beta_0;\infty) \bigcap [0;\infty)$.
Then: $\mathrm{DF}_\varepsilon = \mathrm{DF}_\varepsilon \bigcap [0;\infty) = (\beta_0;\infty) \bigcap [0;\infty) = (\beta_0;\infty)$. □

THEOREM 24.30. Let $\beta_0 \in [0;\infty)$. *For all* $k \in \mathbb{N}$, **let** $n_k := \lfloor e^{\beta_0\cdot k} \rfloor$.
Let $\Sigma := \{(k,j) \in \mathbb{N} \times \mathbb{N} \,|\, j \leqslant n_k\}$.

Define $\varepsilon : \Sigma \to [0..\infty)$ *by:* $\forall (k,j) \in \Sigma$, $\varepsilon(k,j) = k-1$.
Then: $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$.

Proof. We have: as $k \to \infty$, $n_k e^{-\beta_0 \cdot k} \to 1$.
By Theorem 24.25, we get: $\forall k \in \mathbb{N}$, $\#(\,\varepsilon^*[k-1;k)\,) = n_k$.
Then, by Theorem 24.29, we get: $\mathrm{DF}_\varepsilon = (\beta_0;\infty)$. □

THEOREM 24.31. Let Σ *be a set,* $\varepsilon : \Sigma \to [0;\infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \#(\,\varepsilon^*[k-1;k)\,)$.
Let $p \in (1;\infty)$, $\beta_0 \in (0;\infty)$.
Assume: *as* $k \to \infty$, $k^p n_k e^{-\beta_0 \cdot k} \to 1$. *Then:* $\mathrm{DF}_\varepsilon = [\beta_0;\infty)$.

Proof. Since, as $k \to \infty$, $k^p n_k e^{-\beta_0 \cdot k} \to 1$, we get:
$$\#\{k \in \mathbb{N} \mid k^p n_k e^{-\beta_0 \cdot k} = 0\} < \infty.$$
Then $\#\{k \in \mathbb{N} \mid n_k = 0\} < \infty$.
Then $\#\{k \in \mathbb{N} \mid n_k \geqslant 1\} = \infty$, and so $\sum_{k=1}^\infty n_k = \infty$.
Since $\#(\varepsilon^*[0;\infty)) = \sum_{k=1}^\infty [\#(\varepsilon^*[k-1;k))] = \sum_{k=1}^\infty n_k = \infty$,
it follows, from Theorem 24.5, that: $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$.
Since $\mathrm{DF}_\varepsilon \subseteq (0;\infty) \subseteq [0;\infty)$, we get: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = \mathrm{DF}_\varepsilon$.
Since $\beta_0 \in (0;\infty)$, we get: $[\beta_0;\infty) \subseteq (0;\infty)$.
Since $[\beta_0;\infty) \subseteq (0;\infty) \subseteq [0;\infty)$, we get: $[\beta_0;\infty) \bigcap [0;\infty) = [\beta_0;\infty)$.
We have: $\forall \beta \in \mathbb{R}$, $\forall k \in \mathbb{N}$, $[n_k e^{-\beta \cdot k}]/[k^{-p} e^{-(\beta-\beta_0)\cdot k}] = k^p n_k e^{-\beta_0 \cdot k}$.
By hypothesis, as $k \to \infty$, $k^p n_k e^{-\beta_0 \cdot k} \to 1$.
Then: $\forall \beta \in \mathbb{R}$, as $k \to \infty$, $[n_k e^{-\beta \cdot k}]/[k^{-p} e^{-(\beta-\beta_0)\cdot k}] \to 1$.
Then: $\forall \beta \in \mathbb{R}$, $(\sum_{k=1}^\infty [n_k e^{-\beta \cdot k}] < \infty) \Leftrightarrow (\sum_{k=1}^\infty [k^{-p} e^{-(\beta-\beta_0)\cdot k}] < \infty)$.
Also, since $p \in (1;\infty)$, we get:
$$\forall \beta \in \mathbb{R}, \qquad (\,\beta \in [\beta_0;\infty)\,) \Leftrightarrow (\textstyle\sum_{k=1}^\infty [k^{-p} e^{-(\beta-\beta_0)\cdot k}] < \infty).$$
Then: $\forall \beta \in \mathbb{R}$, $(\sum_{k=1}^\infty [n_k e^{-\beta \cdot k}] < \infty) \Leftrightarrow (\,\beta \in [\beta_0;\infty)\,)$.
Then, by Theorem 24.26,
$$\forall \beta \in [0;\infty), \qquad (\beta \in \mathrm{DF}_\varepsilon) \Leftrightarrow (\,\beta \in [\beta_0;\infty)\,).$$
Then: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = [\beta_0;\infty) \bigcap [0;\infty)$.
Then: $\mathrm{DF}_\varepsilon = \mathrm{DF}_\varepsilon \bigcap [0;\infty) = [\beta_0;\infty) \bigcap [0;\infty) = [\beta_0;\infty)$. □

THEOREM 24.32. Let $\beta_0 \in (0;\infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \lfloor k^{-2} e^{\beta_0 \cdot k} \rfloor$.
Let $\Sigma := \{(k,j) \in \mathbb{N} \times \mathbb{N} \,|\, j \leqslant n_k\}$.
Define $\varepsilon : \Sigma \to [0..\infty)$ *by:* $\forall (k,j) \in \Sigma$, $\varepsilon(k,j) = k-1$.
Then: $\mathrm{DF}_\varepsilon = [\beta_0;\infty)$.

Proof. We have: as $k \to \infty$, $k^2 n_k e^{-\beta_0 \cdot k} \to 1$.
By Theorem 24.25, we get: $\forall k \in \mathbb{N}$, $\#(\,\varepsilon^*[k-1;k)\,) = n_k$.
Then, by Theorem 24.31, we get: $\mathrm{DF}_\varepsilon = [\beta_0;\infty)$. $\square$

Let Σ be an infinite set. **Let** $\varepsilon : \Sigma \to [0;\infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \#(\,\varepsilon^*[k-1;k)\,)$.
In many applications, the sequence $n_1, n_2, \ldots$ is subexponential.
By the next theorem, whenever that happens, we get: $\mathrm{DF}_\varepsilon = (0;\infty)$.

THEOREM 24.33. Let Σ *be an infinite set,* $\varepsilon : \Sigma \to [0;\infty)$.
For all $k \in \mathbb{N}$, **let** $n_k := \#(\,\varepsilon^*[k-1;k)\,)$.
Assume: $\forall \beta \in (0;\infty)$, *as* $k \to \infty$, $n_k e^{-\beta \cdot k} \to 0$. *Then:* $\mathrm{DF}_\varepsilon = (0;\infty)$.

Proof. Since $\varepsilon : \Sigma \to [0;\infty)$, we get: $\varepsilon^*[0;\infty) = \Sigma$.
So, since Σ is infinite, we get: $\varepsilon^*[0;\infty)$ is infinite.
It follows, from Theorem 24.5, that: $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$.

Want: $(0;\infty) \subseteq \mathrm{DF}_\varepsilon$.
Given $\beta \in (0;\infty)$, **want:** $\beta \in \mathrm{DF}_\varepsilon$.

Since $\beta \in (0;\infty) \subseteq [0;\infty)$, by Theorem 24.26,
it suffices to show: $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] < \infty$.
Let $\beta' := \beta/2$. Since $\beta \in (0;\infty)$, we get: $\beta' \in (0;\infty)$.
Then, by hypothesis, we have: as $k \to \infty$, $n_k e^{-\beta' \cdot k} \to 0$.
It follows that: $\{n_k e^{-\beta' \cdot k} \,|\, k \in \mathbb{N}\}$ is bounded.
Choose $M \in \mathbb{R}$ s.t., $\forall k \in \mathbb{N}$, $n_k e^{-\beta' \cdot k} \leqslant M$.
Since $\beta' \in (0;\infty)$, it follows that $0 < 1 - e^{-\beta'} < 1$
and that $e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots = e^{-\beta'}/(1 - e^{-\beta'})$.
Then: $e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots < \infty$.
Then: $M \cdot (e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots) < \infty$.
Then: $\sum_{k=1}^{\infty} [n_k e^{-\beta \cdot k}] = \sum_{k=1}^{\infty} [n_k e^{-2\beta' \cdot k}]$
$= \sum_{k=1}^{\infty} [(n_k e^{-\beta' \cdot k}) \cdot e^{-\beta' \cdot k}] \leqslant \sum_{k=1}^{\infty} [M e^{-\beta' \cdot k}] = M \cdot \sum_{k=1}^{\infty} [e^{-\beta' \cdot k}]$
$= M \cdot (e^{-\beta'} + e^{-2\beta'} + e^{-3\beta'} + \cdots) < \infty$. $\square$

The next theorem is a corollary of Theorem 24.33.

THEOREM 24.34. Let Σ *be an infinite set,* $\varepsilon : \Sigma \to [0..\infty)$.
Assume: ε *is injective.* *Then:* $\mathrm{DF}_\varepsilon = (0;\infty)$.

Example: **Let** $\Sigma := [0..\infty)$. **Define** $\varepsilon : \Sigma \to \mathbb{R}$ by: $\forall \sigma \in \Sigma$, $\varepsilon(\sigma) = \sigma$.
Then, $\forall k \in \mathbb{N}$, $\varepsilon^*[k-1;k) = \{k-1\}$, and so $\#(\,\varepsilon^*[k-1;k)\,) = 1$.
Then, by Theorem 24.34, we get: $\mathrm{DF}_\varepsilon = (0;\infty)$.

DEFINITION 24.35. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Then: $\boxed{\mathrm{IDF}_\varepsilon}$ *denotes the interior in* $\mathbb{R}$ *of* DF_ε.

THEOREM 24.36. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*[0;\infty)$ *is infinite and* $\mathrm{DF}_\varepsilon \neq \varnothing$.
Let $\beta_0 := \inf \mathrm{DF}_\varepsilon$. *Then:* $\mathrm{IDF}_\varepsilon = (\beta_0;\infty)$ *and* $\beta_0 \in [0;\infty)$.

The preceding theorem is a corollary of Theorem 24.22.

THEOREM 24.37. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$. *Assume:* $\mathrm{DF}_\varepsilon \neq \varnothing$.
Then one of the following holds:

	(i)		$\mathrm{IDF}_\varepsilon = \mathbb{R}$
or	*(ii)* $\exists \beta_0 \in [0;\infty)$	*s.t.*	$\mathrm{IDF}_\varepsilon = (\beta_0;\infty)$
or	*(iii)* $\exists \beta_0 \in [0;\infty)$	*s.t.*	$\mathrm{IDF}_\varepsilon = (-\infty;-\beta_0)$.

The preceding theorem is a corollary of Theorem 24.24.

THEOREM 24.38. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Then: $(\ \mathrm{DF}_\varepsilon = \varnothing\ ) \quad \Leftrightarrow \quad (\ \mathrm{IDF}_\varepsilon = \varnothing\ )$.

Proof. Since $\mathrm{IDF}_\varepsilon \subseteq \mathrm{DF}_\varepsilon$, we get:
$(\ \mathrm{DF}_\varepsilon = \varnothing\ ) \quad \Rightarrow \quad (\ \mathrm{IDF}_\varepsilon = \varnothing\ )$.
Want: $(\ \mathrm{DF}_\varepsilon \neq \varnothing\ ) \quad \Rightarrow \quad (\ \mathrm{IDF}_\varepsilon \neq \varnothing\ )$.
Assume $\mathrm{DF}_\varepsilon \neq \varnothing$. **Want:** $\mathrm{IDF}_\varepsilon \neq \varnothing$.
By Theorem 24.37, one of the following is true:

	(i)		$\mathrm{IDF}_\varepsilon = \mathbb{R}$
or	(ii) $\exists \beta_0 \in [0;\infty)$	s.t.	$\mathrm{IDF}_\varepsilon = (\beta_0;\infty)$
or	(iii) $\exists \beta_0 \in [0;\infty)$	s.t.	$\mathrm{IDF}_\varepsilon = (-\infty;-\beta_0)$.
Then:			$\mathrm{IDF}_\varepsilon \neq \varnothing$.

□

THEOREM 24.39. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Then: $(\ \mathrm{IDF}_\varepsilon = \mathbb{R}\ ) \Leftrightarrow (\ \mathrm{DF}_\varepsilon = \mathbb{R}\ ) \Leftrightarrow (\ \Sigma$ *is finite* $)$.

Proof. Since $\mathrm{IDF}_\varepsilon \subseteq \mathrm{DF}_\varepsilon \subseteq \mathbb{R}$, we get:
$(\ \mathrm{IDF}_\varepsilon = \mathbb{R}\ ) \Rightarrow (\ \mathrm{DF}_\varepsilon = \mathbb{R}\ )$.
By Theorem 24.11, $(\ \mathrm{DF}_\varepsilon = \mathbb{R}\ ) \Rightarrow (\ \Sigma$ is finite $)$.
It remains to show: $(\ \Sigma$ is finite $) \quad \Rightarrow \quad (\ \mathrm{IDF}_\varepsilon = \mathbb{R}\ )$.
Assume: Σ is finite. **Want:** $\mathrm{IDF}_\varepsilon = \mathbb{R}$.
Since the interior in $\mathbb{R}$ of $\mathbb{R}$ is equal to $\mathbb{R}$,
it suffices to show: $\mathrm{DF}_\varepsilon = \mathbb{R}$.
Since $\mathrm{DF}_\varepsilon \subseteq \mathbb{R}$, **want:** $\mathbb{R} \subseteq \mathrm{DF}_\varepsilon$.
Given $\beta \in \mathbb{R}$, **want:** $\beta \in \mathrm{DF}_\varepsilon$.

Since Σ is finite, we get: $\sum_{\sigma\in\Sigma}^{\mathrm{SP}}[e^{-\beta\cdot(\varepsilon(\sigma))}] < \infty$.
Since $\Delta_\beta^\varepsilon = \sum_{\sigma\in\Sigma}^{\mathrm{SP}}[e^{-\beta\cdot(\varepsilon(\sigma))}] < \infty$, we get: $\beta \in \mathrm{DF}_\varepsilon$. $\square$

THEOREM 24.40. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\exists \beta_0 \in [0;\infty)$ *s.t.* $\mathrm{IDF}_\varepsilon = (\beta_0;\infty)$. *Then:* ε *is* ∞*-proper.*

Proof. We have: $\mathrm{IDF}_\varepsilon \bigcap [0;\infty) \neq \varnothing$.
So, since $\mathrm{DF}_\varepsilon \supseteq \mathrm{IDF}_\varepsilon$, we get: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) \neq \varnothing$.
Then, by Theorem 24.8, we get: ε is ∞-proper. $\square$

THEOREM 24.41. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\exists \beta_0 \in [0;\infty)$ *s.t.* $\mathrm{IDF}_\varepsilon = (-\infty;-\beta_0)$.
Then: ε *is* $(-\infty)$*-proper.*

Proof. We have: $\mathrm{IDF}_\varepsilon \bigcap (-\infty;0] \neq \varnothing$.
So, since $\mathrm{DF}_\varepsilon \supseteq \mathrm{IDF}_\varepsilon$, we get: $\mathrm{DF}_\varepsilon \bigcap (-\infty;0] \neq \varnothing$.
Then, by Theorem 24.9, we get: ε is $(-\infty)$-proper. $\square$

THEOREM 24.42. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon \neq \varnothing$. *Then:*
(ε *is* ∞*-proper*) *or* (ε *is* $(-\infty)$*-proper*).

Proof. By Theorem 24.37,
one of the following holds:

$$\begin{array}{llll} & \text{(i)} & & \mathrm{IDF}_\varepsilon = \mathbb{R} \\ \text{or} & \text{(ii)} \ \exists \beta_0 \in [0;\infty) & \text{s.t.} & \mathrm{IDF}_\varepsilon = (\beta_0;\infty) \\ \text{or} & \text{(iii)} \ \exists \beta_0 \in [0;\infty) & \text{s.t.} & \mathrm{IDF}_\varepsilon = (-\infty;-\beta_0). \end{array}$$

MORE LATER. $\square$

THEOREM 24.43. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathrm{IDF}_\varepsilon$, $\rho \in [0;\infty)$.
Assume ε *is* ∞*-proper. Then:* $\overline{\mathrm{X}}^\rho \mathrm{S}_\beta^\varepsilon < \infty$.

Proof. USE Theorem 24.36.
MORE LATER. $\square$

We can remove the ∞-properness hypothesis from Theorem 24.43:

THEOREM 24.44. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathrm{IDF}_\varepsilon$, $\rho \in [0;\infty)$.
Then: $\overline{\mathrm{X}}^\rho \mathrm{S}_\beta^\varepsilon < \infty$.

Proof. By Theorem 24.42, we have:
(ε is ∞-proper) or (ε is $(-\infty)$-proper).
USE Theorem 24.19. MORE LATER. $\square$

25. Convergence, complex-differentiation, C^ω results

Recall (§2): the notations $\mathbb{I}_f$ and f^*A.
Recall (§8): for $f : S \to [0;\infty]$, the notation $\sum_{x\in S}^{\mathrm{SP}} [f(x)]$.
Recall (§8): for $f : S \to \mathbb{C}$, the notation $\sum_{x\in S} [f(x)]$.
MOVE TO §8:

THEOREM 25.1. Let *I be a set, $a : I \to [0;\infty]$. For all $i \in I$,* **let** *$a_i := a(i)$.*
Let *$I_1, I_2, \ldots \subseteq I$. Assume $I_1 \subseteq I_2 \subseteq \cdots$ and $I_1 \bigcup I_2 \bigcup \cdots = I$.*
Then: as $n \to \infty$, $\sum_{i\in I_n} a_i \to \sum_{i\in I} a_i$.

THEOREM 25.2. Let *Σ be an infinite set, $\varepsilon : \Sigma \to \mathbb{R}$.*
Assume: $\varnothing \neq \mathrm{DF}_\varepsilon \subseteq (0;\infty)$, $\gamma \in \mathrm{IDF}_\varepsilon$, $\rho \in [0;\infty)$.
For all $n \in \mathbb{N}$, **let** *$\Sigma_n := \varepsilon^*(-\infty; n]$ and* **let** *$\varepsilon_n := \varepsilon|\Sigma_n$.*
Then: ($\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon < \infty$) and (as $n \to \infty$, $\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^{\varepsilon_n} \to \overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon$).

Proof. Since $\mathrm{DF}_\varepsilon \subseteq (0;\infty) \subseteq [0;\infty)$, we get: $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = \mathrm{DF}_\varepsilon$.
Since $\mathrm{DF}_\varepsilon \bigcap [0;\infty) = \mathrm{DF}_\varepsilon \neq \varnothing$, by Theorem 24.8,
we get: ε is ∞-proper.
Let $\beta_0 := \inf \mathrm{DF}_\varepsilon$. By Theorem 24.22, we get: $\mathrm{IDF}_\varepsilon = (\beta_0;\infty)$.
Since $\gamma \in (\beta_0;\infty)$, by Theorem 24.19, we get: $\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon < \infty$.
It remains to show: as $n \to \infty$, $\mathrm{X}^\rho \mathrm{S}_\gamma^{\varepsilon_n} \to \mathrm{X}^\rho \mathrm{S}_\gamma^\varepsilon$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Define $f : \Sigma \to [0;\infty)$ by: $\forall \sigma \in \Sigma$, $f(\sigma) = |\varepsilon_\sigma|^\rho \cdot e^{-\gamma\cdot\varepsilon_\sigma}$.
By Theorem 25.1, as $n \to \infty$, $\sum_{\sigma\in\Sigma_n} [f(\sigma)] \to \sum_{\sigma\in\Sigma} [f(\sigma)]$.
So, since $\forall n \in \mathbb{N}, \sum_{\sigma\in\Sigma_n} [f(\sigma)] = \overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^{\varepsilon_n}$
and since $\sum_{\sigma\in\Sigma} [f(\sigma)] = \overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon$,
we get: as $n \to \infty$, $\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^{\varepsilon_n} \to \overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon$. □

Recall (§2): the notations $\Re(z)$ and $\Im(z)$.
Note: $\forall z \in \mathbb{C}$, $|e^z| = e^{\Re(z)}$.
Also, $\forall S \subseteq \mathbb{R}$, $\Re^* S = \{x + y\sqrt{-1} \,|\, x \in S\}$.

THEOREM 25.3. Let *U be an open subset of $\mathbb{C}$, $g, h : U \to \mathbb{C}$.*
Let *$f_1, f_2, \ldots : U \to \mathbb{C}$ all be complex-differentiable on U.*
Assume, as $n \to \infty$, we have:
both $f_n \to g$ pointwise on U and $f_n' \to h$ uniformly on U.
Then: g is complex-differentiable on U and $g' = h$ on U.

Theorem 25.3 is a standard result about
commuting of limit and differentiation. We omit proof.

It will be helpful to extend Definition 24.14 to $\mathbb{C}$:

DEFINITION 25.4. Let *Σ be a set,* $\varepsilon:\Sigma\to\mathbb{C}$, $z\in\mathbb{C}$.
For all $\sigma\in\Sigma$, **let** $\varepsilon_\sigma:=\varepsilon(\sigma)$.
Then, $\forall\rho\in[0;\infty)$, *the* $\boxed{\rho\text{-}\mathbf{absolute\text{-}exponent}\ (z,\varepsilon)\text{-}\mathbf{sum}}$ *is:*

$$\boxed{\overline{\mathrm{X}}^\rho\mathrm{S}_z^\varepsilon} \;:=\; \sum\nolimits_{\sigma\in\Sigma}^{\mathrm{SP}}\big[\,|\varepsilon_\sigma|^\rho\cdot|e^{-z\cdot\varepsilon_\sigma}|\,\big] \;\in\; [0;\infty].$$

Also, $\forall\rho\in[0..\infty)$, *if* $\overline{\mathrm{X}}^\rho\mathrm{S}_z^\varepsilon<\infty$,
then the $\boxed{\rho\text{-}\mathbf{exponent}\ (z,\varepsilon)\text{-}\mathbf{sum}}$ *is:*

$$\boxed{\mathrm{X}^\rho\mathrm{S}_z^\varepsilon} \;:=\; \sum\nolimits_{\sigma\in\Sigma}\big[\,\varepsilon_\sigma^\rho\cdot e^{-z\cdot\varepsilon_\sigma}\,\big] \;\in\; \mathbb{C}.$$

We have: $\forall$set Σ, $\forall\varepsilon:\Sigma\to\mathbb{R}$, $\forall z\in\mathbb{C}$, $\forall\rho\in[0;\infty)$,

$$\overline{\mathrm{X}}^\rho\mathrm{S}_z^\varepsilon \;=\; \overline{\mathrm{X}}^\rho\mathrm{S}_{\Re(z)}^\varepsilon.$$

By Theorem 24.44, we have:

$\forall$set Σ, $\forall\varepsilon:\Sigma\to\mathbb{R}$, $\forall\beta\in\mathrm{IDF}_\varepsilon$, $\forall$real $\rho\geqslant 0$, $\overline{\mathrm{X}}^\rho\mathrm{S}_\beta^\varepsilon<\infty$.

Then: $\forall$set Σ, $\forall\varepsilon:\Sigma\to\mathbb{R}$, $\forall z\in\Re^*\mathrm{IDF}_\varepsilon$, $\forall\rho\in[0;\infty)$,

$\overline{\mathrm{X}}^\rho\mathrm{S}_z^\varepsilon\;<\;\infty$, and so $\mathrm{X}^\rho\mathrm{S}_z^\varepsilon$ is defined.

DEFINITION 25.5. Let *Σ be a set,* $\varepsilon:\Sigma\to\mathbb{R}$, $\rho\in[0..\infty)$.
For all $\sigma\in\Sigma$, **let** $\varepsilon_\sigma:=\varepsilon(\sigma)$.
Then $\boxed{\mathrm{X}^\rho\mathrm{S}_\bullet^\varepsilon}$: $\mathrm{IDF}_\varepsilon\to\mathbb{R}$ *is defined by:*

$\forall\beta\in\mathrm{IDF}_\varepsilon$, $(\mathrm{X}^\rho\mathrm{S}_\bullet^\varepsilon)(\beta)=\mathrm{X}^\rho\mathrm{S}_\beta^\varepsilon$.

Also, $\boxed{\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon}$: $\Re^*\mathrm{IDF}_\varepsilon\to\mathbb{C}$ *is defined by:*

$\forall z\in\Re^*\mathrm{IDF}_\varepsilon$, $(\mathrm{X}^\rho\mathrm{S}_\bullet^\varepsilon)(z)=\mathrm{X}^\rho\mathrm{S}_z^\varepsilon$.

THEOREM 25.6. Let *Σ be a set,* $\varepsilon:\Sigma\to\mathbb{R}$.
Assume: $\varnothing\neq\mathrm{DF}_\varepsilon\subseteq(0;\infty)$. **Let** $\gamma\in\mathrm{IDF}_\varepsilon$.
For all $n\in\mathbb{N}$, **let** $\Sigma_n:=\varepsilon^*(-\infty;n]$ *and* **let** $\varepsilon_n:=\varepsilon|\Sigma_n$.
Then: as $n\to\infty$, $\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^{\varepsilon_n}\to\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon$ *uniformly on* $\Re^*(\gamma;\infty)$.

Proof. **Given** $\delta>0$, **want:** $\exists n_0\in\mathbb{N}$ s.t., $\forall n\in[n_0..\infty)$,

$$|\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon-\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^{\varepsilon_n}|<\delta \quad \text{on } \Re^*(\gamma;\infty).$$

Let $\beta_0:=\inf\mathrm{DF}_\varepsilon$. Then $\mathrm{IDF}_\varepsilon=(\beta_0;\infty)$.
On $\Re^*(\gamma;\infty)$, $|\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon-\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^{\varepsilon_n}|\leqslant\overline{\mathrm{X}}^\rho\mathrm{S}_\gamma^\varepsilon-\overline{\mathrm{X}}^\rho\mathrm{S}_\gamma^{\varepsilon_n}$.
Also, by Theorem 25.2, we get: as $n\to\infty$, $\overline{\mathrm{X}}^\rho\mathrm{S}_\gamma^\varepsilon-\overline{\mathrm{X}}^\rho\mathrm{S}_\gamma^{\varepsilon_n}\to 0$.
Then: as $n\to\infty$, $\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^{\varepsilon_n}\to\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon$ uniformly on $\Re^*(\gamma;\infty)$. $\square$

THEOREM 25.7. Let *Σ be a set,* $\varepsilon:\Sigma\to\mathbb{R}$.
Assume: Σ is finite. **Let** $\rho\in[0..\infty)$.
Then: $\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon$ *is complex-differentiable on* $\mathbb{C}$.
and $(\mathrm{X}^\rho\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon)'=-\mathrm{X}^{\rho+1}\mathrm{S}_{\bullet\mathbb{C}}^\varepsilon$ *on* $\mathbb{C}$.

Proof. For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Since Σ is finite, we have:
$$\forall z \in \mathbb{C}, \qquad (\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})(z) = \textstyle\sum_{\sigma\in\Sigma} [\varepsilon_\sigma^\rho \cdot e^{-z\cdot\varepsilon_\sigma}]$$
Since Σ is finite, we may differentiate term-by-term, yielding:
$$\forall z \in \mathbb{C}, \qquad (\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})'(z) = \textstyle\sum_{\sigma\in\Sigma} [\varepsilon_\sigma^\rho \cdot e^{-z\cdot\varepsilon_\sigma} \cdot (-\varepsilon_\sigma)]$$
Thus $\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ is complex-differentiable on $\mathbb{C}$.
It remains to show: $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ on $\mathbb{C}$.
Since $\forall z \in \mathbb{C}$, we have: $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})'(z) = \sum_{\sigma\in\Sigma} [\varepsilon_\sigma^\rho \cdot e^{-z\cdot\varepsilon_\sigma} \cdot (-\varepsilon_\sigma)]$
$$= -\textstyle\sum_{\sigma\in\Sigma} [\varepsilon_\sigma^{\rho+1} \cdot e^{-z\cdot\varepsilon_\sigma}] = (-X^{\rho+1} S^\varepsilon_{\bullet\mathbb{C}})(z),$$
we conclude: $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ on $\mathbb{C}$. $\square$

In Theorem 25.7, we assumed Σ was finite.
We next investigate what happens without that assumption:

THEOREM 25.8. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varnothing \neq \mathrm{DF}_\varepsilon \subseteq (0;\infty)$. **Let** $\rho \in [0..\infty)$.
Then: $\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ *is complex-differentiable on* $\Re^*\mathrm{IDF}_\varepsilon$.
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ *on* $\Re^*\mathrm{IDF}_\varepsilon$.

Proof. **Let** $\beta_0 := \inf \mathrm{DF}_\varepsilon$. Then $\mathrm{IDF}_\varepsilon = (\beta_0;\infty)$.
For all $n \in \mathbb{N}$, **let** $\Sigma_n := \varepsilon^*(-\infty;n]$ and **let** $\varepsilon_n := \varepsilon|\Sigma_n$.
Given $z \in \Re^*(\beta_0;\infty)$, **want:** $\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ is complex-differentiable at z
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})'(z) = -(\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})(z)$.
Let $\beta := \Re(z)$. **Let** $\gamma := (\beta_0+\beta)/2$. Then $\beta_0 < \gamma < \beta$.
It suffices to show: $\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ is complex-differentiable on $\Re^*(\gamma;\infty)$
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ on $\Re^*(\gamma;\infty)$.
By Theorem 25.6, as $n \to \infty$, we have both
$\mathrm{X}^\rho \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}} \to \mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ uniformly on $\Re^*(\gamma;\infty)$
and $\mathrm{X}^{\rho+1} \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}} \to \mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ uniformly on $\Re^*(\gamma;\infty)$.
For all $n \in \mathbb{N}$, since Σ_n is finite, by Theorem 25.7, we see that
$\mathrm{X}^\rho \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}}$ is complex-differentiable at z
and $(\mathrm{X}^\rho \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1} \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}}$ on $\Re^*(\beta_0;\infty)$.
Then, as $n \to \infty$, we have both
$\mathrm{X}^\rho \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}} \to \mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ pointwise on $\Re^*(\gamma;\infty)$
and $(\mathrm{X}^\rho \mathrm{S}^{\varepsilon_n}_{\bullet\mathbb{C}})' \to -\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ uniformly on $\Re^*(\gamma;\infty)$.
Then, by Theorem 25.3, we get:
$\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ is complex-differentiable on $\Re^*(\gamma;\infty)$
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1} \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ on $\Re^*(\gamma;\infty)$. $\square$

Replacing ε with $-\varepsilon$ in the preceding theorem yields:

THEOREM 25.9. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varnothing \neq \mathrm{DF}_\varepsilon \subseteq (-\infty; 0)$. **Let** $\rho \in [0..\infty)$.
Then: $\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ *is complex-differentiable on* $\Re^*\mathrm{IDF}_\varepsilon$.
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1}\mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ *on* $\Re^*\mathrm{IDF}_\varepsilon$.

Finally, we can remove the hypothesis on DF_ε:

THEOREM 25.10. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\rho \in [0..\infty)$.
Then: $\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ *is complex-differentiable on* $\Re^*\mathrm{IDF}_\varepsilon$.
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}})' = -\mathrm{X}^{\rho+1}\mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ *on* $\Re^*\mathrm{IDF}_\varepsilon$.

Proof. Result holds if Σ is finite, so assume Σ is infinite.
If $\mathrm{DF}_\varepsilon = \varnothing$, there is nothing to prove, so assume $\mathrm{DF}_\varepsilon \neq \varnothing$.
By Theorem 24.42, either ε is ∞-proper or ε is $(-\infty)$-proper.
In the first case, since Σ is infinite, we get $\varepsilon^*[0;\infty)$ is infinite,
so, by Theorem 24.5, we get $\mathrm{DF}_\varepsilon \subseteq (0;\infty)$,
so, by Theorem 25.8, we are done.
In the second case, since Σ is infinite, we get $\varepsilon^*(-\infty;0]$ is infinite,
so, by Theorem 25.9, we get $\mathrm{DF}_\varepsilon \subseteq (-\infty;0)$,
so, by Theorem 24.6, we are done. $\square$

Recall (§2): "C^ω" means "real-analytic".

THEOREM 25.11. Let V *be an open subset of* $\mathbb{C}$, $U := V \bigcap \mathbb{R}$.
Let $g : V \to \mathbb{C}$. *Assume* $g_* U \subseteq \mathbb{R}$. **Let** $f := g|U$.
Assume: $g : V \to \mathbb{C}$ *is complex-differentiable on* V.
Then: $f : U \to \mathbb{R}$ *is* C^ω *on* U.
Also, $f' = g'|U$.

Proof. Since complex-differentiable implies complex-analytic implies C^ω,
we get: g is C^ω on V.
Let $\iota : U \to V$ denote the inclusion. Then ι is C^ω.
Then $g \circ \iota$ is C^ω on U. So, since $f = g \circ \iota$, we get: f is C^ω on U.
It remains to show: $f' = g'|U$.
Since $f = g \circ \iota$ and since $\iota' = 1$ on U,
by the Chain Rule, we get: $f' = g' \circ \iota$.
So, since $g' \circ \iota = g'|U$, we get: $f' = g'|U$. $\square$

THEOREM 25.12. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\rho \in [0..\infty)$.
Then: $\mathrm{X}^\rho \mathrm{S}^\varepsilon_\bullet : \mathrm{IDF}_\varepsilon \to \mathbb{R}$ *is* C^ω *on* IDF_ε
and $(\mathrm{X}^\rho \mathrm{S}^\varepsilon_\bullet)' = -\mathrm{X}^{\rho+1}\mathrm{S}^\varepsilon_\bullet$ *on* IDF_ε.

Proof. **Let** $V := \Re^*\mathrm{IDF}_\varepsilon$, $U := V \bigcap \mathbb{R}$. Then $U = \mathrm{IDF}_\varepsilon$.
Let $f := \mathrm{X}^\rho \mathrm{S}^\varepsilon_\bullet$, $g := \mathrm{X}^\rho \mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$. Then $f = g|U$.
Also, $f_* U \subseteq \mathbb{R}$.
By Theorem 25.10, g is complex-differentiable and $g' = -\mathrm{X}^{\rho+1}\mathrm{S}^\varepsilon_{\bullet\mathbb{C}}$ on V.
Then, by Theorem 25.11, f is C^ω and $f' = -\mathrm{X}^{\rho+1}\mathrm{S}^\varepsilon_{\bullet\mathbb{C}}|U$.
FINISH THE PROOF. $\square$

26. Boltzmann averages on countable sets

DEFINITION 26.1. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Assume: $\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_\beta < \infty$. *Then:* $\boxed{\Gamma^\varepsilon_\beta} := \mathrm{X}^1\mathrm{S}^\varepsilon_\beta$.

Let Σ be a countable set, $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
If $\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_\beta < \infty$, then $\Gamma^\varepsilon_\beta = \sum_{\sigma\in\Sigma} [\varepsilon_\sigma \cdot (\widehat{B}^\varepsilon_\beta\{\sigma\})]$,
and so Γ^ε_β is the integral of ε wrt $\widehat{B}^\varepsilon_\beta$.

In the next definition, in order that $\Gamma^\varepsilon_\beta / \Delta^\varepsilon_\beta$ is defined,
we need: both Γ^ε_β is defined and $0 < \Delta^\varepsilon_\beta < \infty$.
We therefore assume $\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_\beta < \infty$, to ensure that Γ^ε_β is defined.
We also assume Σ is nonempty, to ensure that $\Delta^\varepsilon_\beta > 0$.
Finally, we assume $\beta \in \mathrm{DF}_\varepsilon$, to ensure that $\Delta^\varepsilon_\beta < \infty$.

DEFINITION 26.2. Let Σ *be a nonempty set,* $\varepsilon : \Sigma \to \mathbb{R}$, $\beta \in \mathbb{R}$.
Assume: $\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_\beta < \infty$ *and* $\beta \in \mathrm{DF}_\varepsilon$. *Then:* $\boxed{A^\varepsilon_\beta} := \Gamma^\varepsilon_\beta / \Delta^\varepsilon_\beta$.

Note that, by Theorem 24.16, if ε is ∞-proper, then
$$(\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_\beta < \infty) \Rightarrow (\beta \in \mathrm{DF}_\varepsilon).$$
Without ∞-properness, this fails, by Theorem 24.18,
$$(\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_1 < \infty) \not\Rightarrow (1 \in \mathrm{DF}_\varepsilon).$$
By Theorem 24.17, even with ∞-properness,
$$(1 \in \mathrm{DF}_\varepsilon) \not\Rightarrow (\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_1 < \infty).$$

THEOREM 26.3. Let Σ *be a nonempty countable set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume $\mathrm{DF}_\varepsilon \neq \varnothing$. **Let** $\beta \in \mathrm{DF}_\varepsilon$.
Assume $\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_\beta < \infty$. *Then* $|\varepsilon_* B^\varepsilon_\beta|_1 < \infty$ *and* $A^\varepsilon_\beta = M_{\varepsilon_* B^\varepsilon_\beta}$.

Proof. Since $\Sigma \neq \varnothing$, we get: $\Delta^\varepsilon_\beta > 0$.
Since $\beta \in \mathrm{DF}_\varepsilon$, we get: $\Delta^\varepsilon_\beta < \infty$.
By Theorem 8.17, we have:

$\sum_{t\in\mathbb{I}_\varepsilon}^{\mathrm{SP}} \sum_{\sigma\in\varepsilon^*\{t\}}^{\mathrm{SP}} [|t| \cdot (B_\beta^\varepsilon\{\sigma\})] = \sum_{\sigma\in\Sigma}^{\mathrm{SP}} [|\varepsilon_\sigma| \cdot (B_\beta^\varepsilon\{\sigma\})]$.
So, since $|\varepsilon_* B_\beta^\varepsilon|_1 = \sum_{t\in\mathbb{I}_\varepsilon}^{\mathrm{SP}} \sum_{\sigma\in\varepsilon^*\{t\}}^{\mathrm{SP}} [|t| \cdot (B_\beta^\varepsilon\{\sigma\})]$,
we get: $|\varepsilon_* B_\beta^\varepsilon|_1 = \sum_{\sigma\in\Sigma}^{\mathrm{SP}} [|\varepsilon_\sigma| \cdot (B_\beta^\varepsilon\{\sigma\})]$.
THEN: $|\varepsilon_* B_\beta^\varepsilon|_1 = (\overline{\mathrm{X}}^1 \mathrm{S}_\beta^\varepsilon)/\Delta_\beta^\varepsilon$.
ALSO: $0 \leqslant \overline{\mathrm{X}}^1 \mathrm{S}_\beta^\varepsilon < \infty$ and $0 < \Delta_\beta^\varepsilon < \infty$ THEN: $|\varepsilon_* B_\beta^\varepsilon|_1 < \infty$.
THEN: $M_{\varepsilon_* B_\beta^\varepsilon} = \sum_{t\in\mathbb{I}_\varepsilon} [t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\})]$.
Then $0 < \Delta_\beta^\varepsilon < \infty$, so, since $\Delta_\beta^\varepsilon = \widehat{B}_\beta^\varepsilon(\Sigma)$,
we get: $0 < \widehat{B}_\beta^\varepsilon(\Sigma) < \infty$.
For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
Since $\sum_{\sigma\in\Sigma}^{\mathrm{SP}} |\varepsilon_\sigma \cdot (\widehat{B}_\beta^\varepsilon\{\sigma\})| = \sum_{\sigma\in\Sigma}^{\mathrm{SP}} [\, |\varepsilon_\sigma| \cdot e^{-\beta\cdot\varepsilon_\sigma} \,] = \overline{\mathrm{X}}^1 \mathrm{S}_\beta^\varepsilon < \infty$,
dividing by $\widehat{B}_\beta^\varepsilon(\Sigma)$, we get: $\sum_{\sigma\in\Sigma}^{\mathrm{SP}} |\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})| < \infty$.
Then, by Theorem 8.18,
we get: $\forall t \in \mathbb{I}_\varepsilon$, $\sum_{\sigma\in\varepsilon^*\{t\}}^{\mathrm{SP}} |\varepsilon_\sigma \cdot (B_\beta\{\sigma\})\ | < \infty$ and
$\sum_{t\in\mathbb{I}_\varepsilon}^{\mathrm{SP}} | \sum_{\sigma\in\varepsilon^*\{t\}} [\varepsilon_\sigma \cdot (B_\beta\{\sigma\})] | < \infty$ and
$\sum_{t\in\mathbb{I}_\varepsilon} \sum_{\sigma\in\varepsilon^*\{t\}} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})] = \sum_{\sigma\in\Sigma} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})]$.
Also, $A_\beta^\varepsilon = \sum_{\sigma\in\Sigma} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})]$.
Then: $\sum_{t\in\mathbb{I}_\varepsilon} \sum_{\sigma\in\varepsilon^*\{t\}} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})] = A_\beta^\varepsilon$.
So, since $\sum_{t\in\mathbb{I}_\varepsilon} [t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\})] = M_{\varepsilon_* B_\beta^\varepsilon}$,
we want: $\sum_{t\in\mathbb{I}_\varepsilon} [t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\})] = \sum_{t\in\mathbb{I}_\varepsilon} \sum_{\sigma\in\varepsilon^*\{t\}} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})]$.
Want: $\forall t \in \mathbb{I}_\varepsilon$, $t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\}) = \sum_{\sigma\in\varepsilon^*\{t\}} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})]$.
Given $t \in \mathbb{I}_\varepsilon$, **want:** $t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\}) = \sum_{\sigma\in\varepsilon^*\{t\}} [\varepsilon_\sigma \cdot (B_\beta^\varepsilon\{\sigma\})]$.
For all $\sigma \in \varepsilon^*\{t\}$, since $\varepsilon_\sigma = \varepsilon(\sigma) \in \{t\}$, we get: $t = \varepsilon_\sigma$.
Want: $t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\}) = \sum_{\sigma\in\varepsilon^*\{t\}} [\, t \cdot (B_\beta^\varepsilon\{\sigma\})\,]$.
Because $\varepsilon^*\{t\}$ is the disjoint union, over $\sigma \in \varepsilon^*\{t\}$, of $\{\sigma\}$,
we get: $B_\beta^\varepsilon(\varepsilon^*\{t\}) = \sum_{\sigma\in\varepsilon^*\{t\}} [\ B_\beta^\varepsilon\{\sigma\}\ ]$.
Also, $(\varepsilon_* B_\beta^\varepsilon)\{t\}) = B_\beta^\varepsilon(\varepsilon^*\{t\})$.
Then: $t \cdot ((\varepsilon_* B_\beta^\varepsilon)\{t\}) = t \cdot (B_\beta^\varepsilon(\varepsilon^*\{t\})) = \sum_{\sigma\in\varepsilon^*\{t\}} [\, t \cdot (B_\beta^\varepsilon\{\sigma\})]$. □

THEOREM 26.4. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*[0;\infty)$ *is infinite and* $\mathrm{DF}_\varepsilon \neq \varnothing$. **Let** $\beta_0 := \inf \mathrm{DF}_\varepsilon$.
Then: $\forall$*real* $\gamma > \beta_0$, $\forall$*real* $\rho > 0$, $\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon < \infty$.

Proof. **Given** a real $\gamma > \beta_0$ and a real $\rho > 0$, **want:** $\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon < \infty$.
By Theorem 24.12, ε is ∞-proper.
Then, by Theorem 23.4, $\mathbb{I}_\varepsilon$ is bounded below.
By Theorem 24.19, we have: $\overline{\mathrm{X}}^\rho \mathrm{S}_\gamma^\varepsilon < \infty$. □

DEFINITION 26.5. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Then $\boxed{A^\varepsilon_\bullet} : \mathrm{IDF}_\varepsilon \to \mathbb{R}$ *is defined by:* $\quad \forall \beta \in \mathrm{IDF}_\varepsilon, \quad A^\varepsilon_\bullet(\beta) = A^\varepsilon_\beta$.

THEOREM 26.6. Let Σ *be a set.*
Let $\varepsilon : \Sigma \to \mathbb{R}$. *Assume:* $\#\mathbb{I}_\varepsilon \geqslant 2$.
Then: $A^\varepsilon_\bullet$ *is a strictly-decreasing* C^ω*-diffeomorphism*
from IDF_ε *onto* $(\inf \mathbb{I}_{A^\varepsilon_\bullet} ; \sup \mathbb{I}_{A^\varepsilon_\bullet})$.

Proof. For all $\sigma \in \Sigma$, **let** $\varepsilon_\sigma := \varepsilon(\sigma)$.
We have: $\forall \beta \in \mathrm{IDF}_\varepsilon$, $A^\varepsilon_\bullet(\beta) = \dfrac{\sum_{\sigma\in\Sigma} [\varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma}]}{\sum_{\tau\in\Sigma} [e^{-\beta\cdot\varepsilon_\tau}]}$.
We have: $\forall \beta \in \mathrm{IDF}_\varepsilon$, $A^\varepsilon_\bullet(\beta) = \dfrac{\Gamma^\varepsilon_\bullet(\beta)}{\Delta^\varepsilon_\bullet(\beta)}$.
We have: $\forall \beta \in \mathrm{IDF}_\varepsilon$, $\Gamma^\varepsilon_\bullet = \mathrm{X}^1\mathrm{S}^\varepsilon_\bullet$ and $\Delta^\varepsilon_\bullet = \mathrm{X}^0\mathrm{S}^\varepsilon_\bullet$.
By Theorem 25.10, $\mathrm{X}^1\mathrm{S}^\varepsilon_\bullet$ and $\mathrm{X}^0\mathrm{S}^\varepsilon_\bullet$ are both C^ω.
Then: $\Gamma^\varepsilon_\bullet$ and $\Delta^\varepsilon_\bullet$ are both C^ω.
So, since $\Delta^\varepsilon_\bullet \neq 0$ on IDF_ε, we conclude: $A^\varepsilon_\bullet$ is C^ω.
By Theorem 25.12, we have:
$\forall \beta \in \mathrm{IDF}_\varepsilon$, $(\mathrm{X}^1\mathrm{S}^\varepsilon_\bullet)'(\beta) = -(\mathrm{X}^2\mathrm{S}^\varepsilon_\bullet)(\beta)$,
and $\forall \beta \in \mathrm{IDF}_\varepsilon$, $(\mathrm{X}^0\mathrm{S}^\varepsilon_\bullet)'(\beta) = -(\mathrm{X}^1\mathrm{S}^\varepsilon_\bullet)(\beta)$.
Then: $\forall \beta \in \mathrm{IDF}_\varepsilon$, $(\mathrm{X}^1\mathrm{S}^\varepsilon_\bullet)'(\beta) = \sum_{\sigma\in\Sigma} [(-\varepsilon_\sigma^2) \cdot e^{-\beta\cdot\varepsilon_\sigma}]$,
and $\forall \beta \in \mathrm{IDF}_\varepsilon$, $(\mathrm{X}^0\mathrm{S}^\varepsilon_\bullet)'(\beta) = \sum_{\tau\in\Sigma} [(-\varepsilon_\tau) \cdot e^{-\beta\cdot\varepsilon_\tau}]$.
So, by the C^ω-Inverse Function Theorem and
the Mean Value Theorem, **it suffices to show:** $(A^\varepsilon_\bullet)' < 0$ on IDF_ε.
Given $\beta \in \mathrm{IDF}_\varepsilon$, **want:** $(A^\varepsilon_\bullet)'(\beta) < 0$.
Let $P := \sum_{\sigma\in\Sigma} [\, \varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma} \,]$, $P' := \sum_{\sigma\in\Sigma} [\, (-\varepsilon_\sigma^2) \cdot e^{-\beta\cdot\varepsilon_\sigma} \,]$.
Let $Q := \sum_{\tau\in\Sigma} [\, e^{-\beta\cdot\varepsilon_\tau} \,]$, $Q' := \sum_{\tau\in\Sigma} [\, (-\varepsilon_\tau) \cdot e^{-\beta\cdot\varepsilon_\tau} \,]$.
Then $Q > 0$. Also, by the Quotient Rule, $(A^\varepsilon_\bullet)'(\beta) = [QP' - PQ']/Q^2$.
Want: $QP' - PQ' < 0$.
Then, by Theorem 8.20, we have:
$QP' = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (-\varepsilon_\sigma^2) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.
and $PQ' = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (-\varepsilon_\sigma\varepsilon_\tau) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.
Then: $QP' - PQ' = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (-\varepsilon_\sigma^2 + \varepsilon_\sigma\varepsilon_\tau) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.
Interchanging σ and τ, we get:
$QP' - PQ' = \sum_{\tau\in\Sigma} \sum_{\sigma\in\Sigma} [\, (-\varepsilon_\tau^2 + \varepsilon_\tau\varepsilon_\sigma) \cdot e^{-\beta\cdot(\varepsilon_\tau+\varepsilon_\sigma)} \,]$.
By commutativity of addition and multiplication,
adding the last two equations gives:
$2 \cdot (QP' - PQ') = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\, (-\varepsilon_\sigma^2 - \varepsilon_\tau^2 + 2\varepsilon_\sigma\varepsilon_\tau) \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)} \,]$.

Then: $2 \cdot (QP' - PQ') = \sum_{\sigma\in\Sigma} \sum_{\tau\in\Sigma} [\quad -(\varepsilon_\sigma - \varepsilon_\tau)^2 \quad \cdot e^{-\beta\cdot(\varepsilon_\sigma+\varepsilon_\tau)}]$.
Then: $2 \cdot (QP' - PQ') < 0$. Then: $QP' - PQ' < 0$. $\square$

Recall (Theorem 23.3):
If ε is ∞-proper, then $\mathbb{I}_\varepsilon$ has a minimum element, *i.e.*, $\min \mathbb{I}_\varepsilon$ exists.

THEOREM 26.7. Let Σ *be a set,* $\varepsilon : \Sigma \to \mathbb{R}$.
Assume: $\varepsilon^*[0;\infty)$ *is infinite and* $\mathrm{DF}_\varepsilon \neq \varnothing$.
Then: ε *is* ∞*-proper and as* $\beta \to \infty$, $A^\varepsilon_\beta \to \min \mathbb{I}_\varepsilon$.

Proof. By Theorem 24.12, ε is ∞-proper.
It remains to show: as $\beta \to \infty$, $A^\varepsilon_\beta \to \min \mathbb{I}_\varepsilon$.
Let $t_0 := \min \mathbb{I}_\varepsilon$. **Want:** $A^\varepsilon_\beta \to t_0$.
Let $\Sigma' := \Sigma \backslash (\varepsilon^*\{t_0\})$. **Let** $n_0 := \#(\varepsilon^*\{t_0\})$.
Since $\{t_0\} \subseteq (-\infty; t_0]$, we get $\varepsilon^*\{t_0\} \subseteq \varepsilon^*(-\infty; t_0]$.
Since ε is ∞-proper, we get: $\varepsilon^*(-\infty; t_0]$ is finite.
Then $\varepsilon^*\{t_0\}$ is finite. That is, $n_0 < \infty$.
Since $t_0 \in \mathbb{I}_\varepsilon$, we get $\varepsilon^*\{t_0\} \neq \varnothing$, and so $n_0 > 0$. Then $0 < n_0 < \infty$.
For all $\beta \in (\beta_0; \infty)$, we have:

$$\begin{aligned} A^\varepsilon_\beta &= \frac{n_0 \cdot t_0 \cdot e^{-\beta\cdot t_0} + \sum_{\sigma\in\Sigma'} [\varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma}]}{n_0 \cdot e^{-\beta\cdot t_0} + \sum_{\sigma\in\Sigma'} [e^{-\beta\cdot\varepsilon_\sigma}]} \\ &= \frac{n_0 \cdot t_0 \cdot e^{-\beta\cdot t_0} + \sum_{\sigma\in\Sigma'} [\varepsilon_\sigma \cdot e^{-\beta\cdot\varepsilon_\sigma}]}{n_0 \cdot e^{-\beta\cdot t_0} + \sum_{\sigma\in\Sigma'} [e^{-\beta\cdot\varepsilon_\sigma}]} \cdot \frac{e^{\beta\cdot t_0}}{e^{\beta\cdot t_0}} \\ &= \frac{n_0 \cdot t_0 + \sum_{\sigma\in\Sigma'} [\varepsilon_\sigma \cdot e^{-\beta\cdot(\varepsilon_\sigma - t_0)}]}{n_0 + \sum_{\sigma\in\Sigma'} [e^{-\beta\cdot(\varepsilon_\sigma - t_0)}]}. \end{aligned}$$

Let $\beta_1 := \beta_0 + 1$.
Then, for all $\beta \in [\beta_1; \infty)$, for all $\sigma \in \Sigma$, we have

$$|\varepsilon_\sigma \cdot e^{-\beta\cdot(\varepsilon_\sigma - t_0)}| \leqslant |\varepsilon_\sigma| \cdot e^{-\beta_1\cdot(\varepsilon_\sigma - t_0)}$$

and

$$|e^{-\beta\cdot(\varepsilon_\sigma - t_0)}| \leqslant e^{-\beta_1\cdot(\varepsilon_\sigma - t_0)}.$$

We have: $\sum_{\sigma\in\Sigma} [|\varepsilon_\sigma| \cdot e^{-\beta_1\cdot(\varepsilon_\sigma - t_0)}] = \overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_{\beta_1}$.
Also, $\sum_{\sigma\in\Sigma} [e^{-\beta_1\cdot(\varepsilon_\sigma - t_0)}] = \overline{\mathrm{X}}^0 \mathrm{S}^\varepsilon_{\beta_1}$.
By Theorem 26.4, we have: $\overline{\mathrm{X}}^1 \mathrm{S}^\varepsilon_{\beta_1} < \infty$ and $\overline{\mathrm{X}}^0 \mathrm{S}^\varepsilon_{\beta_1} < \infty$.
So, by the Dominated Convergence Theorem, as $\beta \to \infty$,

$$\sum_{\sigma\in\Sigma'} [\varepsilon_\sigma \cdot e^{-\beta\cdot(\varepsilon_\sigma - t_0)}] \to 0$$

and

$$\sum_{\sigma\in\Sigma'} [e^{-\beta\cdot(\varepsilon_\sigma - t_0)}] \to 0.$$

Then: as $\beta \to \infty$, $A^\varepsilon_\beta \to \dfrac{n_0 \cdot t_0 + 0}{n_0 + 0}$.
Then: as $\beta \to \infty$, $A^\varepsilon_\beta \to t_0$. $\square$

Let Σ be a set and **let** $\varepsilon : \Sigma \to [0;\infty)$ be ∞-proper.
Assume: $\varepsilon^*[0;\infty)$ is infinite and $\sup \mathbb{I}_\varepsilon = \infty$ and $\mathrm{DF}_\varepsilon \neq \varnothing$.

Let $\beta_0 := \inf \mathrm{DF}_\varepsilon$. By Theorem 24.21, $(\beta_0;\infty) \subseteq \mathrm{DF}_\varepsilon$.
Even though $\sup \mathbb{I}_\varepsilon = \infty$,
it does NOT necessarily follow that: as $\beta \to (\beta_0)^+$, $A_\beta^\varepsilon \to \infty$.
Here is an example:

THEOREM 26.8. *For all* $k \in \mathbb{N}$, **let** $n_k := \lfloor e^k/k^3 \rfloor$.
Let $\Sigma := \{(k,j) \in \mathbb{N} \times \mathbb{N} \,|\, k \in \mathbb{N}, j \in [1..n_k]\}$.
Define $\varepsilon : \Sigma \to [0..\infty)$ *by:* $\forall k \in \mathbb{N}$, $\forall j \in [1..n_k]$, $\varepsilon(k,j) = k-1$.
Then $\mathbb{I}_{A_\bullet^\varepsilon}$ *is bounded.*

Proof. We have $\mathrm{DF}_\varepsilon = [1;\infty)$, so $\inf \mathrm{DF}_\varepsilon = 1$.
Also, $\Gamma_1^\varepsilon < \infty$ and $0 < \Delta_1^\varepsilon < \infty$, so $A_1^\varepsilon < \infty$.
Also, by the Dominated Convergence Theorem, we have:
as $\beta \to 1^+$, both $\Gamma_\beta^\varepsilon \to \Gamma_1^\varepsilon$ and $\Delta_\beta^\varepsilon \to \Delta_1^\infty$.
Then, as $\beta \to 1^+$, $A_\beta^\varepsilon \to A_1^\varepsilon < \infty$.
Then $\mathbb{I}_{A_\bullet^\varepsilon}$ is bounded. $\square$

Theorem 26.8 leads to an open problem, as follows:
For all $k \in \mathbb{N}$, **let** $n_k := \lfloor e^k/k^3 \rfloor$.
Let $\Sigma := \{(k,j) \in \mathbb{N} \times \mathbb{N} \,|\, k \in \mathbb{N}, j \in [1..n_k]\}$.
Define $\varepsilon : \Sigma \to \mathbb{N}$ by: $\forall k \in \mathbb{N}$, $\forall j \in [1..n_k]$, $\varepsilon(k,j) = k$.
By Theorem 26.6, $A_\bullet^\varepsilon$ is strictly-decreasing, and so
and since as $\beta \to 1^+$, $A_\beta^\varepsilon \to A_1^\varepsilon$, we get:
$\mathbb{I}_{A_\bullet^\varepsilon}$ is bounded above by A_1^ε.
Let $\alpha \in \mathbb{N}$. Assume: $\alpha > A_1^\varepsilon$. Then: $\alpha \notin \mathbb{I}_{A_\bullet^\varepsilon}$.
Suppose N professors, numbered 1 to N, have states in Σ.
Suppose each state $\sigma \in \Sigma$ has wealth $\varepsilon(\sigma)$.
Suppose the total wealth of all professors is $N\alpha$.
Give equal probability to every dispensation of states.
For each $\sigma_0 \in \Sigma$, we seek a method to approximate
the probability that Professor#N is in state σ_0.
More precisely: For all $n \in \mathbb{N}$,
let $\Omega_n := \{\omega : [1..n] \to \Sigma \,|\, \sum_{\ell=1}^n [\varepsilon(\omega(\ell))] = n\alpha\}$.
Then Ω_N represents the set of all state-dispensations.
Open Problem: For each $\sigma_0 \in \Sigma$,
determine whether
the limit, as $n \to \infty$, of $\nu_{\Omega_n}\{\omega \in \Omega_n \,|\, \omega(n) = \sigma_0\}$ exists,
and, if it does, compute it.
This is a well-defined mathematical problem.

However, since $\alpha \notin \mathbb{I}_{A^\varepsilon_\bullet}$, we cannot solve $A^\varepsilon_\beta = \alpha$ for β,
so our earlier techniques do not immediately apply.

THEOREM 26.9. Let $\beta_0 \in \mathbb{R}$, $I := (\beta_0; \infty)$, $g : I \to \mathbb{R}$.
Assume: g *is differentiable on* I *and* g' *is semi-decreasing on* I.
Assume: *as* $\beta \to (\beta_0)^+$, $g(\beta) \to -\infty$.
Then: *as* $\beta \to (\beta_0)^+$, $g'(\beta) \to \infty$.

Proof. Since $g : I \to \mathbb{R}$ and g is differentiable on I, we get: $g' : I \to \mathbb{R}$.
Since $I = (\beta_0; \infty) \neq \varnothing$ and $g' : I \to \mathbb{R}$, we get: $\mathbb{I}_{g'} \neq \varnothing$.
Since $\varnothing \neq \mathbb{I}_{g'} \subseteq \mathbb{R}$, we get: $\sup \mathbb{I}_{g'} \neq -\infty$. Then $\sup \mathbb{I}_{g'} \in (-\infty; \infty]$.
Let $M := \sup \mathbb{I}_{g'}$. Then $M \in (-\infty; \infty]$.
Since g' is semi-decreasing, we get: as $\beta \to (\beta_0)^+$, $g'(\beta) \to M$.
Want: $M = \infty$. Assume $M \neq \infty$. **Want:** Contradiction.
Since $M \in (-\infty; \infty]$ and since $M \neq \infty$, we get: $M \in \mathbb{R}$.
Let $M_1 := \max\{M, 0\}$. **Let** $\beta_1 := \beta_0 + 1$.
Since, as $\beta \to (\beta_0)^+$, $g(\beta) \to -\infty$,
choose $\gamma \in (\beta_0; \beta_1)$ s.t. $g(\gamma) < (g(\beta_1)) - M_1$.
By the Mean Value Theorem, choose $\xi \in (\gamma; \beta_1)$ s.t.

$$\frac{(g(\beta_1) - (g(\gamma))}{\beta_1 - \gamma} = g'(\xi).$$

Then: $(g(\beta_1) - (g(\gamma)) = (g'(\xi)) \cdot (\beta_1 - \gamma)$.
Since $M = \sup \mathbb{I}_{g'}$, we get: $g'(\xi) \leqslant M$.
Since $\gamma \in (\beta_0; \beta_1)$, we get: $\beta_1 - \gamma > 0$.
Then $(g'(\xi)) \cdot (\beta_1 - \gamma) \leqslant M \cdot (\beta_1 - \gamma)$.
Since $(g(\beta_1) - (g(\gamma)) = (g'(\xi)) \cdot (\beta_1 - \gamma) \leqslant M \cdot (\beta_1 - \gamma)$,
we get: $g(\gamma) \geqslant (g(\beta_1)) - M \cdot (\beta_1 - \gamma)$.
By the choice of γ, we get $\gamma \in (\beta_0; \beta_1)$, and so $\gamma - \beta_0 > 0$.
By the choice of γ, we get:
$(g(\beta_1)) - M_1 > g(\gamma)$.
Since $(g(\beta_1)) - M_1 > g(\gamma) \geqslant (g(\beta_1)) - M \cdot (\beta_1 - \gamma)$,
we get: $M_1 < M \cdot (\beta_1 - \gamma)$.
Since $M \leqslant M_1 < M \cdot (\beta_1 - \gamma) = M \cdot (\beta_0 + 1 - \gamma)$,
we get: $0 < M \cdot (\beta_0 - \gamma)$.
Then: $M \cdot (\gamma - \beta_0) < 0$.
So, since $\gamma - \beta_0 > 0$, we get: $M < 0$.
Recall: $I = (\beta_0; \infty)$ and g is differentiable on I and $\sup \mathbb{I}_{g'} = M$.
Since $\sup \mathbb{I}_{g'} = M < 0$, we get: $g' < 0$ on I.
Then, by the Mean Value Theorem, g is strictly-decreasing on I.

We conclude: $\forall\beta\in(\beta_0;\beta_1)$, $g(\beta)>g(\beta_1)$. Then $g(\gamma)>g(\beta_1)$.
Since $M_1\geqslant 0$, we get: $(g(\beta_1))-M_1\leqslant g(\beta_1)$.
Then $g(\beta_1)<g(\gamma)<(g(\beta_1))-M_1\leqslant g(\beta_1)$, so $g(\beta_1)<g(\beta_1)$.
Contradiction. □

Next, we prove that the pathology observed in Theorem 26.8
does not happen when DF_ε is open in $\mathbb{R}$ and bounded below.
By Theorem 24.33, in typical Boltzmann applications,
$\mathrm{DF}_\varepsilon=(0;\infty)$, and so DF_ε is open in $\mathbb{R}$ and bounded below.

THEOREM 26.10. Let Σ *be a set,* $\varepsilon:\Sigma\to\mathbb{R}$, $\beta_0\in\mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon=(\beta_0;\infty)$. *Then: as* $\beta\to(\beta_0)^+$, $\Delta^\varepsilon_\beta\to\infty$.

Proof. By Theorem 24.40, ε is ∞-proper.
Then, by Theorem 23.4, $\mathbb{I}_\varepsilon$ is bounded below.
Choose $\xi\in\mathbb{R}$ s.t. $\xi+\mathbb{I}_\varepsilon\subseteq(0;\infty)$. **Let** $\widetilde{\varepsilon}:=\varepsilon+\xi$.
Then $\Delta^\varepsilon_\beta=e^{\beta\cdot\xi}\cdot\Delta^{\widetilde{\varepsilon}}_\beta$. **Want:** as $\beta\to(\beta_0)^+$, $\Delta^{\widetilde{\varepsilon}}_\beta\to\infty$.
Otherwise, since $\beta\mapsto\Delta^{\widetilde{\varepsilon}}_\beta$ is strictly-decreasing,
we get $\{\Delta^{\widetilde{\varepsilon}}_\beta\,|\,\beta\in\mathrm{DF}_{\widetilde{\varepsilon}}\}$ is bounded above.
Let M be an upper bound.
Since $\beta_0\notin(\beta_0;\infty)=\mathrm{DF}_{\widetilde{\varepsilon}}$, we get: $\Delta^{\widetilde{\varepsilon}}_\beta=\infty$.
That is, $\sum^{\mathrm{SP}}_{\sigma\in\Sigma}[e^{-\beta\cdot\widetilde{\varepsilon}_\sigma}]=\infty$.
Choose a finite subsum that is $>M$.
Perturb β_0 to a slightly larger β.
If the perturbation is small enough,
then the finite subsum stays $>M$.
Then $\Delta^{\widetilde{\varepsilon}}_\beta\geqslant$ the perturbed finite subsum $>M$,
contradicting that M is an upper bound. □

THEOREM 26.11. Let Σ *be a set,* $\varepsilon:\Sigma\to\mathbb{R}$, $\beta_0\in\mathbb{R}$.
Assume: $\mathrm{DF}_\varepsilon=(\beta_0;\infty)$. *Then: as* $\beta\to(\beta_0)^+$, $A^\varepsilon_\beta\to\infty$.

Proof. **Let** $I:=(\beta_0;\infty)$.
Define $f:I\to(0;\infty)$ by: $\forall\beta\in I$, $f(\beta)=\Delta^\varepsilon_\beta$.
Then $f=\mathrm{X}^0\mathrm{S}^\varepsilon_\bullet$, so, by Theorem 25.12, we get: $f'=-\mathrm{X}^1\mathrm{S}^\varepsilon_\bullet$.
We have: $\forall\beta\in I$, $\mathrm{X}^1\mathrm{S}^\varepsilon_\beta=\Gamma^\varepsilon_\beta$.
Then: $\forall\beta\in I$, $f'(\beta)=-\Gamma^\varepsilon_\beta$.
Define $g:I\to\mathbb{R}$ by: $\forall\beta\in I$, $g(\beta)=-(\ln(f(\beta)))$.
Then: g is differentiable on I and,
by the Chain Rule, $\forall\beta\in I$, $g'(\beta)=-(f'(\beta))/(f(\beta))$.
Then: $\forall\beta\in I$, $g'(\beta)=\Gamma^\varepsilon_\beta/\Delta^\varepsilon_\beta$.

Then: $\quad \forall \beta \in I, \quad g'(\beta) = A^\varepsilon_\beta$.
Then $g' = A^\varepsilon_\bullet$.
Since $\mathrm{DF}_\varepsilon = (\beta_0; \infty)$, we get: $\mathrm{IDF}_\varepsilon = (\beta_0; \infty)$.
Then $I = \mathrm{IDF}_\varepsilon$.
Want: $\quad$ as $\beta \to (\beta_0)^+, \quad g'(\beta) \to \infty$.
By Theorem 26.6, we get: $\quad g'$ is strictly-decreasing on I.
Then g' is semi-decreasing on I.
By Theorem 26.10, we get: $\quad$ as $\beta \to (\beta_0)^+, \quad \Delta^\varepsilon_\beta \to \infty$.
Then: $\quad$ as $\beta \to (\beta_0)^+, \quad f(\beta) \to \infty$.
Then: $\quad$ as $\beta \to (\beta_0)^+, \quad -(\ln(f(\beta))) \to -\infty$.
Then: $\quad$ as $\beta \to (\beta_0)^+, \quad g(\beta) \to -\infty$.
Then, by Theorem 26.9, we get: $\quad$ as $\beta \to (\beta_0)^+, \quad g'(\beta) \to \infty$. $\square$

27. Countably infinite sets of states

MORE LATER

28. Appendix: Python code

Thanks once again to C. Prouty, for writing the Python code to do the Boltzmann computations in this paper:

First code: The GFA and 0, 2, 20 dollar awards, with average 3 dollars.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
def F(beta):
z = np.zeros(3)
z[0] = 1
z[1] = np.exp(-2 * beta)
z[2] = np.exp(-20 * beta)
return z
def G(beta):
z = np.zeros(3)
z[0] = 0
z[1] = 2 * np.exp(-2 * beta)
z[2] = 20 * np.exp(-20 * beta)
return z
def f(beta):
return np.sum(F(beta))
def g(beta):
return np.sum(G(beta))
def bisection(minval, maxval, y, fn):
mid = (maxval + minval) / 2
while((fn(mid) - y) ** 2 > 0.0000001):
if(fn(mid) < y):
maxval = mid
else:
minval = mid
mid = (maxval + minval) / 2
return mid
fn = lambda x: g(x) / f(x)
```

```
target = bisection(-25, 25, 3, fn)
b = 0.07410049 # hard-coded result of bisection
r = F(b) / f(b)
df = pd.DataFrame(r)
df.to_excel("results2.xlsx", index=False)
betas = np.linspace(-25,25,100000)
z = np.zeros(len(betas))
for i in range(len(betas)):
z[i] = fn(betas[i])
plt.plot(betas,z)
plt.show()
```

Second code: The BUA and red bags and blue bags

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
def F(beta):
z = np.zeros(25).reshape(5,5)
for i in range(5):
for j in range(5):
z[i,j] = np.exp(-(i+j)*beta)
z[4,4] = 0
return z
def G(beta):
z = np.zeros(25).reshape(5,5)
for i in range(5):
for j in range(5):
z[i,j] = (i+j) * np.exp(-(i+j)*beta)
z[4,4] = 0
return z
def f(beta):
return np.sum(F(beta))
def g(beta):
return np.sum(G(beta))
def bisection(minval, maxval, y, fn):
```

```
mid = (maxval + minval) / 2
while((fn(mid) - y) ** 2 > 0.0000001):
if(fn(mid) < y):
maxval = mid
else:
minval = mid
mid = (maxval + minval) / 2
return mid
fn = lambda x: g(x) / f(x)
target = bisection(-25, 25, 1, fn)
b = 1.06697083 # hard-coded result of bisection
r = F(b) / f(b)
df = pd.DataFrame(r)
df.to_excel("results5.xlsx", index=False)
betas = np.linspace(-25,25,100000)
z = np.zeros(len(betas))
for i in range(len(betas)):
z[i] = fn(betas[i])
plt.plot(betas, z)
plt.show()
```