1. Some notation

A box around an expression indicates that it is global, meaning that it is fixed to the end of this note.

Let $\mathbb{R}^\ast := \{\infty\} \cup \mathbb{R} \cup \{\infty\}$. For any $p, q \in \mathbb{R}^\ast$, let $[p; q] := \{x \in \mathbb{R}^\ast \mid p \leq x \leq q\}$. Define $\mathbb{P}^\ast := \{\infty\} \cup \mathbb{P} \cup \{\infty\}$. For any $p, q \in \mathbb{P}^\ast$, let $[p..q] := [p; q] \cap \mathbb{P}^\ast$. Define $\mathbb{P}^\ast \cap \mathbb{P}^\ast$ similarly.

For any finite set $F$, let $\#F$ denote the number of elements in $F$.

2. First system of grant awards

Let $\mathbb{N}$ be a large positive integer. Suppose there are $N$ professors, numbered 1 to $N$, who apply, once per year, to the GFA (Grant Funding Agency), seeking funding for the very important work they are doing.

Each year, the GFA has $\$$N to distribute, so the average award will be $\$$1.

The GFA has the rule: every award is 0 or 1 or 10 dollars.

The set of grant distributions is denoted:

$$\Sigma := \left\{ \sigma : [1..N] \to \{0, 1, 10\} \mid \sum_{n \in [1..N]} \sigma(n) = N \right\}.$$

The GFA has set aside $\#\Sigma$ pieces of paper, and writes all the distributions, one on each piece of paper. So, for example, there will be one piece of paper that says:

Professors 1 to $N$ each get $\$$1.

Another piece of paper will say:

Professors 1 to $N - 10$ each get $\$$1 and

Professors $N - 9$ to $N - 1$ each get $\$$0 and

Professor $N$ gets $\$$10.

There are, of course, many, many, many other pieces of paper. A GFA bureaucrat places all the pieces of paper in a big bin, then pulls out one at random and makes the awards as indicated on that piece of paper.
Note that, \( \forall \sigma \in \Sigma, \)
the probability that the grant distribution is \( \sigma \)
is equal to \( 1 / (\#\Sigma). \)

Suppose I am one of those professors.
Problem: Calculate my probability of getting $0.
Then calculate my probability of getting $1.
Then calculate my probability of getting $10.

In the remainder of this note,
we reformulate and then solve this problem.
Spoiler: It’s a Boltzmann distribution.

3. PARTICLES AND ENERGY

Recall that \( N \) is a large positive integer, \( e.g., \) \( 10^{23} \).
Suppose there are \( N \) particles, numbered 1 to \( N \),
each of which has a certain amount of energy.
Suppose the total energy is \( N \) so the average energy of a particle is 1.
Suppose physicists have somehow determined that, for any particle,
its possible energy levels are: 0 or 1 or 10.

Recall: \( \Sigma = \left\{ \sigma : [1..N] \to \{0, 1, 10\} \quad \text{s.t.} \quad \sum_{n \in [1..N]} (\sigma(n)) = N \right\}. \)
Then \( \Sigma \) is the set of energy distributions.

Assume that physicists have somehow determined
this system of particles has a random energy distribution
and that, for this system,
all energy distributions in \( \Sigma \) are equally likely.
That is, physicists tell us: \( \forall \sigma \in \Sigma, \)
the probability that the energy distribution is \( \sigma \) is equal to \( 1 / (\#\Sigma). \)

The equal likelihood of all energy distributions
is a recurring theme in thermodynamics,
and can be explained by the Perron-Frobenius Theorem,
and the principle of “detailed balance”.
For a fun explanation of this,
search for “Coconuts and Islanders” by B. Zhang;
and, in particularly, see the last paragraph of §3.2 therein.
Pick any particle.
Problem: Calculate its probability of having energy level 0.
Then calculate its probability of having energy level 1.
Then calculate its probability of having energy level 10.
Spoiler: It’s a Boltzmann distribution.

The solution to this problem is the same as the solution to the problem about professors and grants.

We will go back to professors and grants. 
It makes no difference, mathematically, but it’s more fun.

4. SECOND AND THIRD SYSTEMS OF GRANT AWARDS

Suppose, in an effort to go paperless, the GFA changes to a new system:

In this second system, instead of all those pieces of paper, the GFA chooses \( p, q, r \geq 0 \) s.t. \( p + q + r = 1 \), and then, for each of the \( N \) professors,
awards \( \$0 \) with probability \( p \),
awards \( \$1 \) with probability \( q \) and
awards \( \$10 \) with probability \( r \).
No professor’s award depends in any way on any other professor’s; the awards are independent.
Recognizing that the average award is intended to be \( \$1 \),
the numbers \( p, q, r \) are chosen subject to the constraint that \( p \cdot 0 + q \cdot 1 + r \cdot 10 = 1 \), \textit{i.e.}, \( q + 10r = 1 \).

Recall: \( \Sigma = \left\{ \sigma : [1..N] \to \{0, 1, 10\} \quad \text{s.t.} \quad \sum_{n \in [1..N]} (\sigma(n)) = N \right\} \).
That is, \( \Sigma \) is the set of all \( \sigma : [1..N] \to \{0, 1, 10\} \) s.t.
the total money paid out is equal to \( N \).

For each \( \sigma : [1..N] \to \{0, 1, 10\} \), let
\[
i_{\sigma} := \#\{n \in [1..N] \mid \sigma(n) = 0\},
\]
\[
j\sigma] := \#\{n \in [1..N] \mid \sigma(n) = 1\},
\]
\[
k\sigma] := \#\{n \in [1..N] \mid \sigma(n) = 10\}.
\]
Note that, $\forall \sigma : \{1..N\} \to \{0, 1, 10\}$, the probability that the grant distribution is $\sigma$
is equal to $p^{i_\sigma}q^{j_\sigma}r^{k_\sigma}$.
Moreover, $\forall \sigma : \{1..N\} \to \{0, 1, 10\}$, the total amount of money payed out by the
distribution $\sigma$ is equal to $i_\sigma \cdot 0 + j_\sigma \cdot 1 + k_\sigma \cdot 10$, i.e., $j_\sigma + 10k_\sigma$;
in particular, the total payout is $N$ iff $j_\sigma + 10k_\sigma = N$.
Then, $\forall \sigma : \{1..N\} \to \{0, 1, 10\}$, we have:

$$\sigma \in \Sigma \iff j_\sigma + 10k_\sigma = N.$$ 

Next, we define $S := \sum_{\sigma \in \Sigma} p^{i_\sigma}q^{j_\sigma}r^{k_\sigma}$. Then $S$ is the probability that $\sigma \in \Sigma$,
i.e., the probability that the total payout is exactly $N$ dollars.

We will eventually estimate $S$ using the Local Limit Theorem.
It turns out that, if $N$ is large, then $S$ is close to zero.
So, under this second system, the probability of paying out exactly $N$ dollars is very small.

On the other hand, the expected payout is $\$1$ per professor, so, each year, the expected total payout is $\$N$.
Congress only allocates $\$N$ per year, so using this system, the GFA will, in almost every year, run a surplus or a deficit, but these will, over time, cancel one another.

Congress, however, is a paragon of fiscal responsibility, and, as soon as it becomes aware of these surpluses and deficits, it insists that the GFA must never again overspend or underspend.
So the GFA changes its system again, as follows.

Under its third system, each year, before announcing any of the awards publicly, the GFA writes out, in an internal electronic memo, a proposal of awards that, for each of the $N$ professors:
awards $\$0$ with probability $p$,
awards $\$1$ with probability $q$ and
awards $\$10$ with probability $r$. 
The proposed awards are independent.
If the sum of proposed awards is not equal to $N$,
  the GFA deems the memo as unacceptable,
  deletes it, and starts over.
Each memo has a probability $S$ of being acceptable,
so the GFA may need to repeat the process many times
  to get to a memo with total exactly equal to $N$.
However, once that happens, it uses that memo, and
  makes the awards public.

Mathematically, we are conditioning on the event $\sigma \in \Sigma$.
So, using the third system, the probability that $\sigma \notin \Sigma$ is 0.
That is, we only accept outcomes in which the total payout is $N$.
Also, for this third system, $\forall \sigma \in \Sigma$, the probability of $\sigma$ is $p^{i_{\sigma}} q^{j_{\sigma}} r^{k_{\sigma}} / S$.
Note that the sum of these probabilities is
$$\sum_{\sigma \in \Sigma} p^{i_{\sigma}} q^{j_{\sigma}} r^{k_{\sigma}} / S = \frac{1}{S} \cdot \sum_{\sigma \in \Sigma} p^{i_{\sigma}} q^{j_{\sigma}} r^{k_{\sigma}} = \frac{1}{S} \cdot S = 1.$$  

This third system is not necessarily equivalent to the first,
  because, in the first system, all the probabilities were $1 / (\# \Sigma)$.
So a new question arises:
  Is it possible to choose $p, q, r \geq 0$ in such a way that
    $p + q + r = 1$ and $q + 10r = 1$ and
    $\forall \sigma \in \Sigma$, $p^{i_{\sigma}} q^{j_{\sigma}} r^{k_{\sigma}} / S = 1 / (\# \Sigma)$?
If yes, then, using that $(p, q, r)$,
  the first and third systems are equivalent.

It will turn out that the answer is yes.
In the next two sections,
  we will show how to compute the only $(p, q, r)$ that works.
Spoiler: It’s a Boltzmann distribution.

In the section after that, we will show that
the answer our earlier question is
  that Boltzmann $(p, q, r)$ that we computed.

That is, we will show:
Under the first and, equivalently, third system of grant distribution,
my probability of getting $\$ 0$ is approximately $p$,
my probability of getting $\$ 1$ is approximately $q$,
and my probability of getting $\$ 10$ is approximately $r$.
The error in these approximations tends to 0, as $N \to \infty$.

Equivalently, for any of the particles,
it its probability of energy 0 is approximately $p$,
it its probability of energy 1 is approximately $q$.
and it its probability of energy 10 is approximately $r$.
The error in these approximations tends to 0, as $N \to \infty$.

5. Computing $p, q, r$ à la Boltzmann

In this section, we recall: $p, q, r \geq 0$ and $p + q + r = 1$.
We assume: $q + 10r = 1$ and $\forall \sigma \in \Sigma, \ p^i q^j r^{k_\sigma} / S = 1 / \(#\Sigma\)$.
We show: there is at most one $(p, q, r)$ that satisfies these conditions.

Define the dot product, $\odot$, on $\mathbb{R}^3$, by:
$$\forall x, y, z, X, Y, Z \in \mathbb{R}, \ (x, y, z) \odot (X, Y, Z) = xx + yy + zz.$$  
For all $u \in \mathbb{R}^3$, let $u^\perp := \{v \in \mathbb{R}^3 | u \odot v = 0\}$.
For all $U \subseteq \mathbb{R}^3$, let $U^\perp := \{v \in \mathbb{R}^3 | \forall u \in U, u \odot v = 0\}$.
For all $u, v \in \mathbb{R}^3$, let $\langle u, v \rangle$ denote the span of $u$ and $v$, i.e.,
$$\langle u, v \rangle := \{au + bv | a, b \in \mathbb{R}\}.$$
Let $V := (1, 1, 1)^\perp \cap (0, 1, 10)^\perp$. Then $V^\perp = \langle (1, 1, 1), (0, 1, 10) \rangle$.

Let $A$ denote the set of all $(i, j, k)$ s.t. $i, j, k \in [0..N]$ and $(i, j, k) \odot (1, 1, 1) = N$ and $(i, j, k) \odot (0, 1, 10) = N$.
Let $D := \{a - b | a, b \in A\}$. Then $D$ spans $V$, so $D^\perp = V^\perp$.

Know: $\forall (i, j, k) \in A, \ p^i q^j r^k / S = 1 / \(#\Sigma\)$.
Equivalently, $\forall (i, j, k) \in A$,
$$(i, j, k) \odot (\ln p, \ln q, \ln r) = (\ln S) - (\ln(\#\Sigma)).$$
Then: $\forall a, b \in A$,
$$a \odot (\ln p, \ln q, \ln r) = (\ln S) - (\ln(\#\Sigma)) = b \odot (\ln p, \ln q, \ln r).$$
Then: $\forall a, b \in A, \ (a - b) \odot (\ln p, \ln q, \ln r) = 0$.
Then: $(\ln p, \ln q, \ln r) \in D^\perp$.
Since $(\ln p, \ln q, \ln r) \in D^\perp = V^\perp = \langle (1, 1, 1), (0, 1, 10) \rangle$,
choose $C > 0$ and $\beta \in \mathbb{R}$ s.t.
$$\ln p, \ln q, \ln r = (\ln C) \cdot (1, 1, 1) - \beta \cdot (0, 1, 10).$$

Then \((p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta}).\)

NOTE: Some readers will recognize this as a Boltzmann distribution.

At this point, \(C\) and \(\beta\) are unknowns.

We now recall that \(p + q + r = 1\) and \(q + 10r = 1\),

and try to solve for \(C\) and \(\beta\).

As \(p + q + r = 1\), we get

\[
C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1,
\]

so

\[
C = \frac{1}{1 + e^{-\beta} + e^{-10\beta}}.
\]

Then \((p, q, r) = \left(\frac{1}{1 + e^{-\beta} + e^{-10\beta}}, e^{-\beta}, e^{-10\beta}\right)\).

Finally, since \(q + 10r = 1\), we get:

\[
e^{-\beta} + 10e^{-10\beta} = 1 + e^{-\beta} + e^{-10\beta},
\]

so \(0 = 1 - 9e^{-10\beta}\),

so \(e^{-10\beta} = 1/9\),

so \(-10\beta = -\ln 9\),

so \(\beta = (\ln 9)/10\).

Then \(\beta = 9^{1/10}\) and \(e^{-10\beta} = 9^{-1}\).

Then \((p, q, r) = \left(\frac{1}{1 + 9^{-1/10} + 9^{-1}}, 9^{-1/10}, 9^{-1}\right)\).

6. Showing the Boltzmann \(p, q, r\) work

In this section, we prove

the converse of the result from the preceding section.

That is, we assume \((p, q, r) = \left(\frac{1}{1 + 9^{-1/10} + 9^{-1}}, 9^{-1/10}, 9^{-1}\right)\),

and we wish to show \(p + q + r = 1\) and \(q + 10r = 1\) and

\[ \forall \sigma \in \Sigma, \ p^\sigma q^\sigma r^\sigma / S = 1 / (\# \Sigma). \]

MORE LATER

7. The original question

Thanks to J. Steif, for showing me how to use

the Local Limit Theorem and the Large Deviations Theorem

to solve the original question. I present his suggestions in this section.

Assume \((p, q, r) = \left(\frac{1}{1 + 9^{-1/10} + 9^{-1}}, 9^{-1/10}, 9^{-1}\right)\).

Then \(p + q + r = 1\) and \(q + 10r = 1\) and

\[ \forall \sigma \in \Sigma, \ p^\sigma q^\sigma r^\sigma / S = 1 / (\# \Sigma). \]

Let \(X_1, \ldots, X_N\) be iid random variables.
Assume: $\forall n \in [1..N],$
\[
\begin{align*}
\Pr[X_n = 0] &= p, \\
\Pr[X_n = 1] &= q, \\
\Pr[X_n = 10] &= r.
\end{align*}
\]
Under the second system of awards, $\forall n \in [1..N],$
Professor $i$ would receive $X_i$ dollars. Let $T := X_1 + \cdots + X_N.$
For the third system of awards,
we also use $X_1, \ldots, X_N,$ but conditioned on $T_N = N.$
Because of our choice of $p, q, r,$ the third system is the same as the first.
We claim that $p, q, r$ are approximate answers
to the original questions we asked.
We wish to show: $\forall n \in [1..N],$
\[
\begin{align*}
\Pr[X_n = 0 | T_N = N] &\approx p, \\
\Pr[X_n = 1 | T_N = N] &\approx q, \\
\Pr[X_n = 10 | T_N = N] &\approx r,
\end{align*}
\]
with error tending to 0 as $N \to \infty.$

MORE LATER