Professors and Grants

1. Introduction

This note is intended as a compliment and complement to B. Zhang’s very enjoyable “Coconuts and Islanders”, which motivates the Boltzmann distribution in the case where every nonnegative integer is a possible energy-level. Here we focus instead on Boltzmann distributions where the only possible energy-levels are 0 and 1 and 10. Taking our cue from “Coconuts and Islanders”, we motivate by story.

Let $N$ be a large positive integer. We analyze three systems for dispensing grant money to $N$ professors. The grant rules stipulate: each professor receives $0$ or $1$ or $10$. Each professor is identified by a number, from 1 to $N$. By a dispensation, we mean a full complement of awards, with a specific amount ($0$ or $1$ or $10$) to Professor#1, a specific amount ($0$ or $1$ or $10$) to Professor#2, etc., up to and including Professor#N.

The first system for awarding grants is very simple: Congress allocates $N$ dollars to award to the $N$ professors. There are many ways to dispense the $N$ dollars. Among all these possible award dispensations, one is selected randomly, giving equal probability to each possible dispensation.

The main problem is to figure out: Using this first system, for a given professor, what is the probability of being awarded $0$? $1$? $10$?

Later, second and third probabilistic award systems are described, each of which depends on three parameters $p,q,r$ satisfying $p,q,r \geq 0$ and $p + q + r = 1$.

The second system uses an iid system of random variables, $X_1, \ldots, X_N$ such that, for all $\ell$, $\Pr[X_\ell = 0] = p$, $\Pr[X_\ell = 1] = q$, $\Pr[X_\ell = 10] = r$.  

For all \( \ell \), the second system pays \( X_\ell \) dollars to Professor\#\( \ell \).
The total dollar payout \( X_1 + \cdots + X_N \) is then random;
- it could be as small as 0 dollars, and
- it could be as large as \( 10N \) dollars.

The **third system**
is obtained from the second, by conditioning on \( X_1 + \cdots + X_N = N \),
so that the total dollar payout is \( N \).

**KEY POINT:** With exactly the right choice of \( p, q, r \),
the first and third systems are shown to be equivalent.
In §6 and §7, we show that this parameter choice is Boltzmann,
meaning: \( (p, q, r) \) is, for some real number \( \beta \),
a scalar multiple of \( (1, e^{-\beta}, e^{-10\beta}) \).

The second and third systems are
accessible by basic tools of probability theory,
while our main problem involves the first system.
However, once we know the first and third systems are equivalent,
we can bring these probabilistic tools to bear on the main problem.
Thanks to J. Steif, for pointing out to me that
the Discrete Local Limit Theorem, which is described in §8,
is the right tool for the main problem, which is solved in §11.

In §13, we show how inequitable
a typical (randomly selected) dispensation is.
In fact, assuming \( N \) is sufficiently large, we will show that:
with probability \( > 99\% \), over half of the professors receive $0.
Thanks to V. Reiner for suggesting
applying Chebyshev’s inequality to a sum of indicator variables
to establish this result.

In the last three sections (§15, §16, §17), we generalize our results to
finite sets that are different from \( \{0, 1, 10\} \).
The main theorems are Theorem 15.3 and Theorem 16.3.

Boltzmann distributions are often motivated by entropy, but,
from our perspective,
what’s special about \( (p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta}) \) is:
For any $i, j, k \geq 0$, we have
\[ p^i q^j r^k = C^{i+j+k} \cdot e^{-\beta (j+10k)}, \]
so $p^i q^j r^k$ depends only on: $i + j + k$ and $j + 10k$.

In the third system of grant awards,
there exists a normalizing $S > 0$ so, for any $i, j, k \geq 0$,
$p^i q^j r^k / S$ is the probability of any dispensation in which
$i$ professors receive $\$0$,
$j$ professors receive $\$1$,
$k$ professors receive $\$10$.

Then that probability, $p^i q^j r^k / S$,
which is equal to $C^{i+j+k} \cdot e^{-\beta (j+10k)}/S$,
depends only on
$i + j + k$, which is the number of professors, and
$j + 10k$, which is the total dollar payout.

So, if the number of professors is constant and
the total dollar payout is constant,
then all the various award dispensations are equally likely,
which (see above) exactly describes the first system.

2. Some notation

A box around an expression indicates that it is global,
meaning that it is fixed to the end of these notes.
Unboxed variables are freed at the end of each section, if not earlier.

Let
\[ \mathbb{R}^* := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}, \quad \mathbb{Z}^* := \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}. \]

For any $s, t \in \mathbb{R}^*$, let
\[ (s; t) := \{x \in \mathbb{R}^* | s < x < t\}, \quad \{s; t\} := \{x \in \mathbb{R}^* | s \leq x < t\}, \]
\[ (s; t) := \{x \in \mathbb{R}^* | s < x \leq t\}, \quad \{s; t\} := \{x \in \mathbb{R}^* | s \leq x \leq t\}. \]

For any $s, t \in \mathbb{R}^*$, let
\[ (s..t) := (s; t) \cap \mathbb{Z}^*, \quad \{s..t\} := \{s; t\} \cap \mathbb{Z}^*, \]
\[ (s..t) := (s; t) \cap \mathbb{Z}^*, \quad \{s..t\} := \{s; t\} \cap \mathbb{Z}^*. \]

Let $\mathbb{N} := [1, \infty)$ be the set of positive integers.

For any finite set $F$, let $\#F$ be the number of elements in $F$.
For any infinite set $F$, let $\#F := \infty$. Then $\#\mathbb{Z} = \infty = \#\mathbb{R}$. 
3. First system of grant awards

Let $\mathbb{N} \in \mathbb{N}$. Think of $N$ as large.

Suppose there are $N$ professors, numbered 1 to $N$, who apply, once per year, to the GFA (Grant Funding Agency), seeking funding for the very important work they are doing.

Each year, Congress authorizes $N$ for the GFA to dispense to the $N$ professors.

The GFA has the rule: every award is 0 or 1 or 10 dollars.

The set of grant-dispensations is represented by:

$$\Omega := \{ \omega : [1..N] \rightarrow \{0, 1, 10\} \text{ s.t. } \sum_{\ell=1}^{N} (\omega(\ell)) = N \}.$$  

The GFA has set aside $\#\Omega$ pieces of paper, and has written down all possible dispensations, one on each piece of paper.

So, for example, there is a piece of paper that says:

Professors 1 to $N$ each get $1$.

Another piece of paper says:

Professors 1 to $N - 10$ each get $1$ and

Professors $N - 9$ to $N - 1$ each get $0$ and

Professor $N$ gets $10$.

There are, of course, many, many, many other pieces of paper.

Each year, a GFA bureaucrat places all the pieces of paper in a big bin, then selects one at random and makes the awards as indicated on that piece of paper.

Under this first system of awarding grants, we have:

$$\forall \omega \in \Omega, \text{ the probability that}$$

the selected grant-dispensation is $\omega$

is equal to $1/(\#\Omega)$.

Suppose I am one of the professors. Here is our main problem:

Calculate my probability of getting $0$.

Then calculate my probability of getting $1$.

Then calculate my probability of getting $10$.

Approximate answers are acceptable.

In the remainder of this note, we reformulate, and then, in §11, solve this problem.

Spoiler: It’s a Boltzmann distribution, approximately.
4. Particles and Energy

Recall that $N \in \mathbb{N}$. Think of $N$ as large.
Suppose there are $N$ particles, numbered 1 to $N$,
each of which has a certain amount of energy.
Suppose the total energy is $N$, dispensed among the $N$ particles.
Suppose physicists have somehow determined that, for any particle,
its possible energy-levels are: 0 or 1 or 10.
Recall: $\Omega = \{ \omega : [1..N] \rightarrow \{0, 1, 10\} \text{ s.t. } \sum_{\ell=1}^{N} (\omega(\ell)) = N \}$.  
Then $\Omega$ represents the set of energy-dispensations.
Assume that physicists have somehow determined that
this system of particles has a random energy-dispensation
and that all energy-dispensations in $\Omega$ are equally likely.
That is, physicists tell us:
\[
\forall \omega \in \Omega, \quad \text{the probability that the energy-dispensation is } \omega \\
\text{is equal to } \frac{1}{\#\Omega}.
\]
The equal likelihood of all energy-dispensations
is a recurring theme in microcanonical-ensemble thermodynamics,
and can often be motivated through
rules of random energy transfer between random pairs of particles.
These rules then lead to a discrete Markov-chain on
the set, $\Omega$, of all energy-dispensations.
If this Markov-chain is irreducible and aperiodic,
and if its transition-matrix is symmetric,
then, by the Perron-Frobenius Theorem,
asymptotically, all energy-dispensations are equally likely.
Note: Symmetry of the transition-matrix is
a special case of “detailed-balance”.
For a well-explained example of all this,
search for “Coconuts and Islanders” by B. Zhang,
and, in particular, see the work leading up to
the last paragraph of §3.2 therein.
In Zhang’s exposition,
instead of particles exchanging energy,
there are islanders exchanging coconuts,
but the principle is exactly the same.
Returing to our $N$ particles, pick any one of them.
Problem: Calculate its probability of having energy-level 0.
Then calculate its probability of having energy-level 1.
Then calculate its probability of having energy-level 10.
Approximate answers are acceptable.
Spoiler: It’s a Boltzmann distribution, approximately.
Except for terminology, this problem is the same as
the main problem (end of §3) about professors and grants.
We will go back to professors and grants.
Mathematically it makes no difference, but it’s more fun.

5. Second and third systems of grant awards

In an effort to go paperless, the GFA changes to a new system:
In this second system, instead of all those pieces of paper,
the GFA chooses \( p, q, r \geq 0 \) s.t. \( p + q + r = 1 \),
and then, for each of the \( N \) professors,
awards $0 with probability \( p \),
$1 with probability \( q \),
$10 with probability \( r \).
No professor’s award depends in any way on any other professor’s;
the awards are independent.
The expected payout, for any professor, is \( p \cdot 0 + q \cdot 1 + r \cdot 10 \) dollars.
Under this second system,
there is no guarantee that the total payout will be \( \$N \),
which is a difficulty that we will address later.
However, recognizing that the average award is intended to be \( \$1 \),
the GFA chooses the numbers \( p, q, r \) subject to the constraint that
\( p \cdot 0 + q \cdot 1 + r \cdot 10 = 1 \), i.e., \( q + 10r = 1 \).
For each function \( \omega : [1..N] \rightarrow \{0, 1, 10\} \), let
\[
\begin{align*}
i_\omega &= \# \{ \ell \in [1..N] \mid \omega(\ell) = 0 \}, \\
j_\omega &= \# \{ \ell \in [1..N] \mid \omega(\ell) = 1 \}, \\
k_\omega &= \# \{ \ell \in [1..N] \mid \omega(\ell) = 10 \};
\end{align*}
\]
that is, \( i_\omega \) is the number of professors awarded $0 and
\( j_\omega \) is the number of professors awarded $1 and
\( k_\omega \) is the number of professors awarded $10.
Then, \( \forall \omega : [1..N] \rightarrow \{0, 1, 10\} \), we have:
the total number of awards is \( i_\omega + j_\omega + k_\omega \)
and the total dollar payout is \( i_\omega \cdot 0 + j_\omega \cdot 1 + k_\omega \cdot 10 \),
Then, \( \forall \omega : [1..N] \to \{0, 1, 10\}, \) we have:
\[
i_\omega + j_\omega + k_\omega = N \quad \text{and} \quad j_\omega + 10k_\omega = \sum_{\ell=1}^N (\omega(\ell)).
\]

Recall: \( \Omega = \left\{ \omega : [1..N] \to \{0, 1, 10\} \; \text{s.t.} \; \sum_{\ell=1}^N (\omega(\ell)) = N \right\}. \)

That is, \( \Omega \) is the set of all payout functions
\[
\omega : [1..N] \to \{0, 1, 10\}
\]
\[
\text{s.t. the total dollar payout is } N.
\]

Then:
\[
\forall \omega : [1..N] \to \{0, 1, 10\}, \quad \omega \in \Omega \iff j_\omega + 10k_\omega = N.
\]

For every \( i, j, k \in [0..N], \)
\[
\text{if } i + j + k = N \quad \text{and} \quad j + 10k = N,
\]
\[
\text{then } \exists \omega \in \Omega \; \text{s.t.} \; (i, j, k) = (i_\omega, j_\omega, k_\omega),
\]
and we can prove this by defining \( \omega : [1..N] \to \{0, 1, 10\} \) by:
\[
\omega = 0 \text{ on } [1..i], \quad \omega = 1 \text{ on } (i..i+j], \quad \omega = 10 \text{ on } (i+j..N].
\]

Let \( \{A\} := \{ (i_\omega, j_\omega, k_\omega) \mid \omega \in \Omega \}. \)

Then \( A \) is the set of all \( i, j, k \in [0..N] \) s.t. \( i+j+k = N \) and \( j + 10k = N. \)

Observe that, under the second system,
\[
\text{each$0$ award is done with probability } p \quad \text{and}
\]
\[
\text{each$1$ award is done with probability } q \quad \text{and}
\]
\[
\text{each$10$ award is done with probability } r.
\]

So, \( \forall \omega : [1..N] \to \{0, 1, 10\}, \) under the second system,
the probability that the grant-dispensation is equal to \( \omega \) is
\[
p^i q^j r^k.
\]

Let \( S := \sum_{\omega \in \Omega} p^i q^j r^k. \)
Then \( S \) is the probability that \( \omega \in \Omega, \)
\[
i.e., \quad \text{the probability that the total payout is exactly } N \text{ dollars.}
\]

Since \( N \) is large, it turns out that \( S \) is close to zero.

So, under this second system,
\[
\text{the probability of paying out exactly } N \text{ dollars}
\]
is very small.

Congress only allocates $N per year for the \( N \) professors.
So, using this second system, the GFA, in each year,
with probability \( 1 - S \approx 1, \) will run a surplus or a deficit.

On the other hand, the expected payout is $1 per professor,
so, each year, the expected total payout is $N.

So these surpluses and deficits should, over time, cancel one another.
Unfortunately, Congress is a paragon of fiscal responsibility, and,
as soon as it becomes aware of these surpluses and deficits, it insists that the GFA never again underspend or overspend. So the GFA changes its system one more time, as follows. Under its third system, each year, before announcing any of the awards publicly, the GFA writes out, in an internal memo, a tentative proposal of awards that, independently, for each of the $N$ professors, awards $\$0$ with probability $p$, $\$1$ with probability $q$, $\$10$ with probability $r$.

If the memo’s total award payout is NOT equal to $\$N$, the GFA deems the memo as unacceptable, deletes it, and starts over. Each memo has a probability $S$ of being acceptable, so, each year, the GFA will likely need to repeat the memo process many times to get to a memo with total payout exactly equal to $\$N$. However, as soon as that happens, the GFA uses that first acceptable memo, and makes the awards public.

Mathematically, we are conditioning on the event $\omega \in \Omega$. So, using the third system, the probability that $\omega \notin \Omega$ is 0. Also, for this third system, $\forall \omega \in \Omega$, the probability of $\omega$ is $p^{\omega} q^{\omega} r^{\omega} / S$.

The sum of these probabilities is 1:

$$\sum_{\omega \in \Omega} \frac{p^{\omega} q^{\omega} r^{\omega}}{S} = \frac{1}{S} \cdot \sum_{\omega \in \Omega} p^{\omega} q^{\omega} r^{\omega} = \frac{1}{S} \cdot S = 1.$$ 

This third system is not necessarily equivalent to the first, because in the first system, all the probabilities were $1 / (\#\Omega)$, whereas, in the third system, they are $p^{\omega} q^{\omega} r^{\omega} / S$.

So a new question arises:

Is it possible to choose $p, q, r \geq 0$ in such a way that

$p + q + r = 1$ and $q + 10r = 1$ and $\forall \omega \in \Omega$, $p^{\omega} q^{\omega} r^{\omega} / S = 1 / (\#\Omega)$?

If yes, then, using that $(p, q, r)$, the first and third systems are equivalent.

We will see that the answer to the new question, in fact, is yes.

In the next two sections, assuming $N \geq 10$,
we will show how to compute the only \((p, q, r)\) that works. 

**Spoiler:** It’s a Boltzmann distribution, exactly.

6. **Computing \(p, q, r\) à la Boltzmann**

As in the preceding section, let \(p, q, r \geq 0\), \(S := \sum_{\omega \in \Omega} p^i \omega q^j \omega r^k \omega\).

We assume: 
\(p + q + r = 1\) and \(q + 10r = 1\).

We also assume: \(\forall \omega \in \Omega, p^i \omega q^j \omega r^k \omega / S = 1 / (\#\Omega)\). 

**We will prove** that, if \(N \geq 10\), then 

there is at most one \((p, q, r)\) that satisfies these conditions, 

specifically, 
\[ (p, q, r) = \left(\frac{1, 9^{-1/10}, 9^{-1}}{1 + 9^{-1/10} + 9^{-1}}\right) \]

**Define** the dot product, \(\odot\), on \(\mathbb{R}^3\), by:

\[
\forall x, y, z, X, Y, Z \in \mathbb{R}, \ (x, y, z) \odot (X, Y, Z) = xX + yY + zZ.
\]

For all \(u \in \mathbb{R}^3\), let \(u^\perp := \{ v \in \mathbb{R}^3 \mid u \odot v = 0 \}\).

For all \(U \subseteq \mathbb{R}^3\), let \(U^\perp := \{ v \in \mathbb{R}^3 \mid \forall u \in U, u \odot v = 0 \}\).

For all \(u, v \in \mathbb{R}^3\), let \(\langle u, v \rangle_{\text{span}}\) denote the \(\mathbb{R}\)-span of \(\{u, v\}\), i.e.,

let \(\langle u, v \rangle_{\text{span}} := \{ au + bv \mid a, b \in \mathbb{R} \}\).

Recall (§3): \(\Omega = \{ \omega : [1..N] \to \{0, 1, 10\} \ s.t. \ \sum_{\ell=1}^{N} (\omega(\ell)) = N \}\).

Recall (§5): \(A = \{(i, j, k) \mid \omega \in \Omega\}\).

Recall (§5): \(A\) is the set of all \((i, j, k)\) s.t. \(i, j, k \in [0..N]\) and 
\(i + j + k = N\) and \(j + 10k = N\).

Then: \(A\) is the set of all \((i, j, k)\) s.t. \(i, j, k \in [0..N]\) and 
\((i, j, k) \odot (1, 1, 1) = N\) and \((i, j, k) \odot (0, 1, 10) = N\).

For all \(a, b \in A\), we have 
\[
\begin{align*}
a \odot (1, 1, 1) &= N = b \odot (1, 1, 1) \\
a \odot (0, 1, 10) &= N = b \odot (0, 1, 10),
\end{align*}
\]

so we get 
\[
(a - b) \odot (1, 1, 1) = 0 \quad \text{and} \quad (a - b) \odot (0, 1, 10) = 0,
\]

so \(a - b \in (1, 1, 1)^\perp \cap (0, 1, 10)^\perp\).

**Let** \(V := (1, 1, 1)^\perp \cap (0, 1, 10)^\perp\).

Then: \(\forall a, b \in A\), \(a - b \in V\).

**Let** \(D := \{ a - b \mid a, b \in A \}\). Then \(D \subseteq V\).

**Let** \(W := \langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}}\).

Since \((1, 1, 1), (0, 1, 10) \in V^\perp\), we get: \(W \subseteq V^\perp\).

Assume \(N \geq 10\). **Let** \(a_1 := (0, N, 0)\), \(a_2 := (9, N - 10, 1)\).

Then \(a_1, a_2 \in A\). **Let** \(d_1 := a_2 - a_1\). Then \(d_1 \in D\).

Since \(W = \langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}}\), we get: \(\dim W = 2\).
Since $d_1 \neq (0, 0, 0)$, we get: $\dim d_1 = 2$.

Since $d_1 \in D \subseteq V$ and $V^\perp \supseteq W$, we get: $d_1^\perp \supseteq D^\perp \supseteq V^\perp \supseteq W$.

So, since $\dim d_1^\perp = 2 = \dim W$, we get: $d_1^\perp = D^\perp = V^\perp = W$.

Then $D^\perp = W$. Recall: $\forall \omega \in \Omega$, $p^i\omega q^j\omega r^k \omega / S = 1/(\#\Omega)$.

So, since $A = \{(i \omega, j \omega, k \omega) | \omega \in \Omega\}$, we get:

\[\forall (i, j, k) \in A, \quad p^i q^j r^k / S = 1/(\#\Omega).\]

Equivalently, $\forall (i, j, k) \in A$, $i \cdot (\ln p) + j \cdot (\ln q) + k \cdot (\ln r) - (\ln S) = -(\ln(\#\Omega))$.

Equivalently, $\forall (i, j, k) \in A$,

\[(i, j, k) \odot (\ln p, \ln q, \ln r) = (\ln S) - (\ln(\#\Omega)) = 0.\]

Then: $\forall a, b \in A$,

\[a \odot (\ln p, \ln q, \ln r) = (\ln S) = b \odot (\ln p, \ln q, \ln r),\]

so we get: $a - b \odot (\ln p, \ln q, \ln r) = 0$.

Then: $\forall d \in D$,

\[d \odot (\ln p, \ln q, \ln r) = 0.\]

Then: $(\ln p, \ln q, \ln r) \in D^\perp$.

Since $(\ln p, \ln q, \ln r) \in D^\perp = W = \langle (1, 1, 1), (0, 1, 10) \rangle_{\text{span}}$.

Choose $\beta > 0$ and $\beta \in \mathbb{R}$ s.t.

\[p, q, r = (\ln C) \cdot (1, 1, 1) - \beta \cdot (0, 1, 10).\]

Then $\ln C = (\ln C, (\ln C) - \beta, (\ln C) - 10\beta)$.

Then $(p, q, r) = (C, Ce^{-\beta}, Ce^{-10\beta})$.

Then $(p, q, r) = C \cdot (1, e^{-\beta}, e^{-10\beta})$.

So, since $p + q + r = 1$, we get $C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1$.

Then $C = \frac{1}{1 + e^{-\beta} + e^{-10\beta}}$. Then $(p, q, r) = (1, e^{-\beta}, e^{-10\beta})$.

So, since $q + 10r = 1$, we get: $\frac{e^{-\beta} + 10e^{-10\beta}}{1 + e^{-\beta} + e^{-10\beta}} = 1$.

Then $e^{-\beta} + 10e^{-10\beta} = 1 + e^{-\beta} + e^{-10\beta}$. Then $0 = 1 - 9e^{-10\beta}$.

Then $e^{-10\beta} = 9^{-1}$. Then $e^{-\beta} = 9^{-1/10}$.

Then $\beta = (\ln 9)/10$. Also, $C = \frac{1}{1 + 9^{-1/10} + 9^{-1}}$.

Also, $(p, q, r) = \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}}$.

So this is the only $(p, q, r)$ that can possibly work.

In the next section, we show that it does work.

7. Showing the Boltzmann $p, q, r$ work

In this section, we prove

the converse of the result from the preceding section.
That is, we let \( (p, q, r) := \frac{(1, 9^{-1/10}, 9^{-1})}{1 + 9^{-1/10} + 9^{-1}} \) and \( S := \sum_{\omega \in \Omega} p^q q^{i\omega} r^{k\omega} \), and we wish to show: \( p + q + r = 1 \) and \( q + 10r = 1 \) and \( \forall \omega \in \Omega, \ p^q q^{i\omega} r^{k\omega} / S = 1 / (\#\Omega) \).

Let \( \beta := (\ln 9)/10 \). Then \( e^{-\beta} = 9^{-1/10} \). Then \( e^{-10\beta} = 9^{-1} \).

Then \( (p, q, r) = \frac{(1, e^{-\beta}, e^{-10\beta})}{1 + e^{-\beta} + e^{-10\beta}} \).

Let \( C := \frac{1}{1 + e^{-\beta} + e^{-10\beta}} \).

Then \( (p, q, r) = C \cdot (1, e^{-\beta}, e^{-10\beta}) \).

Let \( K := C^N \cdot e^{-\beta \cdot N} \).

Claim: \( \forall \omega \in \Omega, \ p^q q^{i\omega} r^{k\omega} = K \).

Proof of Claim: Given \( \omega \in \Omega \), want: \( p^q q^{i\omega} r^{k\omega} = K \).

Recall (§5): \( i_\omega + j_\omega + k_\omega = N \) and \( j_\omega + 10k_\omega = \sum_{\ell=1}^{N} (\omega(\ell)) \).

By definition of \( \Omega \), since \( \omega \in \Omega \), we get: \( \sum_{\ell=1}^{N} (\omega(\ell)) = N \).

Then: \( j_\omega + 10k_\omega = N \).

Recall: \( (p, q, r) = (C, Ce^{-\beta}, C^{-10\beta}) \).

Then: 
\[
\begin{align*}
p^q q^{i\omega} r^{k\omega} &= C^{i_\omega} \cdot (Ce^{-\beta})^{j_\omega} \cdot (C^{-10\beta})^{k\omega} \\
&= C^{i_\omega + j_\omega + k_\omega} \cdot e^{-\beta \cdot (j_\omega + 10k_\omega)} \\
&= C^N \cdot e^{-\beta \cdot N} = K.
\end{align*}
\]
End of proof of Claim.

By definition of \( S \), we have: \( S = \sum_{\omega \in \Omega} p^q q^{i\omega} r^{k\omega} \).

So, by the Claim, we get: \( S = (\#\Omega) \cdot K \). Then \( K/S = 1/(\#\Omega) \).

We have \( 10/9 = 1 + (1/9) \). That is, \( 10 \cdot 9^{-1} = 1 + 9^{-1} \).

So, since \( e^{-10\beta} = 9^{-1} \), we get: \( 10e^{-10\beta} = 1 + e^{-10\beta} \).

Then: \( e^{-\beta + 10e^{-10\beta}} = 1 + e^{-\beta} + e^{-10\beta} \).

By definition of \( C \), we get: \( C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1 \).

Recall: \( (p, q, r) = (C, Ce^{-\beta}, C^{-10\beta}) \).

Since \( p + q + r = C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1 \)
and since \( q + 10r = C \cdot (e^{-\beta} + 10e^{-10\beta}) = C \cdot (1 + e^{-\beta} + e^{-10\beta}) = 1 \),

it remains only to show: \( \forall \omega \in \Omega, \ p^q q^{i\omega} r^{k\omega} / S = 1 / (\#\Omega) \).

Given \( \omega \in \Omega \), want: \( p^q q^{i\omega} r^{k\omega} / S = 1 / (\#\Omega) \).

By the Claim, we get: \( p^q q^{i\omega} r^{k\omega} = K \).

Recall: \( K/S = 1/(\#\Omega) \).

Then: \( p^q q^{i\omega} r^{k\omega} / S = K/S = 1/(\#\Omega) \).

8. The Discrete Local Limit Theorem

**Definition 8.1.** Let \( E \subseteq \mathbb{Z} \).

By \( E \) is [residue-constrained] we mean:
By \( E \) is residue-unconstrained, we mean:

\( E \) is not residue-constrained.

Since \( \emptyset \subseteq 2 \cdot \mathbb{Z} + 1 \), we get: \( \emptyset \) is residue-constrained.

For all \( b \in \mathbb{Z} \), since \( \{b\} \subseteq 2 \cdot \mathbb{Z} + b \), we get: \( \{b\} \) is residue-constrained.

Then: \( \forall \) residue-unconstrained \( E \subseteq \mathbb{R} \), we have: \( \#E \geq 2 \).

We have: \( \{0, 2, 10\} \subseteq 2\mathbb{Z} + 0 \) and \( \{1, 3, 11\} \subseteq 2\mathbb{Z} + 1 \),
so \( \{0, 2, 10\} \) and \( \{1, 3, 11\} \) are both residue-constrained.

Here is a test for residue-unconstrainedness:
Let \( E \subseteq \mathbb{Z} \). Assume \( \#E \geq 2 \). Let \( \varepsilon_0 \in E \).

Then: (\( E \) is residue-unconstrained) iff (\( \gcd(E - \varepsilon_0) = 1 \)).

By this test, we see that:
\( \{0, 1, 10\} \) and \( \{2, 4, 9\} \) and \( \{3, 9, 13, 18\} \) are all residue-unconstrained.

By convention, in this note, any countable set is given its discrete Borel structure.

A measure \( \mu \) on a nonempty countable set \( \Gamma \) is therefore completely determined by
the function \( t \mapsto \mu\{t\} : \Gamma \to [0; \infty] \),
because: \( \forall \Gamma_0 \subseteq \Gamma \), we have \( \mu(\Gamma_0) = \sum_{t \in \Gamma_0} \mu\{t\} \).

**DEFINITION 8.2.** Let \( \Gamma \) be a nonempty countable set.
Then \( \mathcal{M}(\Gamma) \) denotes the set of measures on \( \Gamma \),
and \( \mathcal{FM}(\Gamma) := \{\mu \in \mathcal{M}(\Gamma) \mid \mu(\Gamma) < \infty\} \),
and \( \mathcal{FM}^\times(\Gamma) := \{\mu \in \mathcal{M}(\Gamma) \mid 0 < \mu(\Gamma) < \infty\} \),
and \( \mathcal{P}(\Gamma) := \{\mu \in \mathcal{M}(\Gamma) \mid \mu(\Gamma) = 1\} \).

Then \( \mathcal{FM}(\Gamma) \) is the set of finite measures on \( \Gamma \).
Also, \( \mathcal{FM}^\times(\Gamma) \) is the set of nonzero finite measures on \( \Gamma \).
Also, \( \mathcal{P}(\Gamma) \) is the set of probability measures on \( \Gamma \).

**DEFINITION 8.3.** Let \( \Gamma \) be a nonempty countable set, \( \mu \in \mathcal{FM}(\Gamma) \).
Let \( n \in \mathbb{N} \). Then \( \mu^n \in \mathcal{FM}(\Gamma^n) \) is defined by:
\( \forall x \in \Gamma^n, \mu^n\{x\} = (\mu\{x_1\}) \cdots (\mu\{x_n\}) \).

The following is a basic fact, whose proof we omit:

Let \( \Gamma \) be a nonempty countable set, \( n \in [2; \infty) \), \( \mu \in \mathcal{FM}(\Gamma) \).
Let \( Z \subseteq \Gamma^n \), \( X \subseteq \Gamma^{n-1} \), \( Y \subseteq \Gamma \). Assume that:
under the standard bijection \( \Gamma^n \leftrightarrow \Gamma^{n-1} \times \Gamma \),
we have: \( Z \leftrightarrow X \times Y \).

Then:
\[
\mu^n(Z) = (\mu^{n-1}(X)) \cdot (\mu(Y)).
\]

The countable sets that are of interest here
all carry the discrete topology.

We therefore define:

**Definition 8.4.** Let \( \Gamma \) be a nonempty countable set, \( \mu \in \mathcal{M}(\Gamma) \).
Then: the **support of \( \mu \)** is:
\[
S_\mu := \{ t \in \Gamma \mid \mu \{ t \} \neq 0 \}.
\]

**Definition 8.5.** Let \( \Gamma \subseteq \mathbb{R} \) be nonempty and countable.
Let \( \mu \in \mathcal{P}(\Gamma) \).
Then: the **mean of \( \mu \)** is:
\[
M_\mu := \sum_{t \in \Gamma} [ t \cdot (\mu(t))].
\]
Also, the **variance of \( \mu \)** is:
\[
V_\mu := \sum_{t \in \Gamma} [(t - M_\mu)^2 \cdot (\mu(t))].
\]

**Theorem 8.6.** Let \( \Gamma \subseteq \mathbb{R} \) be nonempty and countable, \( \mu \in \mathcal{P}(\Gamma) \).
Assume: \( \sum_{t \in \Gamma} (|t| \cdot (\mu(t))) < \infty \).
Then: \( (\#S_\mu \geq 2) \iff (V_\mu > 0) \).
Also: \( (\sum_{t \in \Gamma} t^2 \cdot (\mu(t))) < \infty \) \( \iff (V_\mu < \infty) \).

The preceding result is basic. We omit its proof.

We also omit the proof of the following elementary facts.

Let \( \Gamma \subseteq \mathbb{R} \) be nonempty and countable. Let \( \mu \in \mathcal{F} \mathcal{M}(\Gamma) \).
If \( \#S_\mu < \infty \), then both \( \sum_{t \in \Gamma} (|t| \cdot (\mu(t))) < \infty \) and \( \sum_{t \in \Gamma} (t^2 \cdot (\mu(t))) < \infty \).

Since \( \forall t \in \mathbb{Z}, \ |t| \leq t^2 \),
we get: \( \forall \text{nonempty } \Gamma \subseteq \mathbb{Z}, \ \forall \mu \in \mathcal{F} \mathcal{M}(\Gamma) \),
\[
(\sum_{t \in \Gamma} t^2 \cdot (\mu(t))) < \infty \ \Rightarrow \ (\sum_{t \in \Gamma} (|t| \cdot (\mu(t))) < \infty).
\]

**Definition 8.7.** For all \( \alpha \in \mathbb{R} \), for all \( v > 0 \),
define \( \Phi^v_\alpha : \mathbb{R} \to (0; \infty) \) by: \( \forall t \in \mathbb{R} \), \( \Phi^v_\alpha(t) = \frac{\exp(- (t - \alpha)^2 / (2v))}{\sqrt{2\pi}v} \).

Note: \( \Phi^v_\alpha \) is a PDF of a normal variable with mean \( \alpha \) and variance \( v \).

The next result is the Discrete Local Limit Theorem,
stated in measure-theoretic terms:

**Theorem 8.8.** Let \( \Gamma \subseteq \mathbb{Z} \), \( \mu \in \mathcal{P}(\Gamma) \), \( E := S_\mu \).
Assume: \( E \) is residue-unconstrained.
Assume: \( \sum_{t \in E} t^2 \cdot (\mu(t))) < \infty \). Let \( \alpha := M_\mu \), \( v := V_\mu \).
Let $t_1, t_2, \ldots \in \mathbb{Z}$. Then, as $n \to \infty$, 
\[
\sqrt{n} \cdot \left[ \left( \mu^n \{ f \in E^n \mid f_1 + \cdots + f_n = t_n \} \right) - \left( \Phi^{nv}_{n\alpha}(t_n) \right) \right] \to 0.
\]

Next is another version of the Discrete Local Limit Theorem;
this one is stated in probability-theoretic terms:

**THEOREM 8.9.** Let $E \subseteq \mathbb{Z}$ be residue-unconstrained.
Let $X_1, X_2, \ldots$ be a $\mathbb{Z}$-valued iid sequence of $L^2$ random-variables.
Assume: $\forall n \in \mathbb{N}, \{ t \in \mathbb{Z} \mid \Pr[X_n = t] > 0 \} = E$.
Let $\alpha \in \mathbb{R}, v > 0$. Assume: $\forall n \in \mathbb{N}, E[X_n] = \alpha$ and $\text{Var}[X_n] = v$.
Let $t_1, t_2, \ldots \in \mathbb{Z}$. Then, as $n \to \infty$, 
\[
\sqrt{n} \cdot \left[ \Pr[X_1 + \cdots + X_n = t_n] \right) - \left( \Phi^{nv}_{n\alpha}(t_n) \right] \to 0.
\]

For a good exposition of this theorem and its proof,
search on “Terence Tao Local Limit Theorem”.
Visit the website, and then expand “read the rest of this entry”;
and then scroll down to “– 2. Local limit theorems –”.

Here is an application of Theorem 8.9:

**THEOREM 8.10.** Let $E \subseteq \mathbb{Z}$ be residue-unconstrained.
Let $X_1, X_2, \ldots$ be a $\mathbb{Z}$-valued iid sequence of $L^2$ random-variables.
Assume: $\forall n \in \mathbb{N}, \{ t \in \mathbb{Z} \mid \Pr[X_n = t] > 0 \} = E$.
Let $\alpha \in \mathbb{R}, v > 0$. Assume: $\forall n \in \mathbb{N}, E[X_n] = \alpha$ and $\text{Var}[X_n] = v$.
Let $t_1, t_2, \ldots \in \mathbb{Z}$. Assume: $\{ t_n - n\alpha \mid n \in \mathbb{N} \}$ is bounded.
Then: as $n \to \infty$, 
\[
\sqrt{n} \cdot \left( \Pr[X_1 + \cdots + X_n = t_n] \right) \to 1/\sqrt{2\pi v}.
\]

Equivalently:
\[
\text{as } n \to \infty, \quad \Pr[X_1 + \cdots + X_n = t_n] \text{ is asymptotic to } 1/\sqrt{2\pi nv}.
\]

**Proof.** By Theorem 8.9, want:
\[
\text{as } n \to \infty, \quad \sqrt{n} \cdot \left( \Phi^{nv}_{n\alpha}(t_n) \right) \to 1/\sqrt{2\pi v}.
\]
We have: $\forall n \in \mathbb{N}, \Phi^{nv}_{n\alpha}(t_n) = \frac{\exp(-\frac{(t_n - n\alpha)^2}{2nv})}{\sqrt{2\pi nv}}$.
Then: $\forall n \in \mathbb{N}, \sqrt{n} \cdot \left( \Phi^{nv}_{n\alpha}(t_n) \right) = \frac{\exp(-\frac{(t_n - n\alpha)^2}{2nv})}{\sqrt{2\pi v}}$.
Since $\{ t_n - n\alpha \mid n \in \mathbb{N} \}$ is bounded, we get:
\[
\text{as } n \to \infty, \quad \frac{(t_n - n\alpha)^2}{2nv} \to 0.
\]
Then: $\text{as } n \to \infty, \quad \sqrt{n} \cdot \left( \Phi^{nv}_{n\alpha}(t_n) \right) \to 1/\sqrt{2\pi v}$. \hfill \Box

We record a measure-theoretic version of Theorem 8.10:

**THEOREM 8.11.** Let $\Gamma \subseteq \mathbb{Z}, \quad \mu \in \mathcal{P}(\Gamma), \quad E := S_\mu$.
Assume: $E$ is residue-unconstrained.
Assume: $\sum_{t \in E} (t^2 \cdot (\mu\{t\})) < \infty$. Let $\alpha := M_\mu$, $v := V_\mu$.

Let $t_1, t_2, \ldots \in \mathbb{Z}$. Assume: $\{t_n - n\alpha \mid n \in \mathbb{N}\}$ is bounded.

Then: as $n \to \infty$, $\sqrt{n} \cdot (\mu^n\{f \in E^n \mid f_1 + \cdots + f_n = t_n\}) \to 1/\sqrt{2\pi v}$.

9. **AVERAGE EVENTS HAVE LOW INFORMATION, PARTICULAR CASE**

Suppose, in secret, I flip a coin 1000 times, then reveal to you that the total number of heads was 1000, and then ask you to guess the last flip. The answer is that, since all the coin flips were heads, the last flip must have been a head.

Similarly, if I had told you that the total number of heads was 0, then you would have known that the last flip was a tail.

By contrast, if I had told you that the total number of heads was 500, it seems intuitively clear that you’d have had very little information about the last flip.

We wish to formalize that intuition, and then prove it rigorously.

Let $X_1, X_2, X_3, \ldots$ be iid random variables s.t., $\forall n \in \mathbb{N}$,

$\Pr[X_n = -1] = 1/2,$
$\Pr[X_n = 0] = 1/3,$
$\Pr[X_n = 3] = 1/6.$

Then: $\forall n \in \mathbb{N}$, $\mathbb{E}[X_n] = 0$ and $\text{Var}[X_n] = 2$.

Also, $\forall n \in \mathbb{N}$, $-1 \leq X_n \leq 3$ a.s.

For all $n \in \mathbb{N}$, let $T_n := X_1 + \cdots + X_n$.

Then: $\forall n \in \mathbb{N}$, $-n \leq T_n \leq 3n$ a.s.

Then: $-1000 \leq T_{1000} \leq 3000$ a.s.

Also, $[T_{1000} = -1000] \Rightarrow [X_1 = \cdots = X_{1000} = -1]$, and so $\Pr[X_{1000} = -1 \mid T_{1000} = -1000] = 1$.

Similarly, $\Pr[X_{1000} = 3 \mid T_{1000} = 3000] = 1$.

By contrast, the event $T_{1000} = 0$ would seem to give very little information about $X_{1000}$.

It therefore seems reasonable to expect that

$\Pr[X_{1000} = -1 \mid T_{1000} = 0] \approx 1/2$ and
$\Pr[X_{1000} = 0 \mid T_{1000} = 0] \approx 1/3$ and
$\Pr[X_{1000} = 3 \mid T_{1000} = 0] \approx 1/6$. 


More precisely, we expect that, as \( n \to \infty \),

\[
\begin{align*}
\Pr[X_n = -1 | T_n = 0] & \to 1/2 \\
\Pr[X_n = 0 | T_n = 0] & \to 1/3 \\
\Pr[X_n = 3 | T_n = 0] & \to 1/6.
\end{align*}
\]

We will focus on proving the third of these limits; proofs of the other two are similar.

**We wish to prove:** As \( n \to \infty \), \( \Pr[X_n = 3 | T_n = 0] \to 1/6 \).

By definition of conditional probability,

**we wish to prove:** As \( n \to \infty \), \( \frac{\Pr[(X_n = 3) \& (T_n = 0)]}{\Pr[T_n = 0]} \to 1/6 \).

**Claim:** Let \( n \in \{2, \infty\} \).

Then: \( \Pr[(X_n = 3) \& (T_n = 0)] = (1/6) \cdot (\Pr[T_{n-1} = -3]) \).

**Proof of Claim:** We have: \( T_n = X_1 + \cdots + X_{n-1} + X_n \).

Since \( \Pr[(X_n = 3) \& (T_n = 0)] \)

\[= \Pr[(X_n = 3) \& (X_1 + \cdots + X_{n-1} + X_n = 0)]\]

\[= \Pr[(X_n = 3) \& (X_1 + \cdots + X_{n-1} + 3 = 0)]\]

\[= \Pr[(X_n = 3) \& (X_1 + \cdots + X_{n-1} = -3)],\]

it follows, from independence of \( X_1, \ldots, X_n \), that

\[\Pr[(X_n = 3) \& (T_n = 0)] = (\Pr[X_n = 3]) \cdot (\Pr[X_1 + \cdots + X_{n-1} = -3]).\]

So, since \( \Pr[X_n = 3] = 1/6 \) and \( X_1 + \cdots + X_{n-1} = T_{n-1} \),

we get: \( \Pr[(X_n = 3) \& (T_n = 0)] = (1/6) \cdot (\Pr[T_{n-1} = -3]). \)

**End of proof of Claim.**

By the claim, we wish to prove:

As \( n \to \infty \), \( \frac{(1/6) \cdot (\Pr[T_{n-1} = -3])}{\Pr[T_n = 0]} \to 1/6.\)

**We wish to prove:** As \( n \to \infty \), \( \frac{\Pr[T_{n-1} = -3]}{\Pr[T_n = 0]} \to 1.\)

That is, we wish to prove:

As \( n \to \infty \), \( \Pr[T_{n-1} = -3] \) is asymptotic to \( \Pr[T_n = 0] \).

So the question becomes:

How do we get a handle on the asymptotics, as \( n \to \infty \), of both \( \Pr[T_{n-1} = -3] \) and \( \Pr[T_n = 0] \) ?

The Discrete Local Limit Theorem turns out to be just what we need.

Recall: \( \forall n \in \mathbb{N}, \ E[X_n] = 0 \) and \( T_n = X_1 + \cdots + X_n \).
By Theorem 8.10, as \( n \to \infty, \sqrt{n} \cdot (\Pr[T_n = 0]) \to 1/\sqrt{4\pi}, \)
so, as \( n \to \infty, \Pr[T_n = 0] \) is asymptotic to \( 1/\sqrt{4\pi n} \).

By Theorem 8.10, as \( n \to \infty, \sqrt{n} \cdot (\Pr[T_n = -3]) \to 1/\sqrt{4\pi}. \)
Then, as \( n \to \infty, \sqrt{n-1} \cdot (\Pr[T_{n-1} = -3]) \to 1/\sqrt{4\pi}, \)
so, as \( n \to \infty, \Pr[T_{n-1} = -3] \) is asymptotic to \( 1/\sqrt{4\pi(n-1)}, \)
which is asymptotic to \( 1/\sqrt{4\pi n}, \)
which is asymptotic to \( \Pr[T_n = 0] \).

10. Average events have low information, general result

We generalize the result of the last section:

**THEOREM 10.1.** Let \( E \subseteq \mathbb{Z} \) be residue-unconstrained, \( \varepsilon_0 \in E. \)
Let \( X_1, X_2, \ldots \) be a \( \mathbb{Z} \)-valued iid sequence of \( L^2 \) random-variables.
Assume: \( \forall n \in \mathbb{N}, \{t \in \mathbb{Z} | \Pr[X_n = t] > 0\} = E. \)
Let \( \alpha, P \in \mathbb{R}. \) Assume: \( \forall n \in \mathbb{N}, \ E[X_n] = \alpha \) and \( \Pr[X_n = \varepsilon_0] = P. \)
Then: as \( n \to \infty, \Pr[X_n = \varepsilon_0 | X_1 + \cdots + X_n = n\alpha] \to P. \)

Part of the content of Theorem 10.1 is:
\( \forall \) sufficiently large \( n \in \mathbb{N}, \Pr[X_1 + \cdots + X_n = n\alpha] > 0. \)

**Proof.** Since \( E \) is residue-unconstrained, we get: \( \#E \geq 2. \)
Then: \( \forall n \in \mathbb{N}, \) since \( \#\{t \in \mathbb{Z} | \Pr[X_n = t] > 0\} = \#E \geq 2, \)
we get: \( \operatorname{Var}[X_n] > 0. \)

So, since \( X_1, X_2, \ldots \) is identically distributed,
\textbf{choose} \( v > 0 \) s.t., \( \forall n \in \mathbb{N}, \operatorname{Var}[X_n] = v. \)
For all \( n \in \mathbb{N}, \) \textbf{let} \( T_n := X_1 + \cdots + X_n. \)
\textbf{Want:} as \( n \to \infty, \Pr[X_n = \varepsilon_0 | T_n = n\alpha] \to P. \)

By Theorem 8.10,
\[ \text{as } n \to \infty, \sqrt{n} \cdot (\Pr[T_n = n\alpha]) \to 1/\sqrt{2\pi v}. \]
Also, by Theorem 8.10,
\[ \text{as } n \to \infty, \sqrt{n} \cdot (\Pr[T_n = (n+1)\alpha - \varepsilon_0]) \to 1/\sqrt{2\pi v}, \]
so, as \( n \to \infty, \sqrt{n-1} \cdot (\Pr[T_{n-1} = n\alpha - \varepsilon_0]) \to 1/\sqrt{2\pi v}. \)
Then, as \( n \to \infty, \frac{\sqrt{n-1} \cdot (\Pr[T_{n-1} = n\alpha - \varepsilon_0])}{\sqrt{n} \cdot (\Pr[T_n = n\alpha])} \to 1/\sqrt{2\pi v}. \)
Then, as \( n \to \infty, \frac{\sqrt{n-1} \cdot (\Pr[T_{n-1} = n\alpha - \varepsilon_0])}{\sqrt{n} \cdot (\Pr[T_n = n\alpha])} \to 1. \)
Also, as \( n \to \infty, \frac{\sqrt{n}}{\sqrt{n-1}} \to 1. \)
Multiplying the last two limits together, we get:
as \( n \to \infty \), \( \frac{\Pr[T_{n-1} = n\alpha - \varepsilon_0]}{\Pr[T_n = n\alpha]} \to 1 \).

So, since, \( \forall n \in [2, \infty) \),
\[
\Pr[ X_n = \varepsilon_0 \mid T_n = n\alpha ] = \frac{ \Pr[(X_n = \varepsilon_0) \& (T_n = n\alpha)] }{ \Pr[T_n = n\alpha] } \\
= \frac{ \Pr[(X_n = \varepsilon_0) \& (T_{n-1} + X_n = n\alpha)] }{ \Pr[T_n = n\alpha] } \\
= \frac{ \Pr[(X_n = \varepsilon_0) \& (T_{n-1} + \varepsilon_0 = n\alpha)] }{ \Pr[T_n = n\alpha] } \\
= \frac{ \Pr[(X_n = \varepsilon_0) \& (T_{n-1} = n\alpha - \varepsilon_0)] }{ \Pr[T_n = n\alpha] } \\
= \frac{ (\Pr[X_n = \varepsilon_0]) \cdot (\Pr[T_{n-1} = n\alpha - \varepsilon_0]) }{ \Pr[T_n = n\alpha] } \\
= \frac{ P \cdot \Pr[T_{n-1} = n\alpha - \varepsilon_0] }{ \Pr[T_n = n\alpha] },
\]

we get: as \( n \to \infty \), \( \Pr[ X_n = \varepsilon_0 \mid T_n = n\alpha ] \to P \). \( \square \)

Recall (§8): \( \forall \)nonempty countable set \( \Gamma \),
\[
\mathcal{M}(\Gamma) \quad \text{is} \quad \text{the set of measures on} \ \Gamma
\]
and \( \mathcal{F}\mathcal{M}^\times(\Gamma) \quad \text{is} \quad \text{the set of nonzero finite measures on} \ \Gamma
\]
and \( \mathcal{P}(\Gamma) \quad \text{is} \quad \text{the set of probability measures on} \ \Gamma \).

For any countable set \( \Delta \), for any \( \mu \in \mathcal{M}(\Delta) \), for any \( \Gamma \subseteq \Delta \),
the \( \text{restriction of} \ \mu \ \text{to} \ \Gamma \), denoted \( \mu|\Gamma \in \mathcal{M}(\Gamma) \),
is \( \text{defined by:} \ \forall \Gamma_0 \subseteq \Gamma \), \( (\mu|\Gamma)(\Gamma_0) = \mu(\Gamma_0) \).

For any nonempty countable set \( \Gamma \), for any \( \mu \in \mathcal{F}\mathcal{M}^\times(\Gamma) \),
\[
\text{let} \ \mathcal{N}(\mu) := \frac{\mu}{\mu(\Gamma)} \in \mathcal{P}(\Gamma); \ \text{then} \ \forall S \subseteq \Gamma \), \( (\mathcal{N}(\mu))(S) = \frac{\mu(S)}{\mu(\Gamma)} \),
and \( \mathcal{N}(\mu) \) is called the \( \text{normalization of} \ \mu \).

We record a measure-theoretic version of Theorem 10.1:

**THEOREM 10.2.** \( \text{Let} \ \Gamma \subseteq \mathbb{Z} \ \text{be nonempty,} \ \mu \in \mathcal{P}(\Gamma), \ E := S_\mu. \)

**Assume:** \( E \) \( \text{is residue-unconstrained.} \ \text{Let} \ \varepsilon_0 \in E, \ P := \mu(\varepsilon_0). \)

**Assume:** \( \sum_{t \in E} (t^2 \cdot (\mu(t))) < \infty. \ \text{Let} \ \alpha := M_\mu. \)

For all \( n \in \mathbb{N} \), \( \text{let} \ \Omega_n := \{ f \in E^n \mid f_1 + \cdots + f_n = n\alpha \}. \)

Then: as \( n \to \infty \), \( (\mathcal{N}(\mu^n|\Omega_n))(f \in \Omega_n \mid f_n = \varepsilon_0) \to P. \)

Part of the content of Theorem 10.2 is:

\( \forall \)sufficiently large \( n \in \mathbb{N} \), \( \mu^n(\Omega_n) > 0, \)
since, otherwise, \( \mu^n|\Omega_n \) would be the zero measure on \( \Omega_n \), and so \( \mathcal{N}(\mu^n|\Omega_n) \) would not be defined.

11. **The main problem**

We have all we need to answer the main problem (end of §3).

Let \((p, q, r) := (1, 9^{-1/10}, 9^{-1})\).

Again, let’s say I am one of the professors applying to the GFA.

**We will show:** Under the GFA’s first system (§3), my probability of getting $0$ is \( p \), approximately and my probability of getting $1$ is \( q \), approximately and my probability of getting $10$ is \( r \), approximately.

By the work in §7, \( p + q + r = 1 \).

**Let** \( X_1, X_2, X_3, \ldots \) be iid random variables s.t., \( \forall n \in \mathbb{N} \),

\[
\Pr[X_n = 0] = p, \\
\Pr[X_n = 1] = q, \\
\Pr[X_n = 10] = r.
\]

Then: \( \forall n \in \mathbb{N}, \ E[X_n] = q + 10r \).

By the work in §7, \( q + 10r = 1 \).

Then: \( \forall n \in \mathbb{N}, \ E[X_n] = 1 \).

We model the GFA’s second system (§5) by: \( \forall \ell \in [1..N] \), Professor\#\( \ell \) receives \( X_\ell \) dollars.

For all \( n \in \mathbb{N} \), let \( T_n := X_1 + \cdots + X_n \).

We model the GFA’s third system (§5) by: \( \forall \ell \in [1..N] \), Professor\#\( \ell \) receives \( X_\ell \) dollars, conditioned on \( T_N = N \).

**Let** \( S := \sum_{\omega \in \Omega} p^\omega q^{k_\omega} r^{k_\omega} \).

By the work in §7, \( \forall \omega \in \Omega \), \( p^\omega q^{k_\omega} r^{k_\omega} / S = 1 / (\#\Omega) \).

Also, as we observed earlier (end of §5), the third system is equivalent to the first.

For definiteness, let’s assume that I am Professor\#\( N \).

Because the third system is equivalent to the first,

we wish to show:

\[
\Pr[X_N = 0 \mid T_N = N] \approx p \quad \text{and} \\
\Pr[X_N = 1 \mid T_N = N] \approx q \quad \text{and} \\
\Pr[X_N = 10 \mid T_N = N] \approx r.
\]

Recall that the number, \( N \), of professors is assumed large.

So, to be more precise, we wish to show: as \( n \to \infty \),
Pr[ \( X_n = 0 \) \( T_n = n \) ] \( \rightarrow p \) and 
Pr[ \( X_n = 1 \) \( T_n = n \) ] \( \rightarrow q \) and 
Pr[ \( X_n = 10 \) \( T_n = n \) ] \( \rightarrow r \).

Let \( E := \{0, 1, 10\} \). Given \( \varepsilon_0 \in E \), let \( P := \begin{cases} 
p, & \text{if } \varepsilon_0 = 0 
q, & \text{if } \varepsilon_0 = 1 
r, & \text{if } \varepsilon_0 = 10,
\end{cases} \)

\textbf{want:} as \( n \rightarrow \infty \), \( \Pr[X_n = \varepsilon_0 | T_n = n] \rightarrow P \).

By definition of \( X_1, X_2, \ldots \), we get: \( \forall n \in \mathbb{N}, \ Pr[X_n = \varepsilon_0] = P. \)

Recall: \( \forall n \in \mathbb{N}, \ E[X_n] = 1 \).

Then, by Theorem 10.1, we have:
as \( n \rightarrow \infty \), \( \Pr[X_n = \varepsilon_0 | T_n = n] \rightarrow P. \)

\section{12. Probability of two professors getting zero}

Under the GFA’s first system, since \( N \) is large, one would expect:
the award amounts of two different professors
are almost independent.

So, for example,
the probability that two professors both receive zero dollars
should be very close to the square of
the probability that one professor receives zero dollars.

We formalize this statement and prove it, as follows.

Let \((p, q, r) := \left(1, 9^{-1/10}, 9^{-1}\right)\).
Let \( X_1, X_2, X_3, \ldots \) be iid random variables s.t., \( \forall n \in \mathbb{N} \),
\( \Pr[ X_n = 0 ] = p \),
\( \Pr[ X_n = 1 ] = q \),
\( \Pr[ X_n = 10 ] = r \).

For all \( n \in \mathbb{N} \), let \( T_n := X_1 + \cdots + X_n \).

In this section, we will prove:
as \( n \rightarrow \infty \), \( \Pr[ X_{n-1} = 0 = X_n | T_n = n ] \rightarrow p^2. \)

For all \( n \in \mathbb{N} \), define \( \psi_n : \mathbb{Z} \rightarrow \mathbb{R} \) by:
\( \forall t \in \mathbb{Z}, \ \psi_n(t) = \Pr[T_n = t]. \)

For all \( n \in \mathbb{N} \), let \( a_n := \psi_n(n + 2) \), \( z_n := \psi_n(n) \).
Since, \( \forall n \in \mathbb{N} \), we have
both \( z_n = \psi_n(n) = \Pr[T_n = n] \)
and \( \Pr[T_n = n] = \Pr[X_1 + \cdots + X_n = n] \)
By Theorem 8.10, as $n \to \infty$, \( \Pr[X_1 = \cdots = X_n = 1] = q^n > 0 \), we conclude: \( \forall n \in \mathbb{N}, \ z_n = \Pr[T_n = n] > 0 \).

\textbf{Claim:} Let $n \in [3..\infty)$. Then \( \Pr[X_{n-1} = 0 = X_n \mid T_n = n] = p^2 \cdot \frac{a_{n-2}}{z_n} \).

\textbf{Proof of Claim:} We have \( T_n = X_1 + \cdots + X_{n-2} + X_{n-1} + X_n \).

Since \( \Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)] = \Pr[(X_{n-1} = 0 = X_n) \& (X_1 + \cdots + X_{n-2} = n)] = \Pr[(X_{n-1} = 0 = X_n) \& (X_1 + \cdots + X_{n-2} = 0 = n) = n] \), it follows, from independence of \( X_1, \ldots, X_n \), that \( \Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)] = (\Pr[X_{n-1} = 0]) \cdot (\Pr[X_n = 0]) \cdot (\Pr[X_1 + \cdots + X_{n-2} = n]) \).

So, since \( \Pr[X_{n-1} = 0] = p = \Pr[X_n = 0] \) and since \( X_1 + \cdots + X_{n-2} = T_{n-2} \), we get: \( \Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)] = p^2 \cdot \Pr[T_{n-2} = n]. \)

Then \( \Pr[X_{n-1} = 0 = X_n \mid T_n = n] = \frac{\Pr[(X_{n-1} = 0 = X_n) \& (T_n = n)]}{\Pr[T_n = n]} = p^2 \cdot \frac{\psi_{n-2}(n)}{\psi(n)} = p^2 \cdot \frac{a_{n-2}}{z_n}. \)

\textit{End of proof of Claim.}

Because of the Claim, we want to show: as $n \to \infty$, \( p^2 \cdot \frac{a_{n-2}}{z_n} \to p^2. \)

\textbf{Want:} as $n \to \infty$, \( \frac{z_n}{a_{n-2}} \to 1. \)

We compute: \( \forall n \in \mathbb{N}, \ E[X_n] = q + 10r. \)

Recall (§7): $q + 10r = 1$. Then: \( \forall n \in \mathbb{N}, \ E[X_n] = 1. \)

We compute: \( \forall n \in \mathbb{N}, \ Var[X_n] = q + 100r - 1. \)

\textbf{Let} \( v := q + 100r - 1. \) Then: \( \forall n \in \mathbb{N}, \ Var[X_n] = v. \)

Since \( v = (q + 10r - 1) + 90r = 1 + 90r = 90r, \) and since \( r > 0, \) we get: \( v > 0. \)

\textbf{Let} \( \tau := 1/\sqrt{2\pi v}. \) Then \( \tau > 0. \)

By Theorem 8.10, \( \text{as } n \to \infty, \ \sqrt{n} \cdot \Pr[T_n = n + 2] \to 1/\sqrt{2\pi v}. \)

Then: \( \text{as } n \to \infty, \ \sqrt{n} \cdot \psi_n(n + 2) \to \tau. \)

Then: \( \text{as } n \to \infty, \ \sqrt{n} \cdot a_n \to \tau. \)

Then: \( \text{as } n \to \infty, \ \sqrt{n - 2} \cdot a_{n-2} \to \tau. \)

By Theorem 8.10, \( \text{as } n \to \infty, \ \sqrt{n} \cdot \Pr[T_n = n] \to 1/\sqrt{2\pi v}. \)

Then: \( \text{as } n \to \infty, \ \sqrt{n} \cdot \psi_n(n) \to \tau. \)

Then: \( \text{as } n \to \infty, \ \sqrt{n} \cdot z_n \to \tau. \)
Then: as \( n \to \infty \), \( \frac{\sqrt{n} - 2 \cdot a_{n-2}}{\sqrt{n} \cdot z_n} \to \frac{\tau}{\tau} \).

Then: as \( n \to \infty \), \( \frac{\sqrt{n} - 2 \cdot a_{n-2}}{\sqrt{n} \cdot z_n} \to 1 \).

Also, as \( n \to \infty \), \( \frac{a_{n-2}}{z_n} \to 1 \).

Multiplying these last two limits, we get: as \( n \to \infty \), \( \frac{a_{n-2}}{z_n} \to 1 \).

13. FRACTION OF PROFESSORS GETTING A ZERO AWARD

Let \( (p, q, r) := (1, 9^{-1/10}, 9^{-1}) \).

We compute \( (p, q, r) \approx (0.5225, 0.4194, 0.0581) \), accurate to four decimal places.

Let \( X_1, X_2, X_3, \ldots \) be iid random variables s.t., \( \forall n \in \mathbb{N} \),
\[
\begin{align*}
\Pr[X_n = 0] &= p, \\
\Pr[X_n = 1] &= q, \\
\Pr[X_n = 10] &= r.
\end{align*}
\]

For all \( n \in \mathbb{N} \), let \( T_n := X_1 + \cdots + X_n \).

For all \( \ell \in \mathbb{N} \), let \( I_\ell \) be the indicator variable of the event \( X_\ell = 0 \).

For all \( n \in \mathbb{N} \), let \( J_n := (I_1 + \cdots + I_n)/n \).

Using the GFA’s first awards system, the random variable \( J_N \) conditioned on \( T_N = N \) represents the fraction of professors receiving a $0 award.

In this section, we will prove: \( \forall \varepsilon > 0 \),
\[
\text{as } n \to \infty, \quad \Pr\left[ p - \varepsilon < J_n < p + \varepsilon \mid T_n = n \right] \to 1.
\]

From this, it follows that, if \( N \) is sufficiently large, then
\[
\Pr\left[ p - 0.01 < J_N < p + 0.01 \mid T_N = N \right] > 0.99.
\]

Since \( p \approx 0.5225 \), accurate to four decimal places,
we conclude that: \( 0.5 < p - 0.01 \).

So, if \( N \) is sufficiently large, then, under the first system,
\[
\Pr\left[ 0.5 < J_N \mid T_N = N \right] > 0.99,
\]
and so: with probability > 99%, over half of the professors get $0.

Given \( \varepsilon > 0 \), want: as \( n \to \infty \), \( \Pr[p - \varepsilon < J_n < p + \varepsilon \mid T_n = n] \to 1 \).

For all \( n \in \mathbb{N} \), let \( \kappa_n := E\left[ I_n \mid T_n = n \right] \).

Then: \( \forall n \in \mathbb{N} \), \( \kappa_n = \Pr\left[ X_n = 0 \mid T_n = n \right] \),
so, by Theorem 10.1, we get: as \( n \to \infty \), \( \kappa_n \to p \).
So, for all sufficiently large $n$,
we have: $p - (\varepsilon/2) < \kappa_n < p + (\varepsilon/2)$,
and so both $p - \varepsilon < \kappa_n - (\varepsilon/2)$ and $\kappa_n + (\varepsilon/2) < p + \varepsilon$,
and so $\Pr[\kappa_n - (\varepsilon/2) < J_n < \kappa_n + (\varepsilon/2) | T_n = n]$
\[ \leq \Pr[ p - \varepsilon < J_n < p + \varepsilon | T_n = n]. \]

**It therefore suffices to show:**
\[ \text{as } n \to \infty, \quad \Pr[ \kappa_n - (\varepsilon/2) < J_n < \kappa_n + (\varepsilon/2) | T_n = n ] \to 1. \]

We have: $\forall n \in \mathbb{N}$, $T_n$ is invariant under permutation of $X_1, \ldots, X_n$.
Then: $\forall n \in \mathbb{N}, \forall i \in [1..n]$, $E [ I_i | T_n = n ] = E [ I_n | T_n = n ]$.
Then: $\forall n \in \mathbb{N}, \forall i \in [1..n]$, $E [ I_i | T_n = n ] = \kappa_n$.
Since, $\forall n \in \mathbb{N}$, $J_n = (I_1 + \cdots + I_n)/n$, we get:
\[ \forall n \in \mathbb{N}, \quad E [ J_n | T_n = n ] = (\sum_{i=1}^{n} E [ I_i | T_n = n ]) / n. \]
Then: $\forall n \in \mathbb{N}, E [ J_n | T_n = n ] = \kappa_n/n$.
Then: $\forall n \in \mathbb{N}, E [ J_n | T_n = n ] = \kappa_n$.
For all $n \in \mathbb{N}$, let $v_n := \Var [ J_n | T_n = n ]$.
Then, by Chebyshev’s inequality, we have: $\forall n \in \mathbb{N}$,
\[ \Pr[\kappa_n - (\varepsilon/2) < J_n < \kappa_n + (\varepsilon/2) | T_n = n ] \geq 1 - (v_n/(\varepsilon/2)^2). \]

**It therefore suffices to show:** as $n \to \infty$, $v_n \to 0$.
We have: $\forall n \in \mathbb{N}, v_n = \Var [ J_n | T_n = n ]$
\[ = ( E [ J_n^2 | T_n = n ]) - (E [ J_n | T_n = n ])^2. \]
\[ = ( E [ J_n^2 | T_n = n ] ) - \kappa_n^2. \]
So, since, as $n \to \infty$, $\kappa_n^2 \to p^2$,
we want: as $n \to \infty$, $E [ J_n^2 | T_n = n ] \to p^2$.
For all $n \in [2..\infty)$, let $\lambda_n := E [ I_{n-1} \cdot I_n | T_n = n ]$.
Then: $\forall n \in [2..\infty)$, $\lambda_n = \Pr [ X_{n-1} = 0 = X_n | T_n = n ]$.
So, by the result of §12, we get: as $n \to \infty$, $\lambda_n \to p^2$.
For all $n \in \mathbb{N}$, since $I_n$ is an indicator variable, we get: $I_n \in \{0,1\}$ a.s.
Then: $\forall n \in \mathbb{N}$, $I_n = I_n^2$ a.s.
Then: $\forall n \in \mathbb{N}$, $\kappa_n = E [ I_n^2 | T_n = n ]$.
We have: $\forall n \in \mathbb{N}$, $T_n$ is invariant under permutation of $X_1, \ldots, X_n$.
Then: ($\forall n \in \mathbb{N}, \forall i \in [1..n]$, $E [ I_i^2 | T_n = n ] = E [ I_i^2 | T_n = n ]$)
and ($\forall n \in [2..\infty), \forall i, j \in [1..n], \text{ if } i \neq j$, then $E [ I_i \cdot I_j | T_n = n ] = E [ I_{n-1} \cdot I_n | T_n = n ]$).
Then: ($\forall n \in \mathbb{N}, \forall i \in [1..n]$, $E [ I_i^2 | T_n = n ] = \kappa_n$)
and ($\forall n \in [2..\infty), \forall i, j \in [1..n], \text{ if } i \neq j$, then $E [ I_i \cdot I_j | T_n = n ] = \lambda_n$).
For all $n \in \mathbb{N}$, for all $i, j \in [1..n]$, let $c_{ijn} := E [ I_i \cdot I_j | T_n = n ]$.
Then: \( \forall n \in \mathbb{N}, \forall i, j \in [1..n], \ c_{ijn} = \begin{cases} \kappa_n, & \text{if } i = j \\ \lambda_n, & \text{if } i \neq j. \end{cases} \)

Then: \( \forall n \in \mathbb{N}, \ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijn} = n \cdot \kappa_n + (n^2 - n) \cdot \lambda_n. \)

Since \( \forall n \in \mathbb{N}, \ J_n = (I_1 + \cdots + I_n)/n, \) we get:

Then: \( \forall n \in \mathbb{N}, \ E[J_n^2 | T_n = n] = (\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijn})/n^2. \)

Recall: as \( n \to \infty, \) \( \kappa_n \to p \) and \( \lambda_n \to p^2. \)

Recall (§8): For any countable set \( \Gamma, \) \( \mathcal{S}(\Gamma) \) is the set of measures on \( \Gamma \) and \( \mathcal{F}\mathcal{M}^\times(\Gamma) \) is the set of nonzero finite measures on \( \Gamma \) and \( \mathcal{P}(\Gamma) \) is the set of probability measures on \( \Gamma. \)

Recall (§10): For any nonempty countable set \( \Gamma, \) \( \forall \mu \in \mathcal{F}\mathcal{M}^\times(\Gamma), \) \( \mathcal{N}(\mu) \) is the normalization of \( \mu. \)

**DEFINITION 14.1.** Let \( E \subseteq \mathbb{R} \) be nonempty and finite, \( \beta \in \mathbb{R}. \)

The **unnormalized \( \beta \)-Boltzmann distribution on \( E \)** is the measure \( \hat{B}^E_\beta \in \mathcal{F}\mathcal{M}^\times(E) \) defined by:

\[ \forall \varepsilon \in E, \ \hat{B}^E_\beta(\varepsilon) = e^{-\beta \cdot \varepsilon}. \]

Also, the **\( \beta \)-Boltzmann distribution on \( E \)** is

\[ B^E_\beta := \mathcal{N}(\hat{B}^E_\beta) \in \mathcal{P}(E). \]

**Example:** Let \( E := \{2, 4, 9\} \) and let \( \beta \in \mathbb{R}. \)

Then: \( \hat{B}^E_\beta(2) = e^{-2\beta}, \ \hat{B}^E_\beta(4) = e^{-4\beta}, \ \hat{B}^E_\beta(9) = e^{-9\beta}. \)

Let \( C := 1/(e^{-2\beta} + e^{-4\beta} + e^{-9\beta}). \)

Then: \( B^E_\beta(2) = Ce^{-2\beta}, \ B^E_\beta(4) = Ce^{-4\beta}, \ B^E_\beta(9) = Ce^{-9\beta}. \)

Recall (§8): For any countable set \( \Gamma, \) \( \forall \mu \in \mathcal{M}(\Gamma), \) \( S_\mu \) is the support of \( \mu. \)

Recall (§8): For any countable \( \Gamma \subseteq \mathbb{R}, \) \( \forall \mu \in \mathcal{P}(\Gamma), \)

if \( \sum_{q \in \mathbb{Z}} (|q| \cdot \langle \mu(q) \rangle) < \infty, \) then \( M_\mu \) is the mean of \( \mu \) and
Then: 

\[ V_\mu \text{ is the variance of } \mu. \]

Note: \( \forall \) nonempty finite \( E \subseteq \mathbb{R} \), \( \forall \beta \in \mathbb{R} \), we have: \( S_{BE} = E = S_{B\beta}. \)

**THEOREM 14.2.** Let \( E \subseteq \mathbb{R} \) be nonempty and finite. Let \( \varepsilon_0 \in E \), \( \beta, \xi \in \mathbb{R} \). Then: \( B_\beta^{E-\xi}\{\varepsilon_0 - \xi\} = B_\beta^E\{\varepsilon_0\}. \)

**Proof.** We have: \( B_\beta^{E-\xi}\{\varepsilon_0 - \xi\} = \frac{e^{-\beta(\varepsilon_0-\xi)}}{\sum_{\xi \in E} e^{-\beta(\xi-\xi)}} \frac{e^{-\beta\xi}}{e^{-\beta\xi}} = \frac{\sum_{\xi \in E} e^{-\beta(\xi-\xi)}}{e^{-\beta\xi}} = \frac{\sum_{\xi \in E} e^{-\beta\xi}}{e^{-\beta\xi}} = B_{\beta}^{E}\{\varepsilon_0\}. \)

**THEOREM 14.3.** Let \( E \subseteq \mathbb{R} \) be nonempty and finite. Let \( \beta, \xi \in \mathbb{R} \). Then: \( M_{BE} = M_{B\beta}^{E-\xi}. \)

**Proof.** Let \( \lambda := B_\beta^{E-\xi}, \mu := B_\beta^E. \) Want: \( M_\lambda = M_\mu - \xi. \)

We have: \( S_\lambda = E - \xi \) and \( S_\mu = E. \)

By Theorem 14.2, we have: \( \forall \varepsilon \in E, \lambda(\varepsilon - \xi) = \mu(\varepsilon). \)

Since \( \mu = B_\beta^E \in \mathcal{P}(E) \), we get: \( M_\mu(E) = 1. \)

Then: \( M_\lambda = \sum_{\varepsilon \in E} ((\varepsilon - \xi) \cdot (\lambda(\varepsilon - \xi))) = \sum_{\varepsilon \in E} ((\varepsilon - \xi) \cdot (\mu(\varepsilon))) = (\sum_{\varepsilon \in E} (\varepsilon \cdot (\mu(\varepsilon)))) - (\sum_{\varepsilon \in E} (\xi \cdot (\mu(\varepsilon)))) = (\sum_{\varepsilon \in E} (\varepsilon \cdot (\mu(\varepsilon)))) - \xi \cdot (\sum_{\varepsilon \in E} (\mu(\varepsilon))) = M_\mu - \xi \cdot (\mu(E)) = M_\mu - \xi \cdot 1 = M_\mu - \xi. \)

**THEOREM 14.4.** Let \( E \subseteq \mathbb{R} \) be nonempty and finite. Then:

\[ \text{as } \beta \to \infty, \quad M_{BE} \to \min E \]

\[ \text{and as } \beta \to -\infty, \quad M_{BE} \to \max E. \]

The proof is a matter of bookkeeping, best explained by example:

If \( E = \{2, 4, 9\} \), then, \( \forall \beta \in \mathbb{R}, \quad M_{BE} = \frac{2e^{-2\beta} + 4e^{-4\beta} + 9e^{-9\beta}}{e^{-2\beta} + e^{-4\beta} + e^{-9\beta}} \), and so:

\[ \text{as } \beta \to \infty, \quad M_{BE} \to 2 \]

\[ \text{and as } \beta \to -\infty, \quad M_{BE} \to 9. \]

**THEOREM 14.5.** Let \( E \subseteq \mathbb{R} \). Assume: \( 2 \leq \#E < \infty. \)

Define \( A: \mathbb{R} \to \mathbb{R} \) by: \( \forall \beta \in \mathbb{R}, \quad A(\beta) = M_{BE}. \)

Then: \( A \) is a strictly-decreasing \( C^\omega \)-diffeomorphism from \( \mathbb{R} \) onto \( (\min E; \max E). \)
Proof. Let $\kappa := \#E$. Then: $2 \leq \kappa < \infty$.

Choose distinct $\varepsilon_1, \ldots, \varepsilon_\kappa \in \mathbb{R}$ s.t. $E = \{\varepsilon_1, \ldots, \varepsilon_\kappa\}$.

Then: $\forall \beta \in \mathbb{R}$, $A(\beta) = \sum_{j=1}^\kappa \varepsilon_j \cdot e^{-\beta \varepsilon_j}$. Then $A : \mathbb{R} \to \mathbb{R}$ is $C^\omega$.

So, by Theorem 14.4 and the $C^\omega$-Inverse Function Theorem and the Mean Value Theorem, it suffices to show: $A' < 0$ on $\mathbb{R}$.

Given $\beta \in \mathbb{R}$, want: $A'(\beta) < 0$.

Let $P := \sum_{j=1}^\kappa [\varepsilon_j \cdot e^{-\beta \varepsilon_j}]$, $P' := \sum_{j=1}^\kappa [(-\varepsilon_j^2) \cdot e^{-\beta \varepsilon_j}]$.

Let $Q := \sum_{j=1}^\kappa [e^{-\beta \varepsilon_j}]$, $Q' := \sum_{j=1}^\kappa [(-\varepsilon_j) \cdot e^{-\beta \varepsilon_j}]$.

Then $Q > 0$. Also, by the Quotient Rule, $A'(\beta) = [QP' - PQ']/Q^2$.

Want: $QP' - PQ' < 0$.

We have: $QP' = \sum_{i=1}^\kappa \sum_{j=1}^\kappa [(-\varepsilon_i^2) \cdot e^{-\beta (\varepsilon_i + \varepsilon_j)}]$.

We have: $QP' = \sum_{i=1}^\kappa \sum_{j=1}^\kappa [(-\varepsilon_i \varepsilon_j) \cdot e^{-\beta (\varepsilon_i + \varepsilon_j)}]$.\n
Then: $QP' - PQ' = \sum_{i=1}^\kappa \sum_{j=1}^\kappa [(-\varepsilon_i^2 + \varepsilon_i \varepsilon_j) \cdot e^{-\beta (\varepsilon_i + \varepsilon_j)}]$.

Interchanging $i$ and $j$, we get:

\[ QP' - PQ' = \sum_{j=1}^\kappa \sum_{i=1}^\kappa [(-\varepsilon_j^2 + \varepsilon_j \varepsilon_i) \cdot e^{-\beta (\varepsilon_j + \varepsilon_i)}], \]

so, by commutativity of addition and multiplication, we get:

\[ QP' - PQ' = \sum_{i=1}^\kappa \sum_{j=1}^\kappa [(-\varepsilon_j^2 + \varepsilon_j \varepsilon_i) \cdot e^{-\beta (\varepsilon_j + \varepsilon_i)}]. \]

The last three equations are formulas for $QP' - PQ'$.

Adding the first and third, we get:

\[ 2 \cdot (QP' - PQ') = \sum_{i=1}^\kappa \sum_{j=1}^\kappa [(-\varepsilon_i^2 - \varepsilon_j^2 + 2\varepsilon_i \varepsilon_j) \cdot e^{-\beta (\varepsilon_i + \varepsilon_j)}]. \]

Then: $2 \cdot (QP' - PQ') = \sum_{i=1}^\kappa \sum_{j=1}^\kappa [-(\varepsilon_i - \varepsilon_j)^2 \cdot e^{-\beta (\varepsilon_i + \varepsilon_j)}].$

Then: $2 \cdot (QP' - PQ') < 0$. Then: $QP' - PQ' < 0$. □

**DEFINITION 14.6.** Let $E \subseteq \mathbb{R}$. Assume $2 \leq \#E < \infty$.

Define a bijection $A : \mathbb{R} \to (\min E; \max E)$ by

\[ \forall \beta \in \mathbb{R}, \ A(\beta) = M_{B^\beta_E}. \]

Then, $\forall \alpha \in (\min E; \max E)$, the $\alpha$-Boltzmann parameter on $E$ is

\[ \text{BP}_\alpha^E := A^{-1}(\alpha). \]

So the $\alpha$-Boltzmann parameter on $E$ is the unique $\beta \in \mathbb{R}$ s.t. $M_{B^\beta_E} = \alpha$.

**Example:** Computations at the end of §6 show:

\[ \forall \beta \in \mathbb{R}, \ \text{if} \ \frac{e^{-\beta} + 10e^{-10\beta}}{1 + e^{-\beta} + 10e^{-10\beta}} = 1, \ \text{then} \ \beta = (\ln 9)/10. \]

Then $\text{BP}_1^{(0.1,10)} = (\ln 9)/10$.

**THEOREM 14.7.** Let $E \subseteq \mathbb{R}$. Assume $2 \leq \#E < \infty$.

Let $\alpha \in (\min E; \max E)$. Let $\xi \in \mathbb{R}$. Then $\text{BP}_{\alpha-\xi}^E = \text{BP}_\alpha^E$. 

Proof. Let \( \beta := \text{BP}_\alpha^E \).  \textbf{Want:} \( \text{BP}_{\alpha^{-\xi}}^E = \beta \).

By Theorem 14.3, we get: \( M_{\beta^{-\xi}}^E = M_{\beta^{-\xi}}^E - \xi \).

Since \( \beta = \text{BP}_\alpha^E \), we get: \( M_{\beta^{-\xi}}^E = \alpha \).

Since \( M_{\beta^{-\xi}}^E = M_{\beta^{-\xi}}^E - \xi = \alpha - \xi \), we get: \( \text{BP}_{\alpha^{-\xi}}^E = \beta \). \( \square \)

15. Residue-unconstrained sets

In this section, we generalize the result of §11 from \( \{0, 1, 10\} \) to arbitrary residue-unconstrained sets.

Our main theorem is Theorem 15.3 below.

**DEFINITION 15.1.** Let \( F \) be a nonempty finite set.

Then \( \nu_F \in \mathcal{P}(F) \) is defined by: \( \forall f \in F, \ \nu_F(f) = 1/(\#F) \).

Let \( \nu_\emptyset : \{\emptyset\} \to \{-1\} \) be defined by: \( \nu_\emptyset(\emptyset) = -1 \).

**THEOREM 15.2.** Let \( F \) be a nonempty finite set, \( \theta \in \mathcal{P}(F) \).

Assume: \( \forall f, g \in F, \ \theta(f) = \theta(g) \). Then: \( \theta = \nu_F \).

Proof. Since \( F \) is nonempty, choose \( f_0 \in F \). Let \( b := \theta(f_0) \).

Then: \( \forall f \in F, \ b = \theta(f) \). Then \( \sum_{g \in F} (\theta(g)) = (\#F) \cdot b \).

Since \( \theta \in \mathcal{P}(F) \), we get: \( \theta(F) = 1 \).

Since \( (\#F) \cdot b = \sum_{g \in F} (\theta(g)) = \theta(F) = 1 \), we get: \( b = 1/(\#F) \).

Since \( \forall g \in F, \ \theta(g) = b = 1/(\#F) = \nu_F(g) \), we get: \( \theta = \nu_F \). \( \square \)

**THEOREM 15.3.** Let \( E \subseteq \mathbb{Z} \) be finite and residue-unconstrained.

Let \( \alpha \in \mathbb{Z} \). Assume \( \alpha \in (\min E; \max E) \). Let \( \beta := \text{BP}_\alpha^E \).

For all \( n \in \mathbb{N} \), let \( \Omega_n := \{ f \in E^n \mid f_1 + \cdots + f_n = n\alpha \} \).

Let \( \varepsilon_0 \in E \). Then: as \( n \to \infty, \ \nu_{\Omega_n} \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} \to B_{\beta}^E \{ \varepsilon_0 \} \).

Recall: \( \nu_\emptyset(\emptyset) = -1 \).

So, since \( B_{\beta}^E \{ \varepsilon_0 \} > 0 \), part of the content of this theorem is:

\( \forall \) sufficiently large \( n \in \mathbb{N} \), \( \Omega_n \not= \emptyset \).

See Claim 2 in the proof below.

Example: Suppose \( E = \{0, 1, 10\} \) and \( \alpha = 1 \).

Then \( \Omega_N \) represents the set of all GFA dispensations to the N professors.

Since \( \varepsilon_0 \in E \), we get: \( \varepsilon_0 = 0 \) or \( \varepsilon_0 = 1 \) or \( \varepsilon_0 = 10 \).

Since \( \nu_{\Omega_N} \) gives equal probability to each dispensation,

\( \nu_{\Omega_N} \) represents the GFA’s first system for awarding grants.
Since $\beta = \text{BP}_\alpha^E = \text{BP}^{(0,1,10)}_1$, we get: $\beta = (\ln 9)/10$.

Let $\varepsilon_0 \in E$. Since $N$ is large, by Theorem 15.3, we get:

$$\nu_{\alpha_N}\{f \in \Omega_N \mid f_N = \varepsilon_0\} \approx B^E_\beta\{\varepsilon_0\}.$$ 

So, if I am the $N$th professor, then, under the first system, my probability of receiving $\varepsilon_0$ dollars is approximately equal to the $\beta$-Boltzmann-probability in $E$ of $\varepsilon_0$.

Thus Theorem 15.3 reproduces the result of §11.

Proof. Since $\beta = \text{BP}_\alpha^E$, we get: $M_{\text{BP}_\alpha^E} = \alpha$.

Let $\mu := B^E_\beta$. Then: $M_\mu = \alpha$.

For all $n \in \mathbb{N}$, define $\psi_n : \mathbb{Z} \to \mathbb{R}$ by:

$$\forall t \in \mathbb{Z}, \quad \psi_n(t) = \mu^n\{f \in E^n \mid f_1 + \cdots + f_n = t\}.$$ 

Then: $\forall n \in \mathbb{N}$, $\psi_n(n\alpha) = \mu^n(\Omega_n)$.

Since $E$ is finite and residue-unconstrained, we get: $2 \leq \#E < \infty$.

Since $\mu = B^E_\beta$ and $S_{B^E_\beta} = E$, we get: $S_\mu = E$.

Since $\#S_\mu = \#E \geq 2$, by Theorem 8.6, we get: $V_\mu > 0$.

Let $v := V_\mu$. Then $v > 0$. Then $1/\sqrt{2\pi v} > 0$.

Let $\tau := 1/\sqrt{2\pi v}$. Then $\tau > 0$.

Claim 1: As $n \to \infty$, $\sqrt{n} \cdot (\psi_n(n\alpha)) \to \tau$.

Proof of Claim 1: For all $n \in \mathbb{N}$, let $t_n := n\alpha$.

Then: $\forall n \in \mathbb{N}$, $\psi_n(n\alpha) = \mu^n\{f \in E^n \mid f_1 + \cdots + f_n = t_n\}$.

So, by Theorem 8.11, we get: as $n \to \infty$, $\sqrt{n} \cdot (\psi_n(n\alpha)) \to \tau$.

End of proof of Claim 1.

Since $\tau > 0$, by Claim 1, choose $n_0 \in \mathbb{N}$ s.t.

$$\forall n \in [n_0, \infty), \quad \sqrt{n} \cdot (\psi_n(n\alpha)) > 0.$$ 

Claim 2: Let $n \in [n_0, \infty)$.

Then: $\mu^n(\Omega_n) > 0$.

Proof of Claim 2: By the choice of $n_0$, we get: $\sqrt{n} \cdot (\psi_n(n\alpha)) > 0$.

Then: $\psi_n(n\alpha) > 0$. Then: $\mu^n(\Omega_n) = \psi_n(n\alpha) > 0$.

End of proof of Claim 2.

Since $\mu = B^E_\beta$, we get: $\mu \in \mathcal{P}(E)$.

Then: $\forall n \in \mathbb{N}$, $\mu^n \in \mathcal{P}(E^n)$, so $\mu^n(\Omega_n) \leq 1$.

So, by Claim 2, $\forall n \in [n_0, \infty)$, $0 < \mu^n(\Omega_n) \leq 1$.

Also, we have: $\forall n \in \mathbb{N}$, $(\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n)$.
Then: \( \forall n \in [n_0..\infty), \quad 0 < (\mu^n|\Omega_n)(\Omega_n) \leq 1. \)

Then: \( \forall n \in [n_0..\infty), \quad \mu^n|\Omega_n \in \mathcal{F}\mathcal{M}^x(\Omega_n). \)

Then: \( \forall n \in [n_0..\infty), \quad \mathcal{N}(\mu^n|\Omega_n) \in P(\Omega_n). \)

Claim 3: Let \( n \in [n_0..\infty). \) Then: \( \mathcal{N}(\mu^n|\Omega_n) = \nu_{\Omega_n}. \)

Proof of Claim 3: Let \( \theta := \mathcal{N}(\mu^n|\Omega_n), \quad F := \Omega_n. \) Want: \( \theta = \nu_F. \)

By Theorem 15.2, given \( f, g \in F, \) want: \( \theta\{f\} = \theta\{g\}. \)

By Claim 2, we have: \( \mu^n(\Omega_n) > 0. \)

Since \( (\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n) \) and \( \theta = \mathcal{N}(\mu^n|\Omega), \) we get: \( \theta = \frac{\mu^n(\Omega_n)}{\mu^n(\Omega_n)}. \)

Want: \( \frac{(\mu^n|\Omega_n)\{f\}}{\mu^n(\Omega_n)} = \frac{(\mu^n|\Omega_n)\{g\}}{\mu^n(\Omega_n)}. \)

Want: \( (\mu^n|\Omega_n)\{f\} = (\mu^n|\Omega_n)\{g\}). \)

Since \( f, g \in \Omega_n, \) we get: \( (\mu^n|\Omega_n)\{f\} = \mu^n\{f\} \) and \( (\mu^n|\Omega_n)\{g\} = \mu^n\{g\}. \)

Want: \( \mu^n\{f\} = \mu^n\{g\}. \)

Let \( C := 1/(\hat{B}_\beta^F(E)). \) Then \( \mathcal{N}(\hat{B}_\beta^E) = C \cdot \hat{B}_\beta^E. \)

By definition of \( \hat{B}_\beta^E, \) we have: \( \forall \varepsilon \in E, \quad \hat{B}_\beta^E\{\varepsilon\} = e^{-\beta\varepsilon}. \)

So, since \( \mu = B_\beta^E = \mathcal{N}(\hat{B}_\beta^E) = C \cdot \hat{B}_\beta^E, \) we get: \( \forall \varepsilon \in E, \quad \mu\{\varepsilon\} = Ce^{-\beta\varepsilon}. \)

Since \( f \in \Omega_n, \) by definition of \( \Omega_n, \) we get: \( f_1 + \cdots + f_n = n\alpha. \)

Since \( g \in \Omega_n, \) by definition of \( \Omega_n, \) we get: \( g_1 + \cdots + g_n = n\alpha. \)

Since \( f_1 + \cdots + f_n = n\alpha = g_1 + \cdots + g_n, \) we get: \( C^n e^{-\beta(f_1+\cdots+f_n)} = C^n e^{-\beta(g_1+\cdots+g_n)}. \)

Then: \( (Ce^{-\beta f_1}) \cdots (Ce^{-\beta f_n}) = (Ce^{-\beta g_1}) \cdots (Ce^{-\beta g_n}). \)

Then: \( \mu\{f_1\} \cdots (\mu\{f_n\}) = (\mu\{g_1\}) \cdots (\mu\{g_n\}). \)

Then: \( \mu^n\{f\} = \mu^n\{g\}. \)

End of proof of Claim 3.

Let \( P := \mu\{\varepsilon_0\}. \)

By Theorem 10.2, as \( n \to \infty, \quad (\mathcal{N}(\mu^n|\Omega_n))\{f \in \Omega_n | f_n = \varepsilon_0\} \to P. \)

Then, by Claim 3, as \( n \to \infty, \quad \nu_{\Omega_n} \{f \in \Omega_n | f_n = \varepsilon_0\} \to P. \)

So, since \( P = \mu\{\varepsilon_0\} = B_\beta^E\{\varepsilon_0\}, \) we get:

as \( n \to \infty, \quad \nu_{\Omega_n} \{f \in \Omega_n | f_n = \varepsilon_0\} \to B_\beta^E\{\varepsilon_0\}. \)

16. Sets with 0

In this section, we develop a version of Theorem 15.3 that applies to any subset \( E \in \mathbb{Z} \) such that \( 0 \in E. \)
Think about dispensing, to $N$ professors, grants of sizes $\$0, \$2, \$20$. If the average grant is to be $\$2$,
then, by changing our units of money to “double-dollars”,
we can use our earlier $0, 1, 10$ result.

If awards are $\$0, \$2, \$20$, with average award $\$3$, and if $N$ is odd,
then the money simply cannot be dispensed,
because every dispensation has even total dollars, whereas $3N$ is odd.

Finally, suppose awards are $\$0, \$2, \$20$, with average award $\$3$,
and assume $N$ large and even.

Let $E := \{0, 2, 20\}$, $\alpha := 3$.

Let $g := \gcd(E)$, $\delta := \gcd(\alpha, g)$, $\gamma := g/\delta$.

Then: $g = 2$, $\delta = 1$, $\gamma = 2$.

Let $\beta := \text{BP}_E$, $\mu := \text{B}_E^E$.

Under the GFA’s first system of dispensation, for any professor,
what is the approximate probability of receiving $\$0$? $\$2$? $\$20$?

According to Theorem 16.3, below, the answers are $\mu\{0\}$, $\mu\{2\}$, $\mu\{20\}$.

We will need a variant of the Discrete Local Limit Theorem:

**THEOREM 16.1.** Let $\Gamma \subseteq \mathbb{Z}$, $\mu \in \mathcal{P}(\Gamma)$, $E := S_\mu$.

Assume: $0 \in E$ and $\#E \geq 2$. Let $g := \gcd(E)$.

Assume: $\sum_{t \in E} (t^2 \cdot (\mu\{t\})) < \infty$. Let $\alpha := M_\mu$, $v := V_\mu$.

Let $t_1, t_2, \ldots \in g\mathbb{Z}$. Then, as $n \to \infty$,

$$
\sqrt{n} \cdot \left[ (\mu^n\{f \in E^n \mid f_1 + \cdots + f_n = t_n\}) - g \cdot (\Phi^{nv}_{n\alpha}(t_n)) \right] \to 0.
$$

Theorem 16.1 is a measure-theoretic version of the following:

**THEOREM 16.2.** Let $E \subseteq \mathbb{Z}$.

Assume: $0 \in E$ and $\#E \geq 2$. Let $g := \gcd(E)$.

Let $X_1, X_2, \ldots$ be a $\mathbb{Z}$-valued iid sequence of $L^2$ random-variables.

Assume: $\forall n \in \mathbb{N}$, $\{t \in \mathbb{Z} \mid \Pr[X_n = t] > 0\} = E$.

Let $\alpha, v \in \mathbb{R}$. Assume: $\forall n \in \mathbb{N}$, $E[X_n] = \alpha$ and $\text{Var}[X_n] = v$.

Let $t_1, t_2, \ldots \in g\mathbb{Z}$. Then, as $n \to \infty$,

$$
\sqrt{n} \cdot \left[ (\Pr[X_1 + \cdots + X_n = t_n]) - g \cdot (\Phi^{nv}_{n\alpha}(t_n)) \right] \to 0.
$$

**Proof.** Let $E' := E/g$, $\alpha' := \alpha/g$, $v' := v/g^2$.

Then: $E' \subseteq \mathbb{Z}$ and $E'$ is residue-unconstrained.

For all $n \in \mathbb{N}$, let $X'_n := X_n/g$, $t'_n := t_n/g$.

Then: $\forall n \in \mathbb{N}$, $\{t' \in \mathbb{Z} \mid \Pr[X'_n = t'] > 0\} = E'$.

Also, $\forall n \in \mathbb{N}$, $E[X'_n] = \alpha'$ and $\text{Var}[X'_n] = v'$ and $t'_n \in \mathbb{Z}$.
By Theorem 8.9, we get: as $n \to \infty$,
\[ \sqrt{n} \cdot \left( \Pr[X'_1 + \cdots + X'_n = t'_n] \right) \to 0. \]
For all $n \in \mathbb{N}$, we have: $(X'_1 + \cdots + X'_n = t'_n)$ if $(X_1 + \cdots + X_n = t_n)$. For all $n \in \mathbb{N}$, we have: $\Phi_{n\alpha}(t'_n) = g \cdot (\Phi_{n\alpha}(t_n))$.

Then: as $n \to \infty$,
\[ \sqrt{n} \cdot \left( \Pr[X_1 + \cdots + X_n = t_n] \right) - g \cdot (\Phi_{n\alpha}(t_n)) \to 0. \]

**THEOREM 16.3.** Let $E \subseteq \mathbb{Z}$ be finite. Assume $0 \in E$. Let $\alpha \in \mathbb{Z}$.

Assume: $\alpha \in \left\{ \min E, \max E \right\}$. Let $\beta := BP_{\alpha}^E$.

For all $n \in \mathbb{N}$, let $\Omega_n := \left\{ f \in E^n \mid f_1 + \cdots + f_n = n\alpha \right\}$.
Let $g := \gcd(E)$. Let $\delta := \gcd\{\alpha, g\}$. Let $\gamma := g/\delta$. Let $\varepsilon_0 \in E$.

Then: as $m \to \infty$, $\nu_{\gamma m} \{ f \in \Omega_{\gamma m} \mid f_{\gamma m} = \varepsilon_0 \} \to B^E_{\beta}(\varepsilon_0)$.

Part of the content of Theorem 16.3 is:

\[ \forall \text{sufficiently large } m \in \mathbb{N}, \quad \Omega_{\gamma m} \neq \emptyset. \]

See Claim 2 in the proof below.

**Proof.** Since $\beta = BP_{\alpha}^E$, we get: $M_{BP_{\alpha}^E} = \alpha$.

Let $\mu := B^E_{\beta}$. Then: $M_{\mu} = \alpha$. Since $\alpha = (\min E; \max E)$, we get: $\min E \neq \max E$. Then $\# E \geq 2$.

Then $\gcd(E) \in \mathbb{N}$, so, since $g = \gcd(E)$, we get: $g \in \mathbb{N}$.
Then, since $\alpha \in \mathbb{Z}$ and since $\delta = \gcd\{\alpha, g\}$, we get:
\[ \delta \in \mathbb{N} \quad \text{and} \quad \alpha/\delta \in \mathbb{Z} \quad \text{and} \quad g/\delta \in \mathbb{N}. \]

Since $\gamma = g/\delta$ and since $g/\delta \in \mathbb{N}$, we get: $\gamma \in \mathbb{N}$.

For all $n \in \mathbb{N}$, define $\psi_n : \mathbb{Z} \to \mathbb{R}$ by:
\[ \forall t \in \mathbb{Z}, \quad \psi_n (t) = \mu^n \{ f \in E^n \mid f_1 + \cdots + f_n = t \}. \]
Then: $\forall n \in \mathbb{N}, \quad \psi_n (n\alpha) = \mu^n (\Omega_n).$
Then: $\forall m \in \mathbb{N}, \quad \psi_{\gamma m}(\gamma m\alpha) = \mu^{\gamma m}(\Omega_{\gamma m}).$
Since $\mu = B^E_{\beta}$ and since $S_{B^E_{\beta}} = E$, we get: $S_{\mu} = E$.
Since $\# S_{\mu} = \# E \geq 2$, by Theorem 8.6, we get: $V_{\mu} > 0$.

Let $v := V_{\mu}$. Then $v > 0$. Then $g/\sqrt{2\pi v} > 0$.
Let $\tau := g/\sqrt{2\pi v}$. Then $\tau > 0$.

For all $n \in \mathbb{N}$, let $\phi_n := g \cdot \Phi_{n\alpha}$.
Then: $\forall n \in \mathbb{N}, \forall t \in \mathbb{R}, \quad \phi_n (t) = g \cdot \exp(- (t - n\alpha)^2 / (2nv)) / \sqrt{2\pi n v}$.
Then: $\forall n \in \mathbb{N}, \quad \phi_n (n\alpha) = \tau / \sqrt{n}$.
Then: $\forall n \in \mathbb{N}, \quad \sqrt{n} \cdot \left( \phi_n (n\alpha) \right) = \tau$.
Then: $\forall m \in \mathbb{N}, \quad \sqrt{\gamma m} \cdot (\phi_{\gamma m}(\gamma m\alpha)) = \tau$.
Recall: $\alpha/\delta \in \mathbb{Z}$.

So, since $\gamma \alpha = (g/\delta) \cdot \alpha = g \cdot (\alpha/\delta)$, we get: $\gamma \alpha \in g\mathbb{Z}$.
Then: \( \forall m \in \mathbb{N}, \gamma m \alpha \in g \mathbb{Z} \).

**Claim 1:** As \( m \to \infty \), \( \sqrt{\gamma m} \cdot (\psi_{\gamma m}(\gamma m \alpha)) \to \tau \).

**Proof of Claim 1:** By Theorem 16.1, \( \forall t_1, t_2, \ldots \in g \mathbb{Z} \),

as \( n \to \infty \), \( \sqrt{n} \cdot [(\psi_n(t_n)) - (\phi_n(t_n))] \to 0 \).

Then: \( \forall u_1, u_2, \ldots \in g \mathbb{Z} \),

as \( m \to \infty \), \( \sqrt{\gamma m} \cdot [(\psi_{\gamma m}(u_m)) - (\phi_{\gamma m}(u_m))] \to 0 \).

Then, as \( m \to \infty \), \( \sqrt{\gamma m} \cdot [(\psi_{\gamma m}(\gamma m \alpha)) - (\phi_{\gamma m}(\gamma m \alpha))] \to 0 \).

So, since \( \forall m \in \mathbb{N} \),

we get: as \( m \to \infty \), \( \sqrt{\gamma m} \cdot (\psi_{\gamma m}(\gamma m \alpha)) \to \tau \).

**End of proof of Claim 1.**

Since \( \tau > 0 \), by Claim 1, \textbf{choose} \( m_0 \in \mathbb{N} \) s.t.

\[ \forall m \in [m_0, \infty), \sqrt{\gamma m} \cdot (\psi_{\gamma m}(\gamma m \alpha)) > 0. \]

**Claim 2:** \textbf{Let} \( m \in [m_0, \infty). \) Then: \( \mu^{\gamma m}(\Omega_{\gamma m}) > 0. \)

**Proof of Claim 2:** By the choice of \( m_0 \), \( \sqrt{\gamma m} \cdot (\psi_{\gamma m}(\gamma m \alpha)) > 0. \)

Then: \( \psi_{\gamma m}(\gamma m \alpha) > 0. \) Then: \( \mu^{\gamma m}(\Omega_{\gamma m}) = \psi_{\gamma m}(\gamma m \alpha) > 0. \)

**End of proof of Claim 2.**

Since \( \mu = B_{\beta}^E \), we get: \( \mu \in \mathcal{P}(E) \).

Then: \( \forall n \in \mathbb{N}, \mu^n \in \mathcal{P}(E^n), \text{ so } \mu^n(\Omega_n) \leq 1. \)

Then: \( \forall m \in \mathbb{N}, \mu^{\gamma m}(\Omega_{\gamma m}) \leq 1. \)

So, by Claim 2, \( \forall m \in [m_0, \infty), 0 < \mu^{\gamma m}(\Omega_{\gamma m}) \leq 1. \)

Also: \( \forall m \in [m_0, \infty), (\mu^{\gamma m}|\Omega_{\gamma m})(\Omega_{\gamma m}) = \mu^{\gamma m}(\Omega_{\gamma m}). \)

Then: \( \forall m \in [m_0, \infty), 0 < (\mu^{\gamma m}|\Omega_{\gamma m})(\Omega_{\gamma m}) \leq 1. \)

Then: \( \forall m \in [m_0, \infty), \mu^{\gamma m}|\Omega_{\gamma m} \in \mathcal{F} \mathcal{M}^\omega(\Omega_{\gamma m}). \)

Then: \( \forall m \in [m_0, \infty), \mathcal{N}(\mu^{\gamma m}|\Omega_{\gamma m}) \in \mathcal{P}(\Omega_{\gamma m}). \)

**Claim 3:** \textbf{Let} \( m \in [m_0, \infty). \) Then: \( \mathcal{N}(\mu^{\gamma m}|\Omega_{\gamma m}) = \nu_{\Omega_{\gamma m}}. \)

**Proof of Claim 3:** \textbf{Let} \( n := \gamma m. \) \ \textbf{Want:} \( \mathcal{N}(\mu^n|\Omega_n) = \nu_{\Omega_n}. \)

\textbf{Let} \( \theta := \mathcal{N}(\mu^n|\Omega_n), F := \Omega_n. \) \ \textbf{Want:} \( \theta = \nu_F. \)

By Theorem 15.2, \textbf{given} \( f, g \in F, \) \ \textbf{want:} \( \theta(f) = \theta(g). \)

By Claim 2, we have: \( \mu^{\gamma m}(\Omega_{\gamma m}) > 0. \) Then \( \mu^n(\Omega_n) > 0. \)

Since \( (\mu^n|\Omega_n)(\Omega_n) = \mu^n(\Omega_n) \) and \( \theta = \mathcal{N}(\mu^n|\Omega), \) we get: \( \theta = \frac{\mu^n|\Omega_n}{\mu^n(\Omega_n)}. \)

\textbf{Want:} \( \frac{(\mu^n|\Omega_n)(f)}{\mu^n(\Omega_n)} = \frac{(\mu^n|\Omega_n)(g)}{\mu^n(\Omega_n)}. \)
Want: \((\mu^n|\Omega_n)(f) = (\mu^n|\Omega_n)(g)\).

Since \(f, g \in \Omega_n\), we get: \((\mu^n|\Omega_n)(f) = \mu^n(f)\) and \((\mu^n|\Omega_n)(g) = \mu^n(g)\).

Want: \(\mu^n\{f\} = \mu^n\{g\}\).

Let \(C := 1/(\hat{B}_E^E(E))\). Then \(\mathcal{N}(\hat{B}_E^E) = C \cdot \hat{B}_E^E\)

By definition of \(\hat{B}_E^E\), we have: \(\forall \varepsilon \in E, \hat{B}_E^E(\varepsilon) = e^{-\beta \varepsilon}\).

So, since \(\mu = B_E^E = \mathcal{N}(\hat{B}_E^E) = C \cdot \hat{B}_E^E\),

we get: \(\forall \varepsilon \in E, \mu(\varepsilon) = C e^{-\beta \varepsilon}\).

Since \(f \in \Omega_n\), by definition of \(\Omega_n\), we get: \(f_1 + \cdots + f_n = n\alpha\).

Since \(g \in \Omega_n\), by definition of \(\Omega_n\), we get: \(g_1 + \cdots + g_n = n\alpha\).

Since \(f_1 + \cdots + f_n = n\alpha = g_1 + \cdots + g_n\),
we get: \(C^m e^{-\beta (f_1 + \cdots + f_n)} = C^m e^{-\beta (g_1 + \cdots + g_n)}\).

Then: \(Ce^{-\beta f_1} \cdots (Ce^{-\beta f_n} = Ce^{-\beta g_1} \cdots (Ce^{-\beta g_n})\).

Then: \((\mu\{f_1\}) \cdots (\mu\{f_n\}) = (\mu\{g_1\}) \cdots (\mu\{g_n\})\).

Then: \(\mu^n\{f\} = \mu^n\{g\}\).

End of proof of Claim 3.

We have: \(\forall m \in \mathbb{N}, (\mu^{\gamma m}|\Omega_{\gamma m})(\Omega_{\gamma m}) = \mu^{\gamma m}(\Omega_{\gamma m})\).

So, by definition of normalization, we have: \(\forall m \in [m_0, \infty), \forall S \subseteq \Omega_{\gamma m}, (\mathcal{N}(\mu^{\gamma m}|\Omega_{\gamma m}))(S) = \frac{\mu^{\gamma m}(S)}{\mu^{\gamma m}(\Omega_{\gamma m})}\).

and so, by Claim 3, \(\nu_{\gamma m}(S) = \frac{\mu^{\gamma m}(S)}{\mu^{\gamma m}(\Omega_{\gamma m})}\).

We therefore want: As \(m \to \infty\), \(\mu^{\gamma m}\{f \in \Omega_{\gamma m} | f_{\gamma m} = \varepsilon_0\} \to B_E^E(\varepsilon_0)\).

Recall: \(\forall m \in \mathbb{N}, \psi_{\gamma m}(\gamma m\alpha) = \mu^{\gamma m}(\Omega_{\gamma m})\). Also, recall: \(\mu = B_E^E\).

We wish to show: As \(m \to \infty\), \(\frac{\mu^{\gamma m}\{f \in \Omega_{\gamma m} | f_{\gamma m} = \varepsilon_0\}}{\psi_{\gamma m}(\gamma m\alpha)} \to \mu(\varepsilon_0)\).

Recall: \(\forall n \in \mathbb{N}, \Omega_n = \{f \in E^n | f_1 + \cdots + f_n = n\alpha\}\).

Since \(g = \gcd(E)\), we get: \(E \subseteq g\mathbb{Z}\). Recall: \(\forall m \in \mathbb{N}, \gamma m\alpha \in g\mathbb{Z}\).

So, since \(\varepsilon_0 \in E \subseteq g\mathbb{Z}\), we get: \(\forall m \in \mathbb{N}, \gamma m\alpha - \varepsilon_0 \in g\mathbb{Z}\).

For all \(n \in \mathbb{N}\), let \(s_n := n\alpha - \varepsilon_0\).

Then: \(\forall m \in \mathbb{N}, s_{\gamma m} \in g\mathbb{Z}\).

Claim 4: Let \(n \in [2, \infty)\).

Then: \(\mu^n\{f \in \Omega_n | f_n = \varepsilon_0\} = (\psi_{n-1}(s_n)) \cdot (\mu(\varepsilon_0))\).

Proof of Claim 4:

Because \(\{f \in \Omega_n | f_n = \varepsilon_0\} = \{f \in E^n | (f_1 + \cdots + f_{n-1} + f_n = n\alpha) \& (f_n = \varepsilon_0)\}\)
\[
\begin{align*}
= \{ f \in E^n \mid (f_1 + \cdots + f_{n-1} + \varepsilon_0 = n\alpha) \& (f_n = \varepsilon_0) \} \\
= \{ f \in E^n \mid (f_1 + \cdots + f_{n-1} = n\alpha - \varepsilon_0) \& (f_n = \varepsilon_0) \} \\
= \{ f \in E^n \mid (f_1 + \cdots + f_{n-1} = s_n) \& (f_n = \varepsilon_0) \},
\end{align*}
\]
we conclude: under the standard bijection \( E^n \leftrightarrow E^{n-1} \times E \), the subset \( \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} \subseteq E^n \) corresponds to \( \{ f \in E^{n-1} \mid f_1 + \cdots + f_{n-1} = s_n \} \times \{ \varepsilon_0 \} \).

Then \( \mu^n \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} = (\mu^{n-1} \{ f \in E^{n-1} \mid f_1 + \cdots + f_{n-1} = s_n \} \} \cdot (\mu \{ \varepsilon_0 \}) = (\psi_{n-1}(s_n) \cdot (\mu \{ \varepsilon_0 \})) \).

\textit{End of proof of Claim 4.}

Since \( \gamma \in \mathbb{N}, \) by Claim 4, we have: \( \forall m \in [2..\infty), \)
\( \mu^\gamma \{ f \in \Omega_{\gamma m} \mid f_{\gamma m} = \varepsilon_0 \} = (\psi_{\gamma m-1}(s_{\gamma m}) \cdot (\mu \{ \varepsilon_0 \}) \cdot (\mu \{ \varepsilon_0 \}) \).

\textbf{Want:} As \( m \to \infty, \)
\( \frac{\psi_{\gamma m}(\gamma m\alpha)}{\psi_{\gamma m}(\gamma m\alpha)} \to 1. \)

For all \( n \in \mathbb{N}, \) by definition of \( \phi_n, \) we get
\( \phi_n(s_{n+1}) = g \cdot \frac{\exp(-(s_{n+1} - n\alpha)^2/(2nv))}{\sqrt{2\pi nv}}. \)

Then: \( \forall n \in \mathbb{N}, \sqrt{n} \cdot [\phi_n(s_{n+1})] = g \cdot \frac{\exp(-(s_{n+1} - n\alpha)^2/(2nv))}{\sqrt{2\pi v}}. \)

For all \( n \in \mathbb{N}, \) by definition of \( s_{n+1}, \) we have:
\( s_{n+1} - n\alpha = \alpha - \varepsilon_0. \)

Then: \( \text{as} \ n \to \infty, \quad -(s_{n+1} - n\alpha)^2/(2nv) \to 0. \)

Then: \( \text{as} \ n \to \infty, \quad \sqrt{n} \cdot [\phi_n(s_{n+1})] \to g \cdot \frac{1}{\sqrt{2\pi v}}. \)

Recall: \( \tau = g/\sqrt{2\pi v}. \)

Then: \( \text{as} \ n \to \infty, \quad \sqrt{n} \cdot [\phi_n(s_{n+1})] \to \tau. \)

Then: \( \text{as} \ n \to \infty, \quad \sqrt{n-1} \cdot [\phi_{n-1}(s_n)] \to \tau. \)

Then: \( \text{as} \ m \to \infty, \quad \sqrt{\gamma m - 1} \cdot [\phi_{\gamma m-1}(s_{\gamma m})] \to \tau. \)

Recall: \( \forall m \in \mathbb{N}, \quad s_{\gamma m} \in g\mathbb{Z}. \)

By Theorem 16.1, \( \forall t_1, t_2, \ldots \in g\mathbb{Z}, \)
\( \text{as} \ n \to \infty, \quad \sqrt{n} \cdot [\psi_n(t_n)] \to (\phi_n(t_n)]. \)

Then: \( \forall u_1, u_2, \ldots \in g\mathbb{Z}, \)
\( \text{as} \ m \to \infty, \quad \sqrt{\gamma m - 1} \cdot [\psi_{\gamma m-1}(u_m)] \to (\phi_{\gamma m-1}(u_m)]\to 0. \)

Then, \( \text{as} \ m \to \infty, \quad \sqrt{\gamma m - 1} \cdot [\psi_{\gamma m-1}(s_{\gamma m})] \to (\phi_{\gamma m-1}(s_{\gamma m})]^2 \to 0. \)

So, since \( \text{as} \ m \to \infty, \quad \sqrt{\gamma m - 1} \cdot (\phi_{\gamma m-1}(s_{\gamma m})] \to \tau, \)
we get: \( \text{as} \ m \to \infty, \quad \sqrt{\gamma m - 1} \cdot (\psi_{\gamma m-1}(s_{\gamma m})] \to \tau. \)
By Claim 1, as \( m \to \infty \),
\[
\sqrt{\gamma m} \cdot (\psi_{\gamma m} (\gamma m \alpha)) \to \tau.
\]
Dividing the last two limits, we get:
\[
as m \to \infty, \quad \frac{\sqrt{\gamma m - 1} \cdot (\psi_{\gamma m - 1} (s_{\gamma m}))}{\sqrt{\gamma m} \cdot (\psi_{\gamma m} (\gamma m \alpha))} \to 1.
\]
Also, as \( m \to \infty \),
\[
\frac{\sqrt{\gamma m}}{\sqrt{\gamma m - 1}} \to 1.
\]
Multiplying the last two limits, we get:
\[
as m \to \infty, \quad \frac{\psi_{\gamma m - 1} (s_{\gamma m})}{\psi_{\gamma m} (\gamma m \alpha)} \to 1.
\]
\[\square\]

17. General sets

Let \( E \subseteq \mathbb{Z} \). Assume \( \#E < \infty \).
Let \( \alpha \in \mathbb{Z} \). Assume \( \alpha \in (\min E; \max E) \).
For all \( n \in \mathbb{N} \), let \( \Omega_n := \{ f \in E^n \mid f_1 + \cdots + f_n = n\alpha \} \).
Let \( \varepsilon_0 \in E \). In this section, we compute the asymptotics,
as \( n \to \infty \), of \( \nu_{\Omega_n} \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} \).

Since \( \alpha \in (\min E; \max E), \) we get \( \min E \neq \max E \), so \( \min E \geq 2 \).
Let \( E' := E - \varepsilon_0 \). Let \( \alpha' := \alpha - \varepsilon_0 \).
We have: \( \#E' = \#E \). Then \( 2 \leq \#E' < \infty \).
Let \( g := \gcd(E'), \quad \delta := \gcd(\alpha', g), \quad \gamma := g/\delta \).
Then: \( g, \delta, \gamma \in \mathbb{N} \). Let \( \beta := \text{BP}^E_{\alpha} \).
By Theorem 14.7, we get:
\[ \beta = \text{BP}^{E'}_{\alpha'} \).

Define \( H : \mathbb{N} \to \mathbb{R} \) by: \( \forall n \in \mathbb{N}, \ H(n) = \begin{cases} B^E_{\beta} (\varepsilon_0), & \text{if } n \in \gamma \mathbb{N} \\ -1, & \text{if } n \notin \gamma \mathbb{N} \end{cases} \).

In this section, we will prove:
as \( n \to \infty \), \( \nu_{\Omega_n} \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} \) is asymptotic to \( H(n) \).

Also, \( E' \subseteq \mathbb{Z} \) and \( \alpha' \in \mathbb{Z} \) and \( \alpha' \in (\min E'; \max E') \).
Since \( \varepsilon_0 - \varepsilon_0 \in E - \varepsilon_0 = E' \), we get: \( 0 \in E' \).
Let \( \varepsilon'_0 := 0 \). Then: \( \varepsilon'_0 \in E' \).
For all \( n \in \mathbb{N} \), let \( \Omega'_n := \{ f' \in (E')^n \mid f'_1 + \cdots + f'_n = n\alpha' \} \).
The bijection \( \varepsilon' \leftrightarrow \varepsilon - \varepsilon_0 : E \leftrightarrow E' \)
induces, for all \( n \in \mathbb{N} \), a bijection \( E^n \leftrightarrow (E')^n \);
under this bijection, \( \Omega_n \) corresponds to \( \Omega'_n \), and
\( \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} \) corresponds to \( \{ f' \in \Omega'_n \mid f'_n = \varepsilon'_0 \} \).
Then: \( \forall n \in \mathbb{N}, \ \nu_{\Omega_n} \{ f \in \Omega_n \mid f_n = \varepsilon_0 \} = \nu_{\Omega'_n} \{ f' \in \Omega'_n \mid f'_n = \varepsilon'_0 \} \).
Want: as \( n \to \infty \), \( \nu_{\Omega_n'} \{ f' \in \Omega_n' \mid f'_n = \varepsilon'_0 \} \) is asymptotic to \( H(n) \).

By Theorem 14.2, we have: \( B_{\beta}^E \{ \varepsilon'_0 \} = B_{\tilde{\beta}}^E \{ \varepsilon_0 \} \).

Then: \( \forall n \in \mathbb{N}, \quad H(n) = \begin{cases} B_{\beta}^E \{ \varepsilon'_0 \} & \text{if } n \in \gamma N \\ -1, & \text{if } n \notin \gamma N. \end{cases} \)

By Theorem 16.3, as \( m \to \infty \), \( \nu_{\Omega_{\gamma_m}} \{ f' \in \Omega_{\gamma_m} \mid f'_m = \varepsilon'_0 \} \to B_{\beta}^E \{ \varepsilon'_0 \} \).

It suffices to show: \( \forall n \in \mathbb{N} \setminus (\gamma N), \nu_{\Omega_n} \{ f' \in \Omega_n' \mid f'_n = \varepsilon'_0 \} = -1. \)

Given \( n \in \mathbb{N} \setminus (\gamma N) \), \( \nu_{\Omega_n} \{ f' \in \Omega_n' \mid f'_n = \varepsilon'_0 \} = -1. \)

Then: \( B_{\beta}^E \{ \varepsilon'_0 \} \) if \( n \in \gamma N \)

Given \( f' \in (E')^n \), want: \( f'_1 + \cdots + f'_n = n \alpha' \).

Since \( f'_1, \ldots, f'_n \in E' \) and \( g = \gcd(E'), f'_1 + \cdots + f'_n \in g \mathbb{Z} \).

It therefore suffices to show: \( n \alpha' \notin g \mathbb{Z} \).

Recall: \( \delta = \gcd \{ \alpha', g \} \). Then: \( \delta' = \gcd \{ \alpha'/\delta, g/\delta \} \).

Let \( \lambda := \alpha'/\delta \). Recall: \( \gamma = g/\delta \). Then: \( 1 = \gcd \{ \lambda, \gamma \} \).

That is, \( \gamma \) and \( \lambda \) are relatively prime.

Assume: \( n \alpha' \in g \mathbb{Z} \). Want: Contradiction.

Since \( \delta | g | n \alpha' \), we get: \( (g/\delta) | (n \alpha'/\delta) \). Then: \( \gamma | n \lambda \).

So, since \( \gamma \) and \( \lambda \) are relatively prime, we get: \( \gamma | n \).

However, since \( n \in \mathbb{N} \setminus (\gamma N) \subseteq \mathbb{Z} \setminus (\gamma \mathbb{Z}) \), we get: \( \gamma \notin n \). Contradiction.