Kodaira dimensions of low dimensional manifolds

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1. The holomorphic Kodaira dimension κ^h
2. The topological Kod dim κ^t for manifolds up to dimension 3
3. The symplectic Kod dim κ^s for 4−manifolds
4. Relative Kod dim for a symplectic pair

Kodaira dimension type invariants

Roughly speaking, a Kodaira dimension type invariant on a class of $n-$dimensional manifolds
is a numerical invariant taking values in the finite set

$$\{-\infty, 0, 1, \cdots, \lfloor \frac{n}{2} \rfloor\},$$

where $\lfloor x \rfloor$ is the largest integer bounded by x.

holomorphic Kodaira dimension κ^h

Let us first recall the original Kodaira dimension in complex geometry.

Definition

Suppose (M, J) is a complex manifold of real dimension $2m$. The holomorphic Kodaira dimension $\kappa^h(M, J)$ is defined as follows:

$$\kappa^h(M,J) = \begin{cases} -\infty & \text{if } P_l(M,J) = 0 \text{ for all } l \geq 1, \\ 0 & \text{if } P_l(M,J) \in \{0,1\}, \text{ but } \not\equiv 0 \text{ for all } l \geq 1, \\ k & \text{if } P_l(M,J) \sim cl^k;\ c > 0. \end{cases}$$

Here $P_l(M, J)$ is the l−th plurigenus of the complex manifold (M, J) defined by $P_l(M, J) = h^0(\mathcal{K}_J^{\otimes l})$, with $\mathcal{K}_J$ the canonical bundle of (M, J).

κ^t in dimensions 0 and 1

There are other situations where a similar notion can be defined. Let M be a closed, smooth, oriented manifold. To begin with, we make the following definition for logical compatibility.

Definition

If $M = \emptyset$, then its Kodaira dimension is defined to be $-\infty$.

The only closed connected 0−dimensional manifold is a point, and the only closed connected 1−dimensional manifold is a circle.

Definition

If M has dimension 0 or 1, then its Kodaira dimension $\kappa^t(M)$ is defined to be 0.

κ^t in dimension 2

The 2−dimensional Kodaira dimension is defined by the positivity of the Euler class. We write $K = -e$.
Suppose M^2 is a 2−dimensional closed, connected, oriented manifold with Euler class $e(M^2)$. Write $K = -e(M^2)$ and define

$$\kappa^t(M^2) = \begin{cases} -\infty & \text{if } K < 0, \\ 0 & \text{if } K = 0, \\ 1 & \text{if } K > 0. \end{cases}$$

It is easy to see that for any complex structure J on M^2, K is its canonical class, and $\kappa^h(M^2, J) = \kappa^t(M^2)$. $\kappa^t(M^2)$ can be further interpreted from other viewpoints:
symplectic structure (K is also the symplectic canonical class),
the Yamabe invariant,
geometric structures and etc.

Yamabe invariant

Recall that the Yamabe invariant is defined in the following way :

$$Y(M) = \sup_{[g]\in\mathcal{C}} \inf_{g\in[g]} \int_M s_g dV_g, \tag{2.1}$$

where g is a Riem metric on M, s_g the scalar curvature, $[g]$ the conformal class of g, and $\mathcal{C}$ the set of conformal classes on M.
A basic fact is that $Y(M) > 0$ if and only if M admits a metric of positive scalar curvature.
Thus $Y(M)$ is non-positive if M does not admit metrics of positive scalar curvature. Furthermore, in this case, another basic fact is that $Y(M)$ is the supremum of the scalar curvatures of all unit volume constant-scalar-curvature metrics on M (such metrics exist due to the resolution of the Yamabe conjecture).
It immediately follows that, in dimension two, the sign of $Y(M^2)$ completely determines $\kappa^t(M^2)$.

Geometries in dimension 3

We move on to dimension 3. In this dimension the definition of the Kodaira dimension is based on geometric structures in the sense of Thurston. Divide the 8 Thurston geometries into 3 categories:

$$\begin{array}{ll} -\infty: & S^3 \text{ and } S^2\times\mathbb{R}; \\ 0: & \mathbb{E}^3, \text{Nil and Sol}; \\ 1: & \mathbb{H}^2\times\mathbb{R}, \widetilde{SL_2(\mathbb{R})} \text{ and } \mathbb{H}^3. \end{array}$$

Given a closed, connected 3−manifold M^3, we decompose it first by a prime decomposition and then further consider a toridal decomposition for each prime summand, such that at the end each piece has a geometric structure either in group (1), (2) or (3) with finite volume.

The following definition was introduced by Weiyi Zhang, based on the Geometrization theorem.

κ^t in dimension 3

For a closed, connected 3−dimensional manifolds M^3, we define its Kodaira dimension as follows:

1. $\kappa^t(M^3) = -\infty$ if for any decomposition, each piece has geometry type in category $-\infty$,
2. $\kappa^t(M^3) = 0$ if for any decomposition, we have at least a piece with geometry type in category 0, but no piece has type in category 1,
3. $\kappa^t(M^3) = 1$ if for any decomposition, we have at least one piece in category 1.

One basic property: If there is a nonzero degree map from M to N, then $\kappa^t(M) \geq \kappa^t(N)$.

Yamabe invariant and κ^t

In this dimension, $Y(M^3)$ is also closely related to geometric structure of M^3, at least when M^3 is irreducible. However, the number $Y(M^3)$ does not completely determine $\kappa^t(M^3)$.

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For $\Sigma_g \times S^1$, it has vanishing Yamabe invariant if $g \geq 1$.
But $\kappa^t(\Sigma_g \times S^1) = 0$ if $g = 1$, $\kappa^t(\Sigma_g \times S^1) = 1$ if $g \geq 2$.
In this case, κ^t is still determined by (2.1) if we distinguish whether the supremum is attainable by a metric.

But this refinement of $Y(M^3)$ will still not determine κ^t in general since a Nil 3−manifold such as a non-trivial S^1−bundle over T^2 has Yamabe invariant 0 which is not attainable by any metric.

Notice that here we use κ^t to denote the Kodaira dimension for smooth manifolds in dimensions $0, 1, 2, 3$.
Here t stands for *topological/smooth*, because in these dimensions homeomorphic manifolds are actually diffeomorphic.

For a possibly disconnected manifold, we define its Kodaira dimension to be the maximum of that of its components.

In summary, we have defined the Kodaira dimension for all closed, oriented manifolds with dimension less than 4.

Minimality in dimension 4

Let M be a closed, oriented smooth 4-manifold.
Let $\mathcal{E}_M$ be the set of cohomology classes whose Poincaré dual are represented by smoothly embedded spheres of self-intersection -1.
M is said to be (smoothly) minimal if $\mathcal{E}_M$ is the empty set.
Equivalently, M is minimal if it is not the connected sum of another manifold with $\overline{\mathbb{CP}^2}$.

Suppose ω is a symplectic form compatible with the orientation. (M, ω) is said to be (symplectically) minimal if $\mathcal{E}_\omega$ is empty, where

$$\mathcal{E}_\omega = \{E \in \mathcal{E}_M | \ E \text{ is represented by an embedded } \omega\text{−symplectic sphere}\}.$$

We say that (N, τ) is a minimal model of (M, ω) if (N, τ) is minimal and (M, ω) is a symplectic blow up of (N, σ).
A basic fact proved using SW theory is: $\mathcal{E}_\omega$ is empty if and only if $\mathcal{E}_M$ is empty. In other words, (M, ω) is symplectically minimal if and only if M is smoothly minimal.

Definition for minimal (M, ω)

For a minimal symplectic 4−manifold (M^4, ω) with symplectic canonical class K_ω, the Kodaira dimension of (M^4, ω) is defined in the following way:

$$\kappa^s(M^4,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } & K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } & K_\omega \cdot K_\omega > 0. \end{cases}$$

Here K_ω is defined as the first Chern class of the cotangent bundle for any almost complex structure compatible with ω.

κ^s well defined

κ^s is well defined since there doesn't exist minimal (M, ω) with

$$K_\omega \cdot [\omega] = 0, \quad K_\omega \cdot K_\omega > 0.$$

Properties:

- κ^s is independent of ω, so it is an oriented differomorphism invariant of M.
- Dorfmeister+Zhang: $\kappa^s = \kappa^h$ whenever both are defined, eg. the Kodaira-Thurston manifolds. It was shown in the 90s by Friedman and Morgan that $\kappa^h(M^4, J)$ (even the plurigenera) only depends on the oriented diffeomorphism type of M^4.
- Liu: $\kappa^s(M) = -\infty$ if and only if M is $\mathbb{CP}^2, S^2 \times S^2$ or an S^2bundle over a Riemann surface of positive genus.

General (M, ω)

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its minimal models.
$\kappa^s(M, \omega)$ is defined for any (M, ω) since

- Minimal models always exist
- Minimal model almost unique up to diffeomorphisms. If (M, ω) has non-diffeomorphic minimal models, then these minimal models have $\kappa^s = -\infty$.
- Diffeomorphic minimal models have the same κ^s.

Basic property:

- κ^s is an oriented differomorphism invariant of M, and $\kappa^s = \kappa^h$ whenever both are defined.

Yamabe invariant and κ^s

LeBrun calculated $Y(M^4)$ when M^4 admits a Kähler structure, from which he concluded that (2.1) completely determines κ^h. As $\kappa^s = \kappa^h$ for a Kähler surface, if M^4 admits a Kähler structure, then

$$\kappa^s(M^4) = \begin{cases} -\infty & \text{if } Y(M^4) > 0, \\ 0 & \text{if } Y(M^4) = 0 \text{ and 0 is attainable by a metric,} \\ 1 & \text{if } Y(M^4) = 0 \text{ and 0 is not attainable,} \\ 2 & \text{if } Y(M^4) < 0. \end{cases} \tag{3.1}$$

However, (3.1) does not determine $\kappa^s(M^4)$ for all symplectic M^4: All T^2-bundle over T^2 have $\kappa^s = 0$, while most of them do not have any zero scalar curvature metrics. But the question of LeBrun still makes sense: if M^4 admits a symplectic structure and $Y(M^4) < 0$, is $\kappa^s(M^4) = 2$?

Virtual Kod dim

A related question is whether we can extend κ^s and κ^h to κ^d for all smooth 4−manifolds (here d stands for differentiable).

κ^s can be extended to virtually symplectic/complex manifolds, manifolds which are finitely covered by symplectic/complex manifolds. This is based on the observation:

κ^s and κ^h are invariant under finite coverings.

If X is finitely covered by a symplectic manifold M, then $\kappa^d(X) := \kappa^s(M)$.

Rk. κ^t is also a covering invariant, but the Yamabe invariant in dimension 4 is not .

Additivity for a fiber bundle

A classical theorem says that the the holomorphic Kodaira dimension κ^h is additive for any holomorphic fiber bundle.

κ^t is also additive for any fiber bundle.

Dorfmeister+Zhang: κ^s is additive for surface bundles over surfaces,
$\kappa^s = \kappa^t(fiber) + \kappa^t(base)$

Ample evidence for the additivity of κ^s for S^1−bundles and mapping tori:
$\kappa^s = \kappa^t$(3-manifold fiber or base)

κ^s under constructions

Fibre-sum along symplectic surfaces

Usher calculated κ^s for positive genus sum.
Dorfmeister estimated κ^s for Genus zero sum, in particular, rational blow-down.

Luttinger surgery along Lagrangian tori
Ho+ L: κ^s unchanged under Luttinger surgery

Useful to show some small manifolds are exotic.

Important families of manifolds in each class

- $\kappa^s = -\infty$: Rational manifolds, ruled manifolds
- $\kappa^s = 0$:
  K3 surface, Enriques surface
  Kodaira-Thurston manifold
- $\kappa^s = 1$:
  Elliptic surfaces $E(n)$ with $n \geq 3$
  For any finitely presented group G, Gompf constructed M_G with $\pi_1(M_G) = G$. M_G can be chosen to have $\kappa^s = 1$.
- $\kappa^s = 2$:
  Surfaces of general type
  Exotic small manifolds of Akhmedov-D. Park, J. Park, Fintushel-Stern, Stipsicz-Oszvath-Szabo, Baldridge-Kirk etc

Main problems in each class

For manifolds in each κ^s class, the smaller κ^s the more we understand.

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  Prove/disprove the Miyaoka-Yau inequality $K_\omega \cdot K_\omega \leq 3\chi(M)$
  Prove a Noether type inequality

Higher Dimension

In higher dimension, Kodaira dimension is only defined for complex manifolds. And κ^h is known not to be a diffeomorphism invariant. Thus we can only expect to have a notion of Kodaira dimension for manifolds with some structures such as complex structures or symplectic structures. There is a proposal to extend κ^s due to Ruan+L.

Maximal surfaces

There is a relative Kodaira dimension.
(M, ω) is a connected, closed symplectic 4−manifold and $F \subset (M, \omega)$ is a symplectic surface.
The adjoint class of F is defined as $K_\omega + [F]$.
We assume that F has no sphere components.
We also assume that F is maximal.
F is called maximal if for any symplectic -1 class E,

$$(K_\omega + [F]) \cdot E \geq 0.$$

Equivalently, $[F] \cdot E > 0$.
The complement of F is minimal.

Constraints on the adjoint class

Lemma

Suppose F is maximal and has no sphere components.
If $\kappa^s(M,\omega) \geq 0$, then

$$(K_\omega + [F]) \cdot [\omega] > 0, \quad (K_\omega + [F])^2 \geq 0.$$

If $\kappa^s(M,\omega) = -\infty$ and $(K_\omega + [F])^2 > 0$, then $(K_\omega + [F]) \cdot [\omega] > 0$.

The case when $\kappa^s(M,\omega) = -\infty$. As $b^+(M) = 1$ in this case, the statement follows from the light cone lemma and $[\omega]^2 \geq 0$.

Corollary

It is impossible

$$(K_\omega + [F]) \cdot \omega = 0 \quad \text{and} \quad (K_\omega + [F])^2 > 0.$$

Definition

Definition

Let $F \subset (M, \omega)$ be a maximal symplectic surface without sphere components. Then the relative Kodaira dimension of (M, F, ω) is defined in the following way: if F is empty, then (M, ω) is necessarily minimal and $\kappa^s(M, \omega, F)$ is defined to be $\kappa^s(M, \omega)$. Otherwise,

$$\kappa^s(M, \omega, F) = \begin{cases} -\infty & \text{if } (K_\omega + [F]) \cdot \omega < 0 \text{ or } \ (K_\omega + [F])^2 < 0, \\ 0 & \text{if } (K_\omega + [F]) \cdot \omega = 0 \text{ and } (K_\omega + [F])^2 = 0, \\ 1 & \text{if } (K_\omega + [F]) \cdot \omega > 0 \text{ and } (K_\omega + [F])^2 = 0, \\ 2 & \text{if } (K_\omega + [F]) \cdot \omega > 0 \text{ and } (K_\omega + [F])^2 > 0. \end{cases}$$

There are exactly 4 cases.

Assume F is a maximal symplectic surface without sphere components in (M, ω), then

$$\kappa^s(M, \omega, F) \geq \kappa^s(M, \omega).$$

Usher's calculation of κ^s for a fiber-sum can be rephrased:

Theorem

Let (M, ω) be a 4−dimensional relatively minimal fiber sum of (M_1, ω_1) and (M_2, ω_2) along connected genus $g \geq 1$ symplectic surfaces $F_i \subset (M_i, \omega_i)$. Then

$$\kappa^s(M, \omega) = \max\{\kappa^s(M_1, \omega_1, F_1), \kappa^s(M_2, \omega_2, F_2)\}. \tag{4.1}$$

Extends to singular pseudo-hol type surfaces
similar to Log Kod dim in algebraic geometry
open symplectic manifolds