AN EXTENSION OF HÖLDER'S THEOREM

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It is a classical result (essentially due to Hölder) that if Γ is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ such that every nontrivial element acts freely then Γ is Abelian. A natural question to ask is what if every nontrivial element has at most N fixed points where N is a fixed natural number. In the case of $N = 1$, we do have a complete answer to this question: it has been proved independently by Solodov (not published), Barbot [4], Kovacevic [6] and Farb-Franks [5] that in this case the group is metaabelian, in fact, it is isomorphic to a subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$.

In [2], we answer this question for an arbitrary N assuming some regularity on the action of the group.

Our main result there are the following two theorems.

Theorem 1. Let $\epsilon \in (0, 1)$ and Γ be a subgroup of $\mathrm{Diff}_+^{1+\epsilon}(I)$ such that every nontrivial element of Γ has at most N fixed points. Then Γ is solvable.

Assuming a higher regularity on the action we obtain a stronger result

Theorem 2. Let Γ be a subgroup of $\mathrm{Diff}_+^2(I)$ such that every nontrivial element of Γ has at most N fixed points. Then Γ is metaabelian.

An important tool in obtaining these results is provided by Theorems B-C from [1]. Theorem B (Theorem C) states that a non-solvable (non-metaabelian) subgroup of $\mathrm{Diff}_+^{1+\epsilon}(I)$ (of $\mathrm{Diff}_+^2(I)$) is non-discrete in the C_0 metric. Existence of C_0-small elements in a group provides effective tools in tackling the problem. Such tools are absent for less regular actions, and for the group $\mathrm{Homeo}_+(I)$, the problem of characterizing subgroups where every non-identity element has at most $N \geq 2$ fixed points still remains open.

In the recent work [3], by strengthening the results of [1] (consequently of [2]), we prove that any irreducible subgroup of $\mathrm{Diff}_+(I)$ where every non-identity elements has at most N fixed points must be affine, thus obtaining the best possible result (a complete classification) even in C^1 regularity. By introducing the concept of semi-archimedean groups, we also show that the above classification picture fails in the continuous category.

References

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