Exotic Smooth Structures on 4-Manifolds

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Outline

Introduction

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Geography


ARCTIC OCEAN
ATLANTIC
OCEAN
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- If X is a minimal complex surface of general type, $2\chi_h - 6 \leq c_1^2(X)$ (Noether inequality), $0 < \chi_h(X)$, and $c_1^2(X) > 0$.

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- M. Hamilton and D. Kotschick: minimal symplectic 4-manifolds with residually finite fundamental groups are irreducible.

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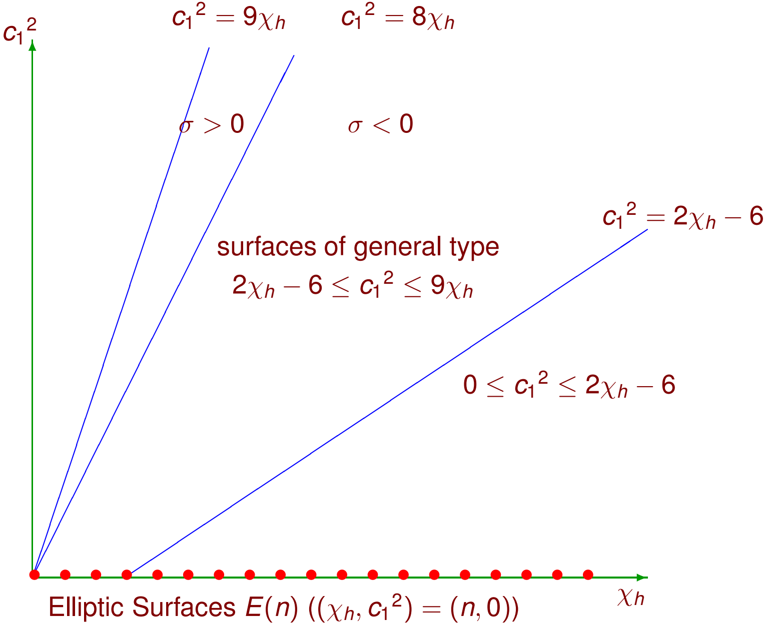
$c_1{}^2$
$c_1{}^2 = 9\chi_h$
$c_1{}^2 = 8\chi_h$
$\sigma > 0$
$\sigma < 0$
$c_1{}^2 = 2\chi_h - 6$
surfaces of general type
$2\chi_h - 6 \leq c_1{}^2 \leq 9\chi_h$
$0 \leq c_1{}^2 \leq 2\chi_h - 6$
χ_h
Elliptic Surfaces $E(n)$ $((\chi_h, c_1{}^2) = (n, 0))$

Figure 1

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- Branched Covers (F. Hirzebruch)

Symplectic Connected Sum

Definition

Let X_1 and X_2 are symplectic 4-manifolds, and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected symplectic submanifolds in them. Suppose that $[F_1]^2 + [F_2]^2 = 0$ and the genera of F_1 and F_2 are equal. Take an orientation-preserving diffemorphism $\psi : F_1 \longrightarrow F_2$ and lift it to an orientation-reversing diffemorphism $\Psi : \partial\nu F_1 \longrightarrow \partial\nu F_2$ between the boundaries of the tubular neighborhoods of νF_i. Using Ψ, we glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary. The 4-manifold $X_1 \#_\Psi X_2$ is called the *(symplectic) connected sum* of X_1 and X_2 along F_1 and F_2, determined by Ψ.

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$$c_1^2(X_1 \#_\Psi X_2) \quad = \quad c_1^2(X_1) + c_1^2(X_2) + 8(g-1),$$

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$$\begin{aligned} c_1^2(X_1 \#_\Psi X_2) &= c_1^2(X_1) + c_1^2(X_2) + 8(g-1), \\ \chi_h(X_1 \#_\Psi X_2) &= \chi_h(X_1) + \chi_h(X_2) + (g-1), \end{aligned}$$

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(ii) If one of the summands X_i (say X_1) admits the structure of an $\mathbb{S}^2$-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.

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(ii) If one of the summands X_i (say X_1) admits the structure of an $\mathbb{S}^2$-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.
(iii) In all other cases, Z is minimal.

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If D_1 and D_2 are smooth disjoint curves in an algebraic surface S, and the homology class of the divisor $D = D_1 - D_2$ is divisible by n, then there exist an algebraic surface X which is a $\mathbb{Z}/n\mathbb{Z}$ cyclic cover of S and ramified over D.

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Let X be a 4-manifold (with $b_2{}^+(X) \geq 1$) which contains a homologically nontrivial torus T of self-intersection 0. Let $N(K)$ be a tubular neighborhood of K in $\mathbb{S}^3$, and let $T \times D^2$ be a tubular neighborhood of T in X.

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Luttinger surgery has been very effective tool recently for constructing exotic smooth structures.

Luttinger Surgery and Sympletic Kodaira Dimension

Definition

For a minimal symplectic 4-manifold (M^4, ω) with symplectic canonical class K_ω, the Kodaira dimension of (M^4, ω) is defined in the following way:

$$\kappa^s(M^4, \omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } \quad K_\omega \cdot K_\omega < 0, \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } \quad K_\omega \cdot K_\omega = 0, \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } \quad K_\omega \cdot K_\omega = 0, \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } \quad K_\omega \cdot K_\omega > 0. \end{cases}$$

If (M^4, ω) is not minimal, its Kodaira dimension is defined to be that of any of its minimal models.

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Theorem (*C.-I. Ho and T.J. Li, 2008*)

The symplectic Kodaira dimension is unchanged under Luttinger surgery.

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Let K be a genus one and K' be any genus $2 \leq g$ fibered knot in $\mathbb{S}^3$. Then there exist an infinite family of minimal symplectic 4-manifolds $V_{KK'}$ cohomology equivalent to $\#_{2g-1}(\mathbb{S}^2 \times \mathbb{S}^2)$.

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Let K be a genus one fibered knot (i.e., the trefoil or the figure eight knot) in $\mathbb{S}^3$ and m a meridional circle to K.

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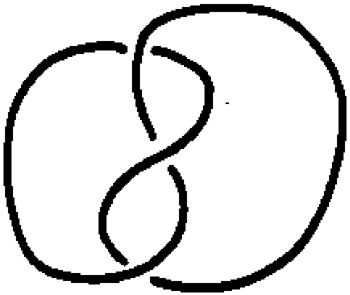

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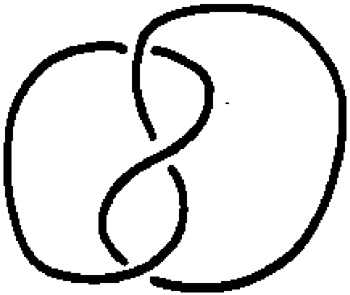
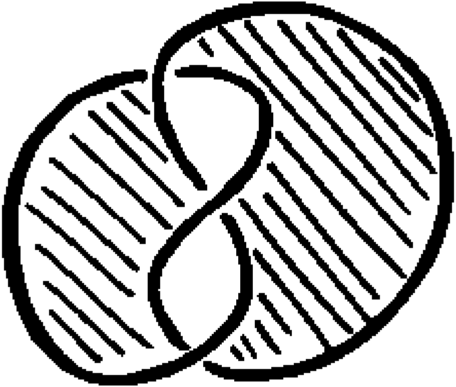


$$\pi_1(\mathbb{S}^3 \setminus K) \quad = \quad < a, b \mid aba = bab >$$



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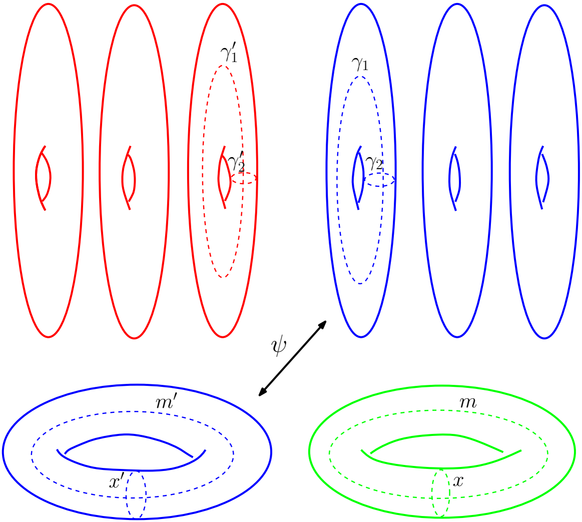

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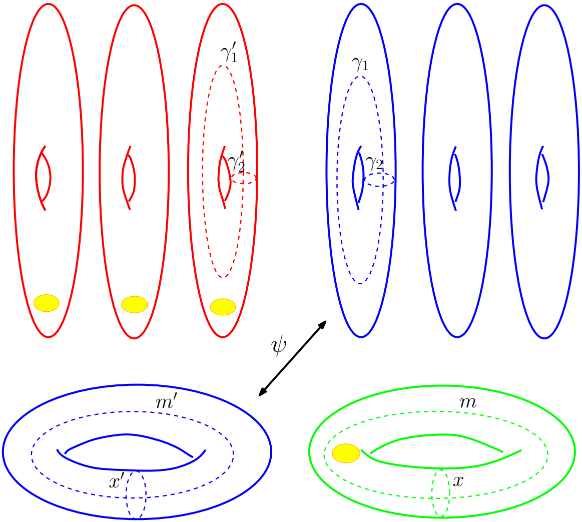

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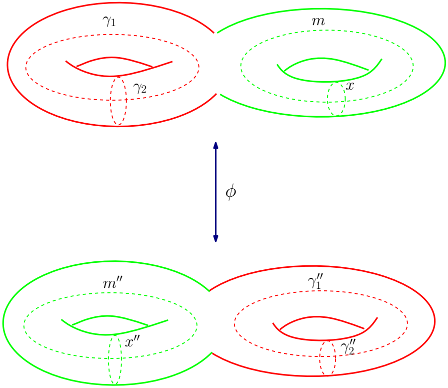

$$\begin{aligned} c_1^2(X_K) &= 2c_1^2(Y_K) + 8(2-1) = 8 \\ \chi_h(X_K) &= 2\chi_h(Y_K) + (2-1) = 1 \end{aligned}$$

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Remark: Using non-fibered genus one *n*-twist knots leads to non-symplectic fake $\mathbb{S}^2 \times \mathbb{S}^2$.

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Remark: Y_K, X_K, $Y_{K'K}$, $V_{K'K}$ serve as an important building blocks in the construction of (small or big) exotic 4-manifolds

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All these construction starts with simply-connected 4-manifolds and apply some surgery techniques that preserves simple connectivity.

We introduced a new technique in [*A.A, 2006, Alg and Geom Topol*], and constructed exotic symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ and exotic $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ using non-simply connected building blocks.

Theorem (*A.A, 2006, Alg and Geom Topol*)

Let M be one of the following 4-manifolds.

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Then there exist an irreducible symplectic 4-manifolds homeomorphic but not diffeomorphic to M.

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- Let $M = \mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$.

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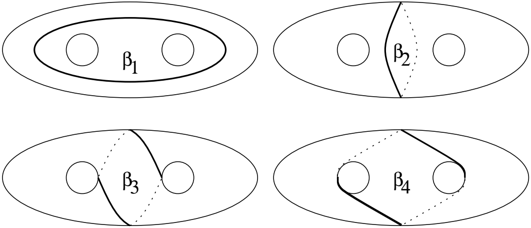
Sketch of Proof Continued

- The global monodromy of Matsumoto's fibration: $(D_{\beta_1} D_{\beta_2} D_{\beta_3} D_{\beta_4})^2 = 1$,



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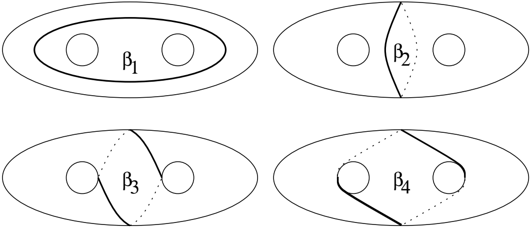

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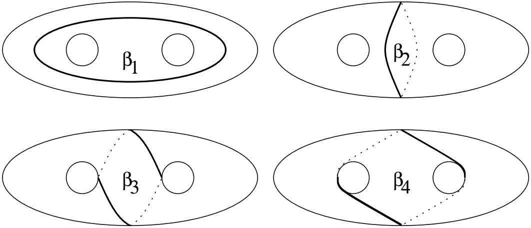

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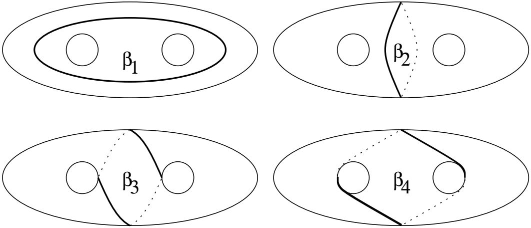

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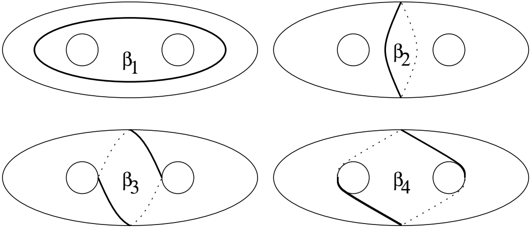

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- X is minimal symplectic by M. Usher's Minimality Theorem, so it cannot be diffemorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$.

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Alternative construction of exotic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ was later given by S. Baldridge - P. Kirk, and R. Fintushel - D. Park - R. Stern.

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- $c_1^2(X) = c_1^2(M) + c_1^2(Y_K) + 8(2-1) = 6$, $\chi_h(X) = \chi_h(M) + \chi_h(Y_K) + (2-1) = 1$.

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Exotic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ of Akhmedov - Park via Luttinger Surgery, (A.A - I. Baykur - D. Park)

- Use 3 copies of the 4-torus, T_1^4, T_2^4 and T_3^4.

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- The two symplectic tori $a_3 \times b_3$ and $c_3 \times d_3$ in T^4_3 intersect at one point. Smooth out intersection point to get a symplectic surface of genus 2. Blow up T^4_3 twice at the self-intersection points to obtain a symplectic genus two surface Σ' of self-intersection zero.

- Take the symplectic fiber sum of $Y = T^2 \times \Sigma_2$ and $Y' = T^4_3 \# 2\overline{\mathbb{CP}}^2$ along the surfaces Σ_2 and Σ', determined by a map that sends the circles c_1, d_1, c_2, d_2 to a_3, b_3, c_3, d_3 in the same order. By Seifert-Van Kampen theorem, the fundamental group of the resulting manifold X' can be seen to be generated by a_1, b_1, c_1, d_1, c_2 and d_2, which all commute with each other. $\pi_1(X')$ is isomorphic to $\mathbb{Z}^6$, $e(X') = 6$ and $\sigma(X') = -2$, which are also the characteristic numbers of $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

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$$(a_1 \times \tilde{c}_1, \tilde{c}_1, -1), \ \ (a_1 \times \tilde{d}_1, \tilde{d}_1, -1), \ \ (\tilde{a}_1 \times c_2, \tilde{a}_1, -1),$$
$$(\tilde{b}_1 \times c_2, \tilde{b}_1, -1), \ \ (c_1 \times \tilde{c}_2, \tilde{c}_2, -1), \ \ (c_1 \times \tilde{d}_2, \tilde{d}_2, -1).$$

We obtain a symplectic 4-manifold X with $\pi_1(X)$ generated by a_1, b_1, c_1, d_1, c_2, d_2 with relations:

$$[b_1, d_1^{-1}] = b_1 c_1 b_1^{-1}, \ \ [c_1^{-1}, b_1] = d_1, \ \ [d_2, b_1^{-1}] = d_2 a_1 d_2^{-1},$$
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X is simply-connected and surgeries do not change the characteristic numbers, we have it homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$. Y is minimal and the exceptional spheres in Y' intersect Σ', Ushers Theorem guarantees that X' is minimal. X is an irreducible symplectic 4-manifold which is not diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

Theorem (*A.A - Doug Park, May 2007, Inven. Math*)

Let M be one of the following 4-manifolds.

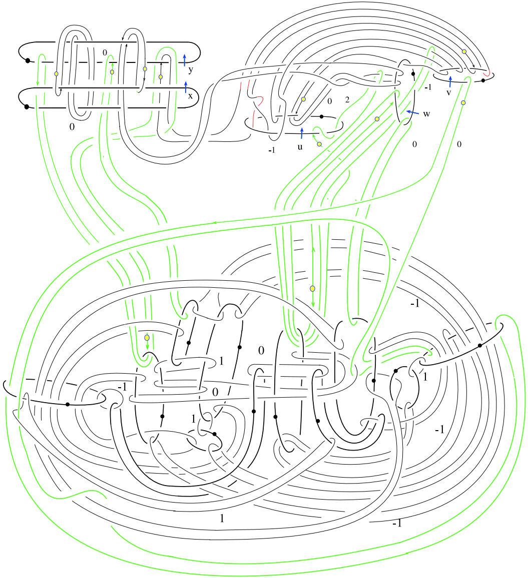
(i) $\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$ *for* $m = 2, 4$*,*

(ii) $3\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ *for* $k = 4, 6, 8, 10$*,*

(iii) $(2n-1)\mathbb{CP}^2 \# 2n\overline{\mathbb{CP}}^2$ *for any integer* $n \geq 3$*.*

Then there exist an irreducible symplectic 4-manifold and an infinite family of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds, all of which are homeomorphic to M.

Handlebody of Akhmedov-Park's exotic $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ by Selman Akbulut



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- ($\sigma = 0$) J. Park constructed the exotic smooth structures on $\#_{2n-1}(S^2 \times S^2)$ for $n \geq 267145kx^2$, where integer k and x are large numbers, which were not explicitly computed.
- There are simply connected symplectic family by A. Stipsicz approaching BMY line, but his examples are not spin.

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There exisit an infinite family of closed simply connected minimal symplectic 4-manifolds $\{M_n | n \in \mathbb{N}\}$ *satisfying* $8.92\chi_h < {c_1}^2(M_n) < 9\chi_h$ *for every* $n \geq 12$, *and* $\lim_{n\to\infty} \frac{{c_1}^2(M_n)}{\chi_h(M_n)} = 9$. *Moreover,* M_n *has* ∞^2*-property for every n and spin if* $n \equiv 4(mod 8)$.

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- Our construction uses the spin complex surfaces of Hirzebruch near BMY line, the elliptic surfaces $E(2n^3)$, and the spin symplectic 4-manifolds S^G of R. Gompf (with $\pi_1(S^G) = G$, and ${c_1}^2(S^G) = 0$).

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- construction (i) uses the small surfaces bundles with non-zero signature by J. Bryan and R. Donagi, and homotopy $K3$ surfaces by R. Fintushel-R. Stern.

Branched Cover Construction of Surface Bundles with Non-Zero Signature, after M. Atiyah, K. Kodaira, F. Hirzebruch

Theorem (*J. Bryan - R. Donagi - A. Stipsicz*)

For any integers $n \geq 2$, there exisit smooth algebraic surface X_n that have signature $\sigma(X_n) = 8/3n(n-1)(n+1)$ and admit two smooth fibrations $X_n \longrightarrow B$ and $X_n \longrightarrow B'$ such that the base and fiber genus are $(3, 3n^3 - n^2 + 1)$ and $(2n^2 + 1, 3n)$ respectively.

Theorem (*J. Bryan - R. Donagi*)

For any pair of integers $g, n \geq 2$, there exisit smooth algebraic surface $X_{n,g}$ that have signature $\sigma(X_n) = 4/3g(g-1)(n^2-1)n^{2g-3}$ and admit two smooth fibrations $X_{n,g} \longrightarrow B$ and $X_{n,g} \longrightarrow B'$ such that the base and fiber genus are $(g(g-1)n^{2g-2} + 1, gn)$ and $(g, g(gn-1)n^{2g-2} + 1)$ respectively.

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Let B be a genus 3 surface and $\tau : B \to B$ be a fixed point free involution of B around the center hole.

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Assume that X is spin, $\sigma(X) = 16s$ and X has a section $\Sigma_b \longrightarrow X$ whose image is a genus b surface of self-intersection $-2t$.

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Assume that X is spin, $\sigma(X) = 16s$ and X has a section $\Sigma_b \longrightarrow X$ whose image is a genus b surface of self-intersection $-2t$.

Symplectically resolve the double point of Σ_f and the image of a section to get a symplectic submanifold of Σ_{f+b} in X of genus $f + b$ and self intersection $2 - 2t$.

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These tori intersect at $(0,0)$ and do not intersect each other anywhere else.

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There are n^4 exceptional curves L_j ($j \in U_n$) resulting from blow-ups. Denote by $\tilde{D}_i$ the proper transforms of D_i after the blow-up

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Also, the inclusion induced homomorphism $\pi_1(F_n) \longrightarrow \pi_1(X_n)$ is surjective.

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$w_2(X_n) \equiv \overline{K} \equiv 0(mod\ 2)$

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Arbitrary fundamental group

Theorem (*R. Gompf*)

Let G be a finitely presented group. There exisit a spin symplectic 4-manifold S^G with $\pi_1 = G$, $c_1{}^2(S^G) = 0$, and $\chi_h(S^G) > 0$. Moreover, S^G contains a symplectic torus T of self-intersection 0 such that the inclusion induced homomorphism $\pi_1(T) \longrightarrow \pi_1(S^G)$ is trivial.

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S^G is the symplectic sum of $\Sigma_g \times T$ and l copies of $K3$ surface along self-intersection 0 tori.

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Our $M_k{}^G$ is the symplectic sum of M_{8n-4} and S^G along the tori.

THANK YOU!