I. Introduction to Riemann Surface Theory

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Example of a 2D orientable surface R in 3D

The Construction

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- To change the coordinate system on R so that this property holds at all points requires the introduction of *isothermal coordinates* on R. This is done be solving a differential equation.

Technical Details

A Riemann surface is a **second countable Hausdorff space** with a complex structure as described above:

A **topological space** is a set X together with a collection of subsets $\{S\}$ of X that satisfies:
(a) $\emptyset \cup X \subset \{S\}$.
(b) The *intersection* of a finite number of subsets of $\{S\}$ lies in $\{S\}$, as well as the *union* of finite or infinitely many subsets.

The elements of $\{S\}$ are called *open sets* and $\{S\}$ itself is called a *topology* of X.

X is **second countable** if there exists a countable collection of open subsets $\{U\} \subset X$ such that any open subset of X is a finite union of elements of some subfamily of $\{U\}$.

A **Hausdorff space** is a topological space in which distinct points have disjoint neighborhoods.

Summary

A Riemann surface is a 1-dim $\mathbb{C}$-manifold $R = \{(u_\alpha, f_\alpha)\}$.

$\{\mathrm{U}_\alpha\}$ are open sets that cover R;

$f_\alpha : U_\alpha \mapsto f_\alpha(U_\alpha) \subset \mathbb{C}$, a homeomorphism,

$f_\alpha \circ {f_\beta}^{-1} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$, conf map of overlaps.

(Gives a "rule" for measuring angles on R)

Riemann Surface Properties:

- necessarily connected,
- has a countable basis, $\{(U_\alpha, f_\alpha)\}$
- Can be closed (no bdry), OR compact with bdry, OR open (non compact).

Topologically determined by:

- Given $O \in R$, $\pi_1(R; O)$, fund. group based at O.
- $H_1(R, \mathbb{Z})$, integral homology basis.
- genus$=$ $\#$"handles".
- boundary or "ideal boundary at ∞'',
- punctures—removal of isolated points on R.

Covering Surfaces; Universal Covering Surface

Let R be a Riemann surface and $O \in R$ a fixed pt.
Set $\tilde{R} = \{(z, \gamma_z) : z \in R \ \gamma_z$,a simple path from O to $\gamma_z\}$,
$(z, \gamma_z) \equiv (z', \alpha_z)$ iff $z = z'$ AND ${\gamma_z}^{-1}(\alpha_z)$ from O to O is retractable to O (homotopic to id.).

$\tilde{R}$ is also a Riemann surface; $\pi : \tilde{R} \to R$ is loc. injective anal. map.
Pick $\tilde{O} \in \tilde{R}$ over O. A s.c. nbhd N of O lifts to a s.c. $\tilde{N}$ of $\tilde{O}$.
Lifts $\tilde{\gamma}' \sim \tilde{\gamma}'_1$ in $\tilde{R}$ iff $\gamma \sim \gamma'$ in R (homotopies).
Fund. gp $\pi_1(\tilde{R}; \tilde{O})$ is isomorphic to a subgp G of $\pi_1(R; O)$.
Conversely, given a subgroup H of $\pi_1(R; O)$, construct a R-cover $(\tilde{R}; \tilde{O})$ s.t. fundamental gp. $\pi_1(\tilde{R}; \tilde{O}) \cong H$ (isomorphic).

Set $H =$, so $\pi_1(\tilde{R}; \tilde{O}) =$. Then $\tilde{R}$ is simply connected! It is the **Universal Covering Surface**: it covers all other covering surfaces.

Every R.S. is conf. equivalent to exactly one of: $\mathbb{S}^2, \mathbb{C}, \mathbb{D}$.
(a)$\tilde{R} \equiv \mathbb{S}^2$ iff $R \equiv \mathbb{S}^2$. **(b)** $\tilde{R} \equiv \mathbb{C}$ iff $R \equiv \mathbb{C}$, OR
R$\equiv \mathbb{C} \setminus 2pts.$, OR $R \equiv$ a torus.
(c) $\tilde{R} \equiv \mathbb{D}$ ALL OTHER CASES.

The Riemann Surface of an Elliptic Curve

Consider the equation $w^2 = (z^2 - 1)(z^2 - 4)$.

Slit the complex plane $\mathbb{C}$ by the two horizontal line segments, $A = [-2, -1], B = [1, 2]$. Denote the top and bottom edge of each of A, B by $\{+, -\}$, respectively. We are going to construct a 2-sheeted parking ramp.

Take two copies of the plane in 3D, one lying over the other; call them A and B. Each sheet with include a point at ∞. A car travelling in copy A meeting a $-$ edge will cross it and enter copy B. A car travelling in copy B, crossing a $+$ edge will enter copy A. The two copies $A \cup B$, *together with their copies of* ∞ form a surface without any boundary. A car can travel over the whole thing forever without leaving.

Each pair of values (z, w) uniquely determines a point of the Riemann surface. The map $(z, w) \to z$ is a two-to-one analytic map of the Riemann surface onto $\mathbb{C} \cup \infty$. There are branch points at the end points of the segments.