Homework 2

MATH 8660 Fall 2019

Due by 11/25/2019

Q1. (circular law)

1. Let G be an $n \times n$ matrix with i.i.d. standard complex Gaussian entries. The eigenvalues (in random exchangeable order) of G have joint density

$$\frac{1}{\pi^n \prod_{i=1}^n k!} \exp\left(-\sum_{k=1}^n |z_k|^2\right) \prod_{i<j} |z_i - z_j|^2,$$

with respect to the Lebesgue measure on $\mathbb{C}^n$. Take this fact as granted.

   (a) Show that the eigenvalues of G form a determinantal point process on $(\mathbb{C}, \pi^{-1} e^{-|z|^2} dz)$ with kernel

$$K_n(z,w) = \sum_{k=0}^{n-1} \phi_k(z)\overline{\phi_k(z)}, \qquad \text{where } \phi_k(z) := \frac{z^k}{\sqrt{k!}}.$$

   (b) Let L_n and $\bar{L}_n$ be the ESD and the expected ESD of $n^{-1/2}G$ respectively. Write down the density of $\bar{L}_n$ with respect to Lebesgue measure on $\mathbb{C}$. Show that $\bar{L}_n \xrightarrow{d} \pi^{-1}\mathbf{1}_{\{|z|\le 1\}} dz$. The limiting distribution is known as *circular law.*

   (c) Prove that for each continuous bounded function f,

$$\mathbf{E}\left(\int f dL_n(z) - \int f d\bar{L}_n(z)\right)^4 = O(n^{-2}).$$

   (Note that to compute the LHS above, you need to know m-correlation functions of the eigenvalues of G for $m \le 4$.) Deduce that almost surely,

$$L_n \xrightarrow{d} \pi^{-1}\mathbf{1}_{\{|z|\le 1\}} dz.$$

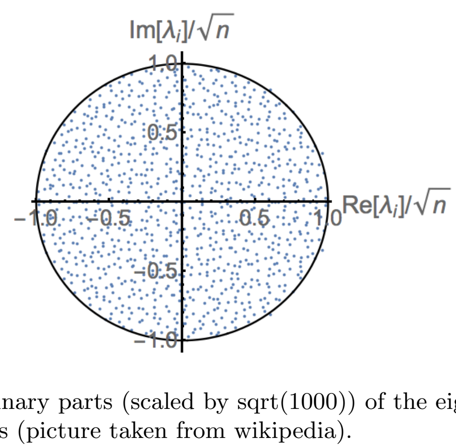

Figure 1: Plot of the real and imaginary parts (scaled by sqrt(1000)) of the eigenvalues of a 1000x1000 matrix with independent, standard normal entries (picture taken from wikipedia).

Q2. (linear statistics of CUE) Recall that the arguments of eigenvalues $\theta_1, \dots, \theta_n$ of a $n \times n$ random Haar distributed unitary matrix (known as the Circular unitary Ensemble (CUE)) forms a determinantal point process with Kernel $K_n(s,t) = \frac{1}{2\pi}\sum_{k=0}^{n-1} e^{ik(s-t)}$ with respect to the Lebesgue measure on $[-\pi, \pi]$. Take $h : [-\pi, \pi] \to \mathbb{R}$ and define $N_n(h) = \sum_{k=1}^{n} h(\theta_k)$.

(a) Show that

$$\mathbf{E}N_n(h) = \frac{n}{2\pi}\int_{-\pi}^{\pi} h(t)dt$$

and if h has the Fourier expansion $h(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt}$, then

$$\mathrm{Var}(N_n(h)) = \sum_{k:|k|\le n} |k||a_k|^2 + n \sum_{k:|k|>n} |a_k|^2.$$

(b) Now take $h = 1_{[-\alpha,\alpha]}$ where $0 < \alpha < \pi$. Show that

$$\mathrm{Var}(\chi_n([-\alpha,\alpha]) = \frac{1}{\pi^2}\log n + O(1).$$

(c) Conclude that

$$\frac{\chi_n([-\alpha,\alpha]) - \frac{n\alpha}{\pi}}{\pi^{-1}\sqrt{\log n}} \xrightarrow{d} N(0,1).$$

Q3. ☹ Using the steepest descent analysis, prove the following asymptotics of the Airy function

$$Ai(x) := \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{zx - z^3/3} dz \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}} \qquad \text{as } x \to \infty.$$

Q4. (non-intersecting random walks) Fix $n \ge 1$. Let $X^1, X^2, \dots, X^n$ be n independent (discrete-time) biased random walks on $\mathbb{Z}$, that is, $X_t^i = X_0^i + \xi_1^i + \dots + \xi_t^i$ for $1 \le i \le n, t \ge 1$ where $\xi_j^i, 1 \le i \le n, j \ge 1$ are i.i.d. random variables with $\mathbf{P}(\xi_1^1 = +1) = p = 1 - \mathbf{P}(\xi_1^1 = -1)$ with $p \in (0,1)$. Let T be a fixed positive integer. Define the transition kernel

$$P_T(x,y) = P(X_T^i = y | X_0^i = x).$$

Let the starting points of these n random walks $X^1, X^2, \dots, X^n$ are given by $x_1 < x_2 < \dots < x_n \in 2\mathbb{Z}$ respectively. Show that for any $y_1 < \dots < y_n \in 2\mathbb{Z}$

$$\mathbf{P}\Big(X_T^1 = y_1, \dots, X_T^n = y_n \text{ and } X_t^1 < \dots < X_t^n \quad \forall 0 \le t \le T\Big) = \det\Big(((p_T(x_i, y_j)))_{1\le i,j\le n}\Big).$$