1. Let X, Y be random variables. We proved in class that if we want to predict Y by $h(X)$, then $h(X) = \mathbb{E}(Y|X)$ minimizes the mean square error $\mathbb{E}[(Y - h(X))^2]$ over all functions h. Note that we can always write $Y = \mathbb{E}(Y|X) + Z$, where $Z = Y - \mathbb{E}(Y|X)$ is called the residual.

   Prove that $\mathrm{Cov}(Z, g(X)) = 0$ for any function g. In words, X does not contain any more information about the residual.

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p$. Let $S_n = X_1 + X_2 + \ldots + X_n$. For each integer k, find $\mathbb{P}(S_n = k)$ in terms of n, k and p.

3. A bag contains 6 red marbles, 3 blue marbles, and 4 green marbles. You pull out a marble, record the color, and replace it in the bag 6 times. You may leave your answers unsimplified.

   (i) What is the probability of getting 4 red, 1 blue and 1 green?

   (ii) What is the probability of getting exactly 4 red marbles?

   (iii) What are the probabilities in (i) and (ii) if you reach in the bag and pull out 6 marbles at once?

4. Let $X \sim \mathrm{Poi}(\lambda)$. Find $\mathbb{E}\left[\frac{1}{1+X}\right]$.