1. Recall the following version of Cauchy-Schwarz inequality that we proved in class: For any two random variables X and Y, we have

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}.$$

Use the above inequality to prove the following two versions of Cauchy-Schwarz inequality.

(i) For any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, show that

$$\Big|\sum_{i=1}^{n} a_i b_i\Big| \leq \left(\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

[Hint: Take X, Y to be discrete random variables with joint p.m.f. $f_{X,Y}(a_i, b_i) = \frac{1}{n}$ for all i.]

(iI) For any $a < b$ and any two bounded functions $f, g : [a, b] \to \mathbb{R}$, show that

$$\Big|\int_a^b f(x)g(x)dx\Big| \leq \left(\int_a^b f(x)^2 dx \cdot \int_a^b g(x)^2 dx\right)^{1/2}.$$

2. Let X and Y be two i.i.d. exp(1) random variables. Recall that p.d.f. of an exp(1) random variable is given by

$$f(x) = e^{-x} \text{ if } x > 0 \quad \text{and } = 0 \text{ if } x \leq 0.$$

Let $L = X - Y$. Our goal is to show that the p.d.f. of L is

$$g(x) = e^{-\frac{1}{2}|x|}, x \in \mathbb{R}.$$

Use the following two ways to prove the above claim.

(i) First compute the CDF of L and then differentiate the CDF to obtain the p.d.f. of L.

(ii) Compute the moment generating function of $X - Y$ and show that it matches with moment generating function of any random variable whose p.d.f. is g.

[Correction (4/13): The above expression of $g(x)$ is incorrect. The correct one should read as $g(x) = \frac{1}{2}e^{-|x|}$.]

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with mean μ and variance σ^2 and the moment generating function ψ, that is, $\psi(t) = \mathbb{E}[e^{tX_i}]$ for $t \in \mathbb{R}$. Let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i\ , \quad \text{and} \quad Z_n = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right).$$

(i) Show that Z_n has mean 0 and variance 1.

(ii) Compute the moment generating function of Z_n in terms of ψ.