1. (20 points) Use the method of Lagrange multipliers to find the extreme values of the function $f(x, y) = xy$ on the ellipse $\frac{x^2}{4} + y^2 = 1$.

$f(x,y) \underset{\text{def}}{=} xy$, $g(x,y) \underset{\text{def}}{=} \frac{x^2}{4} + y^2 - 1 = 0$

Lagrange multipliers :- $\nabla f = \lambda \nabla g$, $g = 0$.

$f_x = y$, $f_y = x$, $g_x = \frac{x}{2}$, $g_y = 2y$.

$y = \lambda . \frac{x}{2}$ — ① , $x = \lambda . 2y$ — ② $\frac{x^2}{4} + y^2 = 1$, — ③

From ②, $x = 2\lambda y$, plug in ①, $y = \lambda^2 y$

$\Rightarrow y = 0$ or $\lambda^2 = 1$ or $\lambda = \pm 1$

Case 1. $y = 0$. then from ①, $x = 0$, but

then $\frac{x^2}{4} + y^2 = 0 \neq 1$

contradicting ③.

Case 2. $\lambda = \pm 1$, From ②, $x = \pm 2y$ — ④

From ③ & ④ $\frac{(\pm 2y)^2}{4} + y^2 = 1$ or, $2y^2 = 1$

or, $y^2 = 1/2$ or $y = \pm 1/\sqrt{2}$

From ④

$x = \left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$, $\left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$, $\left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right)$, $\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$.

$f\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right) = f\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}\right) = 1$ abs. max

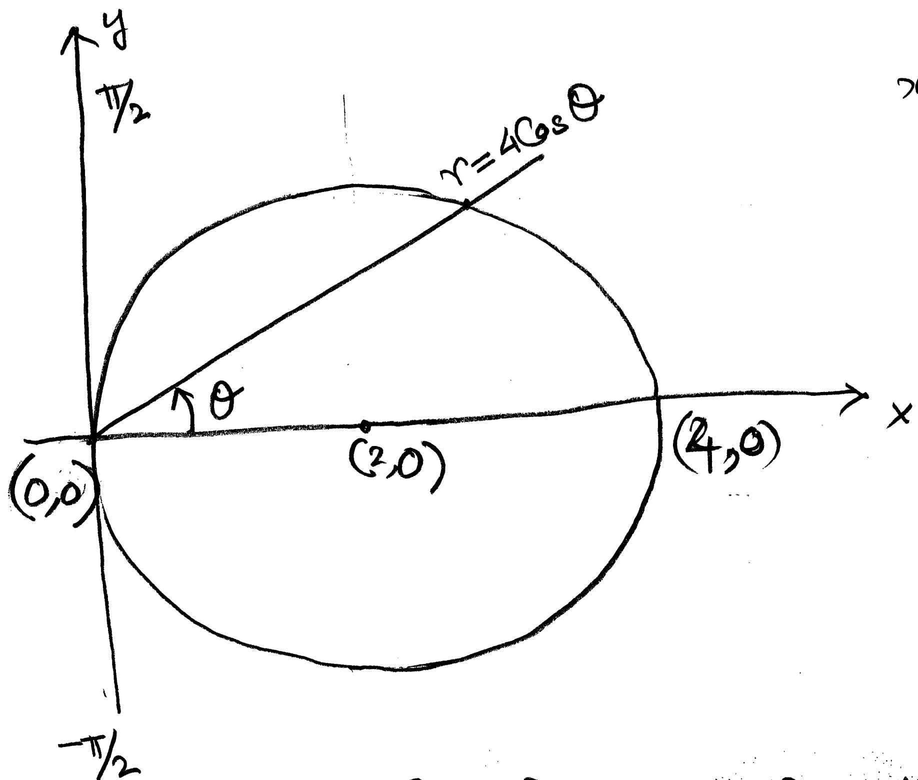
$f\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right) = f\left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right) = -1$ abs. min.

2. (15 points) Transform the following integral into **polar coordinates** with appropriate limits for r and θ where D is a disk enclosed by the circle $x^2 + y^2 = 4x$:

$$\iint_D f(x,y)dA.$$

[Note that you cannot evaluate the integral since the function f is unknown.]

$x^2+y^2=4x$ or, $(x-2)^2+y^2=4$

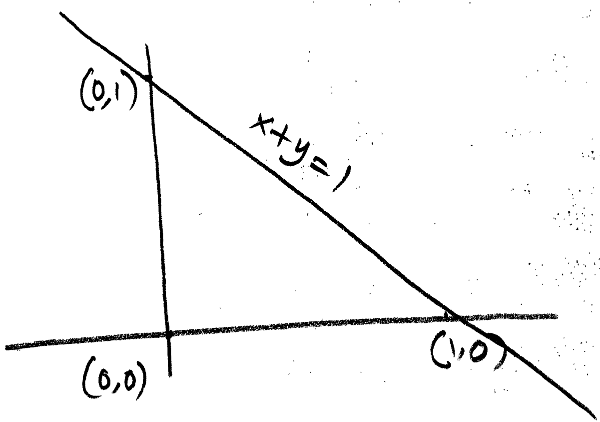


$$x^2+y^2=4x$$
$$r^2=4r\cos\theta$$
$$r=4\cos\theta$$

$D=\{(r,\theta) : -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 4\cos\theta\}$

$$\iint_D f(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

3. (20 points) Find the x-**component** of the center of mass of a triangular lamina D with vertices at $(0,0)$, $(1,0)$ and $(0,1)$ if the density of mass function is $\rho(x,y) = y$.



Mass - $M = \iint_D \rho(x,y)\,.dA$

$= \int_0^1 \int_0^{1-x} y\,.dy\,.dx$

$= \int_0^1 \frac{y^2}{2}\Big|_0^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2 dx$

$= \frac{1}{2}\int_0^1 z^2.dz = \frac{1}{2} \times \frac{z^3}{3}\Big|_0^1 = \frac{1}{6}.$

(but $z = 1-x$)

x-moment:- $M_x = \iint_D x\rho(x,y)\,dA = \int_0^1\int_0^{1-x} xy\,dy.dx$

$= \int_0^1 x\left(\int_0^{1-x} y.dy\right)dx = \frac{1}{2}\int_0^1 x(1-x)^2.dx$

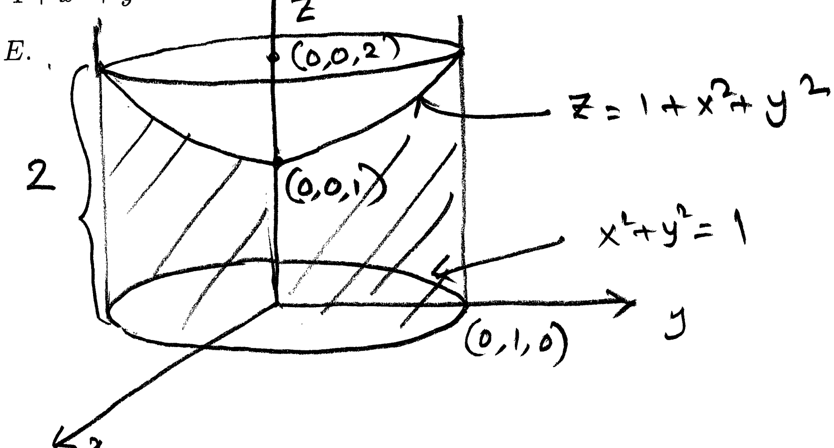
$= \frac{1}{2}\int_0^1 (x-2x^2+x^3).dx = \frac{1}{2}\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)\Big|_0^1$

$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}.$

$\bar{x} = \frac{M_x}{M} = \frac{\frac{1}{24}}{\frac{1}{6}} = \frac{6}{24} = \boxed{\frac{1}{4}}.$

4. (20 points) Consider the solid region E which lies within the cylinder $x^2+y^2=1$, above the xy-plane and below the paraboloid $z=1+x^2+y^2$.

   (a) (5 points) Sketch the solid region E.



   (b) (15 points) Use cylindrical coordinates to compute the volume of E.

$$x^2+y^2 \le 1 \implies r^2 \le 1 \implies r \le 1$$

$$\underbrace{0 \le z}_{\text{above x-y plane}} \le 1+x^2+y^2 \implies 0 \le z \le 1+r^2$$

$$E = \left\{ (r,\theta,z) \;\middle|\; 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1+r^2 \right\}$$

$$\text{Vol}(E) = \int_0^{2\pi}\int_0^1\int_0^{1+r^2} 1.\,dz\, r.dr\, d\theta = 2\pi \int_0^1 (1+r^2)\, r.\,dr$$

$$= 2\pi \int_0^1 (r+r^3)\,dr = 2\pi\left(\frac{r^2}{2}+\frac{r^4}{4}\right)\Big|_0^1$$

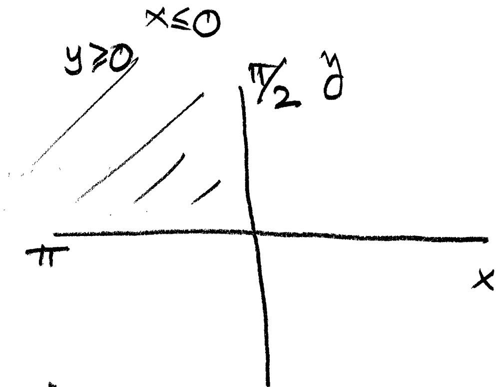
$$= 2\pi\left(\frac{1}{2}+\frac{1}{4}\right) = 2\pi \times \frac{3}{4} = \frac{3\pi}{2}.$$

5. (10 points) Let E be the portion of the ball $x^2 + y^2 + z^2 \leq 4$ that lies in the octant $x \leq 0, y \geq 0, z \geq 0$. Express the solid region E in terms of spherical coordinates.

$x = \rho \sin\phi \cos\theta$, $\quad y = \rho \sin\phi \sin\theta$, $\quad z = \rho \cos\theta$

$x \leq 0$, $y \geq 0 \Rightarrow \theta \in [\pi/2, \pi]$

$z \geq 0 \Rightarrow \phi \in [0, \pi/2]$



$x^2 + y^2 + z^2 \leq 4 \Rightarrow \rho^2 \leq 4 \Rightarrow$

$\rho \in [0, 2]$

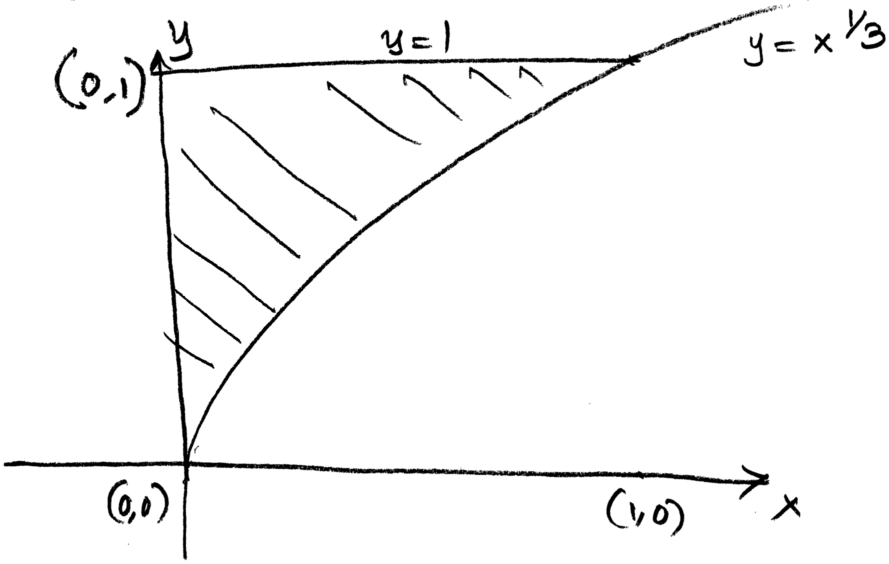
So, $E = \{ (\rho, \theta, \phi) : 0 \leq \rho \leq 2,$

$\pi/2 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi/2 \}$

6. (15 points) Sketch the region of integration and evaluate the integral

$$\int_0^1 \int_{x^{1/3}}^1 \frac{1}{y^4+1} dydx.$$

[Hint: Switch the order of integration.]



$$\text{Integral} = \int_0^1 \int_0^{y^3} \frac{1}{1+y^4} \, dx \, dy = \int_0^1 \frac{1}{1+y^4} \int_0^{y^3} dx \cdot dy$$

$$= \int_0^1 \frac{y^3}{1+y^4} \cdot dy$$

$u = 1 + y^4$

$du = 4 \cdot y^3 \cdot dy$

$$= \frac{1}{4} \int_1^2 \frac{1}{u} \cdot du = \frac{1}{4} \ln|u| \Big|_1^2$$

$$= \frac{1}{4} (\ln 2 - \ln 1)$$

$$= \frac{1}{4} \ln 2$$