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ON THE ASYMPTOTIC CONVERGENCE OF SPLINE COLLOCATION METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. We examine the asymptotic accuracy of the method of collocation for the approximate solution of linear elliptic partial differential equations. Specifically we consider the nodal collocation of a second order equation in the plane with biperiodicity conditions using tensor product smooth splines of odd degree as trial functions. We prove optimal rates of convergence in L^2 for partial derivatives of the approximate solution which are of order at least two in one variable, while the solution itself and its gradient converge in L^2 at rates less than the optimal approximation theoretic results.

1. Introduction. In the method of collocation an approximate solution to a differential equation (or other functional equation) is determined in a given space of trial functions of finite dimension n by the condition that the equation be satisfied at n specified collocation points. This method, appealing for its conceptual simplicity, wide applicability, and ease of implementation, dates back five decades to the work of Kantorovitch [25] and Frazer, Jones and Skan [20]. In [25] Kantorovitch sought the solution of a *partial* differential equation, but collocated only in a single variable for each fixed value of the second variable. (Thus, he used collocation in conjunction with a method of lines procedure.) For several decades, authors taking up his work concentrated on one-dimensional collocation, that is collocation of ordinary differential equations [27], [26, ch. XIV.5.1], [43], [44]. Frazer, Jones and Skan were also primarily concerned with ordinary differential equations, but already in 1938 [19, p. 227] Frazer, Duncan and Collar specifically remark on the direct applicability of the collocation method to partial differential equations. In his monograph of 1951 on the numerical solution of differential equations, Collatz includes discussion and examples of collocation for both ordinary and partial differential equations [16, ch. III.4.1, ch. V.4.2].

The choice of the trial space and collocation points greatly influences the effectiveness of a collocation method. The first three decades of study focused almost exclusively on trial spaces formed from polynomials satisfying the boundary conditions [25], [20], [30], [19, ch. 7.9], [16, ch. III.4.1, ch. V.4.2], [27], [31, ch. VII.17], [26, ch. XIV.5.1], [45], [43], [44]. It was realized quite early that with such trial functions equidistant collocation points are not appropriate,[1] but that rather the Chebyshev points, Gauss points, or similarly constructed points should be used [30], [27], [31, ch. VII.7], [26, ch. XIV.5.1], [45], [43], [44]. This form of collocation, collocation by polynomials at the roots of orthogonal polynomials, is often termed orthogonal collocation. An error analysis was first given by Karpilovskaya in 1953 [27]; see also [26, ch. XIV.5.1], [43]. Ten years later, Karpilovskaya [28] analyzed a collocation method in two dimensions, considering a perturbation of the Laplacian on a square and collocating with trigonometric polynomials of two variables at points on a uniform grid. A generalization to an equation with biharmonic principal part is studied in [29].

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[1] This is not surprising in view of the famous example of Runge showing the failure of interpolation—which is collocation of the identity operator—by polynomials at equally spaced points [12, ch. II].

In the wake of the great surge of interest in spline functions in the nineteen sixties, their use as collocation trial functions was proposed. In light of the superior qualities of spline interpolation and the suitability of splines for computer implementation, such spline collocation is very appealing and a number of researchers began, naturally, with the case of nodal point collocation by cubic splines for a second order two point boundary value problem [11], [1, ch. 2.7], [32], [10], [2], [21]. This method converges as the mesh size tends to zero, but it was soon realized that the maximum error decreases as the square of the mesh size rather than as the fourth power which is achieved by other projection methods using cubic spline trial functions [11]. Nodal collocation by smoothest splines of higher order was also found to be convergent with order two less than optimal [39], [33].

In 1973 de Boor and Swartz published an influential paper [13] analyzing collocation for ordinary differential equations using splines of positive defect (i.e., less than smoothest splines) and collocating at Gauss points in each mesh interval. We follow Prenter [36, ch. 8] in terming these orthogonal spline collocation methods. In the simplest case, a Hermite cubic approximation to the solution of a second order problem is determined by collocation at two Gauss points within each subinterval. Unlike nodal cubic spline collocation, this method converges with fourth order in the maximum norm, although at the expense of larger bandwidth and larger matrices for the same mesh. The past decade witnessed much research on spline collocation for ordinary differential equations, both nodal and orthogonal for quite general problems, splines of arbitrary order, etc. [22], [17], [3], [5], [40], [18], [23], [34], [38].

The theory of cubic spline collocation for partial differential equations is far less developed. In 1972, Cavendish [15] and Ito [24] separately proved second order convergence of the method of bicubic spline collocation (at the nodal points) for a second order elliptic equation on a square. This result, whose proof is reproduced in [36, ch. 8], is a generalization of the convergence results for cubic spline nodal collocation in one dimension. Four years later, Prenter and Russell [37] analyzed the two-dimensional analogue of Hermite cubic orthogonal collocation; that is, they considered the approximation of the solution of a second order elliptic equation on a square using bicubic Hermite trial functions and collocating at the 2×2 product Gauss points in each mesh rectangle. They proved optimal fourth order convergence for this method. In 1980 Percell and Wheeler [35] proved analogous results for orthogonal collocation using tensor product C^1 Hermite trial functions of degree r, $r \geqq 3$.

In the present paper we analyze the method of nodal collocation by tensor product smoothest splines of arbitrary odd order for second order elliptic differential equations. We prove convergence with sharp rates in a variety of norms. In order to describe the results, we first recall the results of Arnold and Wendland [5] concerning the convergence of nodal collocation by odd order splines in one dimension. In this paper, in which is considered a very general class of equations (integral, differential, integrodifferential, etc.), the failure of nodal spline collocation to approximate the values of the solution of a differential equation with optimal order is exposed in a different light. By introducing a new technique of proof that enables the application of the well-developed theory of Galerkin methods, it is shown that the method does converge with optimal order in various Sobolev spaces, but only those with order greater than or equal to the order of the operator. For second order problems, for example, the derivatives of order at least two and no greater than the degree of the splines, are approximated at the optimal order, but the first derivatives and values are not approximated with higher order than the second derivatives, thus accounting for the loss of two orders of convergence for the values of solution.

In the two-dimensional case considered here we prove analogous results. It turns out, however, that the Sobolev spaces are not appropriate. After stating the problems and method precisely in the next section we define the special function spaces we require for the analysis in the § 3. In the final two sections the method is analyzed. For the analysis, we first show that the collocation solution is equal to the approximation produced by a Galerkin method using the splines as trial and test functions paired, however, by a rather unusual bilinear form. In § 4, this method is shown to be stable (that is, the map taking the exact solution to the approximate solution is shown to be bounded independent of the mesh) in a certain specially constructed space. In § 5, we proceed from this point to establish L^2 rates of convergence with respect to the mesh size for each partial derivative of the error. As in the one-dimensional case, optimal order convergence occurs only for derivatives of order at least two.

2. The differential equation and the collocation method. We consider the numerical solution of the second order, linear, partial differential equation

$$Lu(x, y) := \sum_{k+l\leqq 2} a_{kl}(x, y)\, \partial_x^k\, \partial_y^l u(x, y) = f(x, y), \qquad (x, y) \in \mathbb{R}^2. \tag{2.1}$$

The coefficients a_{kl} and the forcing function f are real-valued functions one periodic in each argument. We assume that $f \in C^0(\mathbb{R}^2)$ and $a_{kl} \in C^d(\mathbb{R}^2)$ where d is the degree of the collocating spline functions. Moreover, we assume that the coefficients of the principal part of L have the form

$$a_{kl}(x, y) = -\tilde{a}_{kl} a(x, y), \qquad k + l = 2, \tag{2.2}$$

where $a(x, y)$ is a strictly positive biperiodic function and the constants $\tilde{a}_{kl}$ satisfy the strong ellipticity condition

$$\sum_{k+l=2} \tilde{a}_{kl} x^k y^l \geqq \gamma(x^2 + y^2), \qquad (x, y) \in \mathbb{R}^2, \tag{2.3}$$

for some $\gamma > 0$. The eigenvalues for L (for the biperiodic eigenvalue problem) then form a sequence tending to infinity; we assume that 0 is not among them. That is, we assume that (2.1) has a unique biperiodic solution u which we seek to approximate.

Remark 2.1. The restriction (2.2) on the coefficients of the principal part of L will allow us to reduce to the case of principal part with constant coefficients. The assumption is satisfied, for example, if L is of the form

$$Lu = -\nabla \cdot a \nabla u + b\, \partial_x u + c\, \partial_y u + du$$

where a, b, c and d are sufficiently smooth biperiodic functions with a strictly positive.

Remark 2.2. The choice of the periodic problem rather than, for example, the Dirichlet or Neumann boundary value problem results in three simplifications. First, we may apply elliptic regularity theory to conclude any desired degree of smoothness for u supposing that the coefficients and forcing function are sufficiently smooth. This does not hold for boundary value problems on the square. Second, we utilize the periodicity in defining various Sobolev and related norms in terms of Fourier coefficients. This is convenient but not indispensable. (We do *not*, however, use Fourier techniques to analyze the numerical solution, and will not require a uniform mesh.) Third and most important, we avoid difficulties with collocation conditions at the boundary. Even in one dimension for splines of degree greater than three for a second order problem the selection of boundary conditions for the collocation solution is not straightforward [22]. For partial differential equations the problem is greater. Even in the simplest case, bicubic spline collocation of Poisson's equation on the square with homogeneous Dirichlet data, special collocation conditions must be imposed at the corners [36, ch. 8]. We do not address these matters here.

For the numerical approximation of (2.1), we select mesh points

$$0=x_0<x_1<\cdots<x_M=1, \qquad 0=y_0<y_1<\cdots<y_N=1,$$

and extend these to periodic meshes $\Delta_x=\{x_n\}_{n\in\mathbf{Z}}$, $\Delta_y=\{y_n\}_{n\in\mathbf{Z}}$, with $x_{n+M}-x_n=y_{n+N}-y_n=1$ for all n. For $d\geqq 3$ an odd integer we define $S_d(\Delta_x)$ to be the space of smooth periodic splines of degree d subordinate Δ_x, i.e., the space of 1-periodic functions in $C^{d-1}(\mathbb{R})$ which restrict to polynomials of degree at most d on each of the subintervals (x_{n-1}, x_n). We recall that $\dim S_d(\Delta_x)=M$, $\dim S_d(\Delta_y)=N$, and each space has a basis consisting of periodic B-splines which are supported in $d+1$ consecutive subintervals and their integral translates. Let $\Delta=\Delta_x\times\Delta_y$ denote the product mesh and set $h_\Delta=\max_m \max(x_m-x_{m-1}, y_m-y_{m-1})$. The collocation method we investigate is based on the trial space $\mathcal{M}_d(\Delta)=S_d(\Delta_x)\otimes S_d(\Delta_y)$, spanned by the products $\varphi(x)\psi(y)$, $\varphi\in S_d(\Delta_x)$, $\psi\in S_d(\Delta_y)$. This space has dimension MN and consists of all 1-biperiodic piecewise polynomials of degree at most d in each variable separately subordinate to the product mesh and having continuous derivatives of all orders up to $d-1$ in each variable separately. Implementation and other matters related to periodic and tensor product splines are discussed in [12, ch. XVI], [41, ch. 8] and [12, ch. XVII], [41, ch. 9] respectively.

The collocation method may now be defined. It seeks an approximate solution $u_\Delta\in\mathcal{M}_d(\Delta)$ satisfying the collocation equations

$$Lu_\Delta(x_m, y_n)=f(x_m, y_n), \qquad 1\leqq m\leqq M, \quad 1\leqq n\leqq N. \tag{2.4}$$

3. Preliminaries. Let $\mathcal{D}'_\omega(\mathbb{R})$ denote the space of all 1-periodic real-valued distributions on the real line. Recall [42, ch. VII.1] that to each $v\in\mathcal{D}'_\omega(\mathbb{R})$ is associated the complex sequence $\{\hat{v}(m)\}_{m\in\mathbf{Z}}$ of its Fourier coefficients given by

$$\hat{v}(m)=\int_0^1 v(x, y)\, e^{-2\pi imx}\, dx$$

in case v is a locally integrable function. Using the notation $\underline{m}=\max(2\pi|m|, 1)$ we define for $p\in\mathbf{Z}$ the Hilbert space

$$H^p_\omega(\mathbb{R})=\left\{v\in\mathcal{D}'_\omega(\mathbb{R})\,\middle|\, \|v\|_{H^p_\omega(\mathbb{R})}:=\left[\sum_{m\in\mathbf{Z}}|\hat{v}(m)|^2\underline{m}^{2p}\right]^{1/2}<\infty\right\}.$$

These are the periodic Sobolev spaces. For nonnegative p an equivalent norm on $H^p_\omega(\mathbb{R})$ is

$$v\mapsto\left[\sum_{k=0}^{p}\sum_{m\in\mathbf{Z}}|\hat{v}(m)|^2|2\pi m|^{2k}\right]^{1/2}=\left[\sum_{k=0}^{p}\left\|\frac{d^kv}{dx^k}\right\|^2_{L^2(I)}\right]^{1/2}$$

where I denotes the unit interval.

Similarly for $v\in\mathcal{D}'_\omega(\mathbb{R}^2)$, the space of 1-biperiodic distributions, there is associated a double Fourier series, so that

$$\hat{v}(m, n)=\int_0^1\int_0^1 v(x, y)\, e^{-2\pi i(mx+ny)}\, dx\, dy$$

for locally integrable v. We shall define various Hilbert spaces of the form

$$\left\{v\in\mathcal{D}'_\omega(\mathbb{R}^2)\,\middle|\, \sum_{m,n\in\mathbf{Z}}|\hat{v}(m, n)|^2[\varphi(m, n)]^2<\infty\right\} \tag{3.1}$$

with inner product

$$(v, w) \mapsto \sum_{m,n \in \mathbb{Z}} \hat{v}(m,n) \overline{\hat{w}(m,n)} [\varphi(m,n)]^2.$$

Particular choices of the positive multiplier function $\varphi: \mathbb{Z}^2 \to \mathbb{R}$ determine particular spaces. For $p, q \in \mathbb{Z}$ we define the following spaces by (3.1):

$$H^p \text{ with } \varphi(m,n) = \max(\underline{m}^p, \underline{n}^p),$$

$$H^{p,q} \text{ with } \varphi(m,n) = \underline{m}^p \underline{n}^q,$$

$$H_\cap^{p,q} \text{ with } \varphi(m,n) = \max(\underline{m}^p \underline{n}^q, \underline{m}^q \underline{n}^p),$$

$$H_+^{p,q} \text{ with } \varphi(m,n) = \min(\underline{m}^p \underline{n}^q, \underline{m}^q \underline{n}^p).$$

The spaces $H^p = H_\cap^{p,0}$ are the biperiodic Sobolev spaces. For nonnegative p an equivalent norm is given by

$$v \mapsto \left[\sum_{k=0}^{p} \sum_{l=0}^{p-k} \| \partial_x^k \partial_y^l v \|_{L^2}^2 \right]^{1/2},$$

where $L^2 = L^2(I \times I) = H^0$.

The space $H^{p,q}$ is the Hilbert space tensor product $H^p(I) \otimes H^q(I)$, cf. [6, ch. 12]. For p and q nonnegative the norm in $H^{p,q}$ is equivalent to the norm

$$v \mapsto \left[\sum_{k=0}^{p} \sum_{l=0}^{q} \| \partial_x^k \partial_y^l v \|_{L^2}^2 \right]^{1/2}.$$

Further, $H_\cap^{p,q} = H^{p,q} \cap H^{q,p}$ and $H_+^{p,q} = H^{p,q} + H^{q,p}$ and the given norms are equivalent to the intersection and sum norms [14, ch. 3.2.1].

The H^0 inner product extends compatibly to pairings on $H^p \times H^{-p}$, $H^{p,q} \times H^{-p,-q}$, $H_\cap^{p,q} \times H_+^{-p,-q}$, and $H_+^{p,q} \times H_\cap^{-p,-q}$, for all p, q. These pairing establish isometric isomorphisms of H^p, $H^{p,q}$, $H_\cap^{p,q}$, and $H_+^{p,q}$ onto the dual spaces of H^{-p}, $H^{-p,-q}$, $H_+^{-p,-q}$ and $H_\cap^{-p,-q}$ respectively. We denote the extended inner product by $\langle \cdot, \cdot \rangle$.

Throughout this paper the letters k, l, m, n, p, and q refer to integers. The symbols C and h_0 will be used as generic positive constants. The letter d is reserved for the degree of the splines (an odd integer at least 3), and j for $(d+1)/2 \geqq 2$.

We conclude this section with some basic results on the best approximation by tensor product splines.

THEOREM 3.1. *Let $0 \leqq k$, $l \leqq d$ and let $P_\Delta: H^{k,l} \to \mathcal{M}_d(\Delta)$ denote the orthogonal projection. Then there exists a constant C such that*

$$\| v - P_\Delta v \|_{H^{k,l}} \leqq C h_\Delta^p [\| v \|_{H^{k+p,l}} + \| v \|_{H^{k,l+p}}]$$

whenever $0 \leqq p \leqq d+1-\max(k,l)$ and $v \in H^{k-p,l} \cap H^{k,p+l}$.

Proof. Let $Q_{\Delta_x}: H_\omega^k(\mathbb{R}) \to S_d(\Delta_x)$ and $R_{\Delta_y}: H_\omega^l(\mathbb{R}) \to S_d(\Delta_y)$ denote the orthogonal projections. Then, from well-known results for best spline approximation in one dimension we have

$$\| w - Q_{\Delta_x} w \|_{H_\omega^k(\mathbb{R})} \leqq C h_\Delta^p \| w \|_{H_\omega^{k+p}(\mathbb{R})}, \qquad w \in H_\omega^{k+p}(\mathbb{R}),$$

$$\| w - R_{\Delta_y} w \|_{H_\omega^l(\mathbb{R})} \leqq C h_\Delta^p \| w \|_{H_\omega^{l+p}(\mathbb{R})}, \qquad w \in H_\omega^{l+p}(\mathbb{R}).$$

Therefore

$$\begin{aligned} \| v - P_\Delta v \|_{H^{k,l}} &\leqq \| v - (Q_{\Delta_x} \otimes R_{\Delta_y}) v \|_{H^{k,l}} \\ &\leqq \| [(I - Q_{\Delta_x}) \otimes I] v \|_{H^{k,l}} + \| [Q_{\Delta_x} \otimes (I - R_{\Delta_y})] v \|_{H^{k,l}} \\ &\leqq C h_\Delta^p [\| w \|_{H^{k+p,l}} + \| w \|_{H^{k,l+p}}]. \end{aligned}$$

(I denotes the identity operator. For the tensor product of operators see, e.g., [6, ch. 12].) □

THEOREM 3.2. *Let* $0 \leqq k \leqq d-1$ *and let* $P_\Delta : H_\cap^{k,k+1} \to \mathcal{M}_d(\Delta)$ *denote the orthogonal projection. Then there exists a constant C such that*

$$\|v - P_\Delta v\|_{H_\cap^{k,k+1}} \leqq Ch_\Delta^p \|v\|_{H_\cap^{k,k+p+1}}$$

for all $p \in [0, d-k]$, $v \in H_\cap^{k,k+p+1}$.

Proof. The case $p = 0$ is obvious, so we assume $p \geqq 1$. Let $Q_{\Delta_x} : H^{k+1}(I) \to S_d(\Delta_x)$ denote the orthogonal projection. Then

$$\|w - Q_{\Delta_x} w\|_{H_\omega^k(\mathbb{R})} \leqq Ch_\Delta \|w - Q_{\Delta_x} w\|_{H_\omega^{k+1}(\mathbb{R})} \leqq Ch_\Delta^{q-k} \|w\|_{H_\omega^q(\mathbb{R})}$$

for all $q \in [k+1, d+1]$, $w \in H_\omega^q(\mathbb{R})$. Now

$$\begin{aligned} \|v - (Q_{\Delta_x} \otimes Q_{\Delta_y})v\|_{H^{k,k+1}} &\leqq \|[(I - Q_{\Delta_x}) \otimes Q_{\Delta_y}]v\|_{H^{k,k+1}} + \|[I \otimes (I - Q_{\Delta_y})]v\|_{H^{k,k+1}} \\ &\leqq Ch_\Delta^p[\|v\|_{H^{k+p,k+1}} + \|v\|_{H^{k,k+p+1}}]. \end{aligned}$$

But it is easy to check (using the Fourier series definition of the norms), that

$$\|v\|_{H^{k+p,k+1}} \leqq \|v\|_{H_\cap^{k,k+p+1}},$$

so

$$\|v - (Q_{\Delta_x} \otimes Q_{\Delta_y})v\|_{H^{k,k+1}} \leqq Ch_\Delta^p \|v\|_{H_\cap^{k,k+p+1}}.$$

By symmetry we in fact have

$$\|v - (Q_{\Delta_x} \otimes Q_{\Delta_y})v\|_{H_\cap^{k,k+1}} \leqq Ch_\Delta^p \|v\|_{H_\cap^{k,k+p+1}},$$

and the theorem follows. □

4. Stability analysis. The starting point for our analysis is the realization of the collocation solution u_Δ by a Galerkin procedure. For $p \in \mathbb{Z}$ the integral $v \mapsto \int_0^1 v(x)\,dx$ defines a continuous linear functional on $H_\omega^p(\mathbb{R})$ which we denote either by J_x or J_y. If q is positive the trapezoidal approximations,

$$J_{\Delta_x} v = \sum_{m=1}^{M} \frac{x_{m+1} - x_{m-1}}{2} v(x_m), \qquad J_{\Delta_y} v = \sum_{m=1}^{N} \frac{y_{m+1} - y_{m-1}}{2} v(y_m),$$

are also bounded functionals on $H_\omega^q(\mathbb{R})$. The tensor products $J = J_x \otimes J_y : H^{p,p} \to \mathbb{R}$ and $J_\Delta = J_{\Delta_x} \otimes J_{\Delta_y} : H^{q,q} \to \mathbb{R}$ are thus bounded for such p and q, and are simply the integral over $I \times I$ and the product trapezoidal rule approximation respectively. A key role in the following analysis will be played by the operator $B_\Delta^d : H^{1,1} \to H^{-d,-d}$ defined by

$$B_\Delta^d = J_\Delta + \partial_x^{d+1} \otimes J_{\Delta_y} + J_{\Delta_x} \otimes \partial_y^{d+1} + \partial_x^{d+1}\,\partial_y^{d+1}. \tag{4.1}$$

Thus if v is a smooth biperiodic function,

$$\begin{aligned} B_\Delta^d v(x, y) = &\sum_{m=1}^{M} \sum_{n=1}^{N} \frac{(x_{m+1} - x_{m-1})(y_{n+1} - y_{n-1})}{4} v(x_m, y_n) \\ &+ \sum_{n=1}^{N} \frac{y_{n+1} - y_{n-1}}{2} \partial_x^{d+1} v(x, y_n) + \sum_{m=1}^{M} \frac{x_{m+1} - x_{m-1}}{2} \partial_y^{d+1} v(x_m, y) \\ &+ \partial_x^{d+1}\,\partial_y^{d+1} v(x, y). \end{aligned}$$

The following lemma is analogous to [5, Thm. 2.1.1].

LEMMA 4.1. *A function* $w \in H^{1,1}$ *vanishes at all the nodal points* $(x_m, y_n) \in \Delta$ *if and only if*

$$\langle B_\Delta^d w, v\rangle = 0 \quad \text{for all } v \in \mathcal{M}_d(\Delta). \tag{4.2}$$

Proof. Note that

$$\begin{aligned}\langle B_\Delta^d w, v\rangle &= J_\Delta w \cdot Jv - \int_0^1 (\partial_x \otimes J_{\Delta_y}) w (\partial_x^d \otimes J_y) v \, dx \\ &\quad - \int_0^1 (J_{\Delta_x} \otimes \partial_x) w (J_x \otimes \partial_y^d) v \, dy + \int_0^1 \int_0^1 \partial_x \, \partial_y w \, \partial_x^d \, \partial_y^d v \, dx \, dy.\end{aligned} \tag{4.3}$$

Now for $v \in \mathcal{M}_d(\Delta)$, $(\partial_x^d \otimes J_y)v$ and $(J_x \otimes \partial_y^d)v$ are piecewise constant functions of one variable subordinate to the meshes Δ_x and Δ_y respectively, and $\partial_x^d \, \partial_y^d v$ is a piecewise constant function of two variables subordinate to the product mesh. Therefore the integrations indicated in (4.3) may be performed separately on the individual subintervals and rectangles, and we see that $\langle B_\Delta^d w, v\rangle$ reduces to a linear combination of nodal values of w. Hence if w vanishes on Δ, then (4.2) holds.

Conversely, suppose that (4.2) holds. We first note that each of the four terms on the right-hand side of (4.3) must vanish for $v \in \mathcal{M}_d(\Delta)$. Indeed, taking $v \equiv 1$ we see that $J_\Delta w$ vanishes. Then taking $v(x, y) = \varphi(x)$, $\varphi \in S_d(\Delta_x)$ we see that $(\partial_x \otimes J_{\Delta_y})w$ is orthogonal to $\partial_x^d \varphi$ for all $\varphi \in S_d(\Delta_x)$, and hence to $(\partial_x^d \otimes J_y)v$ for all $v \in \mathcal{M}_d(\Delta)$. Hence the second term of (4.3) vanishes. Similarly the third term vanishes, and consequently also the last.

Next we choose, as in [5], $\varphi_m \in S_d(\Delta_x)$ satisfying

$$\partial_x^d \varphi_m(x) = \begin{cases} -(x_m - x_{m-1})^{-1}, & x_{m-1} < x < x_m, \\ (x_{m+1} - x_m)^{-1}, & x_m < x < x_{m+1}, \\ 0, & x_{m+1} < x < x_{m-1} + 1, \end{cases}$$

and $\psi_n \in S_d(\Delta_y)$ satisfying

$$\partial_y^d \psi_n(y) = \begin{cases} -(y_n - y_{n-1})^{-1}, & y_{n-1} < y < y_n, \\ (y_{n+1} - y_n)^{-1}, & y_n < y < y_{n+1}, \\ 0, & y_{n+1} < y < y_{n-1} + 1. \end{cases}$$

For each fixed m we have

$$\int_0^1 \partial_y \left[\int_0^1 \partial_x w(x, y) \, \partial_x^d \varphi_m(x) \, dx \right] \partial_y^d \psi_n(y) \, dy = 0, \qquad n = 1, \cdots, N-1. \tag{4.4}$$

Setting

$$\begin{aligned}\Phi_m(n) &= \int_0^1 \partial_x w(x, y_n) \, \partial_x^d \varphi_m(x) \, dx \\ &= \frac{w(x_{m+1}, y_n) - w(x_m, y_n)}{x_{m+1} - x_m} - \frac{w(x_m, y_n) - w(x_{m-1}, y_n)}{x_m - x_{m-1}},\end{aligned} \tag{4.5}$$

(4.4) can be rewritten

$$\frac{\Phi_m(n+1) - \Phi_m(n)}{y_{n+1} - y_n} = \frac{\Phi_m(n) - \Phi_m(n-1)}{y_n - y_{n-1}}, \qquad n = 1, \cdots, N-1.$$

Since $\Phi_m(0)=\Phi_m(N)$, it follows that there exists for each m a constant C_m independent of n such that

$$\Phi_m(n)=C_m, \quad \text{for all } m \text{ and } n. \tag{4.6}$$

Next we use the vanishing of the second term on the right-hand side of (4.3) when $v(x,y)=\varphi_m(x)$, $m=1,\cdots,M-1$, in a similar way to infer that

$$\sum_{n=1}^{N} \frac{y_{n+1}-y_{n-1}}{2} w(x_m, y_n)=C, \tag{4.7}$$

with the same constant C for all m. Combining (4.6), (4.5), and (4.7) yields

$$C_m=\sum_{n=1}^{N} \frac{y_{n+1}-y_{n-1}}{2}\Phi_m(n)=0,$$

i.e., $\Phi_m(n)=0$ for all m and n. Applying this successively for $m=1,\cdots,M-1$ with n fixed and again invoking periodicity we see that the nodal value $w(x_m, y_n)$ must be independent of m. Similarly this value must be independent also of n, so w assumes a constant value on Δ. Since $J_\Delta w=0$, w in fact vanishes on Δ. □

As a direct consequence at Lemma 4.1 we have the Galerkin formulation of the collocation equations upon which the subsequent analysis is based. Recalling (2.2) let $\tilde{L}=a^{-1}L$, so that the different operator $\tilde{L}$ has principal part with constant coefficients. Set $A_\Delta^d=B_\Delta^d\tilde{L}$.

THEOREM 4.2. *A function $u_\Delta\in\mathcal{M}_d(\Delta)$ satisfies the collocation equations* (2.4) *if and only if it satisfies the Galerkin equations*

$$\langle A_\Delta^d u_\Delta, v\rangle=\langle A_\Delta^d u, v\rangle, \qquad v\in\mathcal{M}_d(\Delta). \tag{4.8}$$

We now proceed to establish the stability of the Galerkin method (4.8) in the space $H_\cap^{j,j+1}$ where $j=(d+1)/2$. We shall treat the operator A_Δ^d as a small perturbation of the operator $A^d=B^d\tilde{L}$ where

$$B^d=J+\partial_x^{d+1}\otimes J_y+J_x\otimes\partial_y^{d+1}+\partial_x^{d+1}\,\partial_y^{d+1}. \tag{4.9}$$

LEMMA 4.3. *The operator A^d maps $H_\cap^{j,j+1}$ isomorphically onto its dual $H_+^{-j,-j-1}$.*

Proof. It is easily verified that

$$(B^d v)\hat{\ }(m,n)=\underline{m}^{d+1}\underline{n}^{d+1}\hat{v}(m,n)=\underline{m}^{2j}\underline{n}^{2j}\hat{v}(m,n), \qquad v\in\mathcal{D}'_\omega(\mathbb{R}^2). \tag{4.10}$$

Hence B^d is an isometric isomorphism of $H_+^{j,j-1}$ onto $H_+^{-j,-j-1}$ and it suffices to prove that the differential operator $\tilde{L}$ maps $H_\cap^{j,j+1}$ isomorphically onto $H_+^{j,j-1}$.

Decompose $\tilde{L}$ as L_0+L_1, where

$$L_0 u=-\sum_{k+l=2}\tilde{a}_{kl}\,\partial_x^k\,\partial_y^l u+u$$

and L_1 is a first order operator. Now

$$(L_0 u)\hat{\ }(m,n)=\sigma(m,n)\hat{u}(m,n), \qquad m,n\in\mathbb{Z}, \tag{4.11}$$

where

$$\sigma(m,n)=1-\sum_{k+l=2}\tilde{a}_{kl}(2\pi i m)^k(2\pi i n)^l.$$

Using the ellipticity condition (2.3) we find that there exists a constant C such that

$$C^{-1}\max(\underline{m}^2,\underline{n}^2)\leqq\sigma(m,n)\leqq C\max(\underline{m}^2,\underline{n}^2), \qquad m,n\in\mathbb{Z}. \tag{4.12}$$

Since

$$\max(\underline{m}^2, \underline{n}^2)\min(\underline{m}^{j-1}\underline{n}^j, \underline{m}^j\underline{n}^{j-1}) = \max(\underline{m}^j\underline{n}^{j+1}, \underline{m}^{j+1}\underline{n}^j),$$

it follows easily that L_0 maps $H_\cap^{j,j+1}$ isomorphically onto $H_+^{j,j-1}$.

The partial derivative operators ∂_x and ∂_y and the inclusion operator are clearly bounded operators $H_\cap^{j,j+1} \to H^{j,j}$. Also $H^{j,j}$ is compactly included in $H_+^{j,j-1}$ (this follows from the fact that the ratio of the multipliers, $\underline{m}^j\underline{n}^j/\min(\underline{m}^j\underline{n}^{j-1}, \underline{m}^{j-1}\underline{n}^j)$, tends to infinity as $\min(|m|, |n|)$ tends to infinity). Finally, since the coefficients of L are assumed to have d continuous derivatives, multiplication by the coefficients of $\tilde{L}$ map $H_+^{j,j-1}$ boundedly into itself. In all, then, L_1 maps $H_\cap^{j,j+1}$ compactly into $H_+^{j,j-1}$, and consequently the Fredholm alternative holds for $\tilde{L} = L_0 + L_1$. Since by assumption L, and so $\tilde{L}$, has trivial nullspace, $\tilde{L}$ is an isomorphism as claimed. □

The next lemma shows that as the mesh size parameter h_Δ tends to zero A_Δ^d tends to A^d in the operator norm $H_\cap^{j,j+1} \to H_+^{-j,-j-1}$.

LEMMA 4.4. *There exists a constant C such that for every Δ*

$$\|(A_\Delta^d - A^d)w\|_{H_+^{-j,-j-1}} \leqq Ch_\Delta\|w\|_{H_\cap^{j,j+1}}, \qquad w \in H_\cap^{j,j+1}.$$

Proof. By definition $A_\Delta^d - A^d = (B_\Delta^d - B^d)\tilde{L}$. In the previous proof we showed that $\tilde{L}$ maps $H_\cap^{j,j+1}$ isomorphically onto $H_+^{j,j-1}$. Thus it suffices to prove that

$$\|(B_\Delta^d - B^d)w\|_{H_+^{-j,-j-1}} \leqq Ch_\Delta\|w\|_{H_+^{j,j-1}}, \qquad w \in H_+^{j,j+1}.$$

Subtracting (4.9) from (4.1) gives

$$B_\Delta^d - B^d = (J_\Delta - J) + \partial_x^{d+1} \otimes (J_{\Delta_y} - J_y) + (J_{\Delta_x} - J_x) \otimes \partial_y^{d+1}. \tag{4.13}$$

Now it is easy to establish the following one-dimensional estimates:

$$|(J_{\Delta_x} - J_x)\varphi| + |(J_{\Delta_y} - J_y)\varphi| \leqq Ch_\Delta^p\|\varphi\|_{H_\omega^p(\mathbb{R})}, \qquad \varphi \in H_\omega^p(\mathbb{R}), \quad p = 1 \text{ or } 2, \tag{4.14}$$

$$|J_x\varphi| + |J_{\Delta_x}\varphi| + |J_{\Delta_y}\varphi| \leqq C\|\varphi\|_{H_\omega^1(\mathbb{R})}, \qquad \varphi \in H_\omega^1(\mathbb{R}), \tag{4.15}$$

with C independent of Δ. Thus, for $w \in H^{1,1} = H_\omega^1(\mathbb{R}) \otimes H_\omega^1(\mathbb{R})$,

$$\begin{aligned}\|(J_\Delta - J)w\|_{H_+^{-j,-j-1}} &= |(J - J_\Delta)w| \leqq |[(J_{\Delta_x} - J_x) \otimes J_{\Delta_y}]w| + |[J_x \otimes (J_{\Delta_y} - J_y)]w| \\ &\leqq Ch_\Delta\|w\|_{H^{1,1}} \leqq Ch_\Delta\|w\|_{H_+^{j,j-1}}.\end{aligned}$$

It remains to bound the last two terms on the right-hand side of (4.13) in operator norm, and by symmetry it is clearly sufficient to consider one. Now if $\varphi(x, y)$ is independent of y, then one easily verifies that $\|\partial_x^{d+1}\varphi\|_{H_+^{-j,-j-1}} = \|\partial_x^{j-1}\varphi\|_{L^2}$. Thus

$$\begin{aligned}\|[\partial_x^{d+1} \otimes (J_{\Delta_y} - J_y)]w\|_{H_+^{-j,-j-1}} &= \|[\partial_x^{j-1} \otimes (J_{\Delta_y} - J_y)]w\|_{L^2} \\ &\leqq Ch_\Delta\|w\|_{H^{j-1,1}} \leqq Ch_\Delta\|w\|_{H_+^{j,j-1}}.\end{aligned}$$ □

The mapping properties of A_Δ^d follow directly from the previous two lemmas.

THEOREM 4.5. *There exist constants C, $h_0 > 0$ such that for every Δ satisfying $h_\Delta \leqq h_0$, A_Δ^d maps $H_\cap^{j,j+1}$ isomorphically onto $H_+^{-j,-j-1}$ and*

$$C^{-1}\|w\|_{H_\cap^{j,j+1}} \leqq \|A_\Delta^d w\|_{H_+^{-j,-j-1}} \leqq C\|w\|_{H_\cap^{j,j+1}}, \qquad w \in H_\cap^{j,j+1}.$$

Having established the mapping properties of A_Δ^d we now show that the Galerkin projection onto the spline space $\mathcal{M}_d(\Delta)$ is stable uniformly with respect to Δ.

THEOREM 4.6. *There exist constants γ, $h_0 > 0$ such that for every Δ with $h_\Delta \leqq h_0$,*

$$\inf_{0 \neq v \in \mathcal{M}_d(\Delta)} \sup_{0 \neq w \in \mathcal{M}_d(\Delta)} \frac{\langle A_\Delta^d v, w\rangle}{\|v\|_{H_\cap^{j,j+1}}\|w\|_{H_\cap^{j,j+1}}} \geqq \gamma. \tag{4.16}$$

Proof. In light of Lemma 4.4 it suffices to prove (4.16) with A^d in place of A^d_Δ. Furthermore, decomposing $\tilde{L}$ as L_0+L_1 as in the proof of Lemma 4.3, we have $A^d=B^dL_0+B^dL_1$ with $B^dL_1: H^{j,j+1}_\cap \to H^{-j,-j-1}_+$ compact. It follows [4, Thm. 4.9], [7, Lemma 4.2] that it in fact suffices to prove (4.16) with A^d_Δ replaced by B^dL_0.

Now it follows directly from Parseval's formula, (4.10), (4.11), (4.12), and the definition of the norm in $H^{j,j+1}_\cap$ that there exists a positive constant $\gamma'>0$ such that

$$\langle B^dL_0v, v\rangle \geqq \gamma'\|v\|_{H^{j,j+1}_\cap}, \qquad v\in H^{j,j+1}_\cap.$$

Thus Theorem 4.6 is proven. □

5. Convergence analysis. Theorems 4.2, 4.5, and 4.6 imply by standard arguments [8], [9, ch. 5] quasioptimality of the collocation method. In light of the approximation theoretic result of Theorm 3.2 we also infer convergence estimates in the $H^{j,j+1}_\cap$ norm.

THEOREM 5.1. *There exist positive constants h_0 and C such that the collocation equations (2.4) have a unique solution $u_\Delta\in\mathcal{M}_d(\Delta)$ whenever $h_\Delta\leqq h_0$. Moreover*

$$\|u-u_\Delta\|_{H^{j,j+1}_\cap}\leqq C\inf_{v\in\mathcal{M}_d(\Delta)}\|u-v\|_{H^{j,j+1}_\cap}.$$

If in addition $u\in H^{j,k+1}_\cap$ for some $k\in[j,d]$, then

$$\|u-u_\Delta\|_{H^{j,j+1}_\cap}\leqq Ch^{k-j}_\Delta\|u\|_{H^{j,k+1}_\cap}.$$

Theorem 5.1 shows that for $0\leqq p\leqq j$ the derivatives $\partial^{j+1}_y\,\partial^p_y(u-u_\Delta)$ and $\partial^p_x\,\partial^{j+1}_y(u-u_\Delta)$ converge to zero at the optimal rate in L^2 as $h_\Delta\to 0$, that is, as h^{d-j}_Δ for smooth u. This implies in particular such an optimal rate of convergence in the Sobolev space H^{j+1}. We now establish higher rates of convergence for certain derivatives of lower order.

THEOREM 5.2. *For each integer $p\in[2,j]$ and Δ with $h_\Delta\leqq h_0$ we have*

$$\|u-u_\Delta\|_{H^{p,p}}\leqq Ch^{j+1-p}_\Delta\|u-u_\Delta\|_{H^{j,j+1}_\cap}. \tag{5.1}$$

Moreover if $u\in H^{j,k+1}_\cap$ for some $k\in[j,d]$, then

$$\|u-u_\Delta\|_{H^{p,p}}\leqq Ch^{k+1-p}_\Delta\|u\|_{H^{j,k+1}_\cap}.$$

Proof. The second assertion is a consequence of the first and the preceding theorem. To prove (5.1) we use a duality argument involving several cases.

First suppose $p\in[3,j]$. Then a simple variant of the proofs of Lemmas 4.3 and 4.4 show that A^d_Δ maps $H^{p,p}$ isomorphically onto $H^{p-2j,p-2j-2}_+$ for $h_\Delta\leqq h_0$. Moreover it is clear from (4.10) that B^{2p-1} (this operator is defined by (4.9) with d replaced by $2p-1$) maps $H^{p,p}$ isomorphically onto $H^{-p,-p}$. Consequently there exists a unique $w\in H^{2j-p,2j+2-p}_\cap$ (the dual space of $H^{p-2j,p-2j-2)}_+$) such that

$$\langle A^d_\Delta v, w\rangle=\langle v, B^{2p-1}(u-u_\Delta)\rangle, \qquad v\in H^{p,p}. \tag{5.2}$$

Moreover,

$$\|w\|_{H^{2j-p,2j+2-p}_\cap}\leqq C\|u-u_\Delta\|_{H^{p,p}}. \tag{5.3}$$

Setting $v=u-u_\Delta$ in (5.2) we get

$$\begin{aligned}\|u-u_\Delta\|^2_{H^{p,p}}&=\langle u-u_\Delta, B^{2p-1}(u-u_\Delta)\rangle\\&=\langle A^d_\Delta(u-u_\Delta), w\rangle=\inf_{v\in\mathcal{M}_d(\Delta)}\langle A^d_\Delta(u-u_\Delta), w-v\rangle\\&\leqq C\|u-u_\Delta\|_{H^{j,j+1}_\cap}\inf_{v\in\mathcal{M}_d(\Delta)}\|w-v\|_{H^{j,j+1}_\cap}\\&\leqq Ch^{j+1-p}_\Delta\|u-u_\Delta\|_{H^{j,j+1}_\cap}\|w\|_{H^{j,2j+2-p}_\cap}\\&\leqq Ch^{j+1-p}_\Delta\|u-u_\Delta\|_{H^{j,j+1}_\cap}\|u-u_\Delta\|_{H^{p,p}},\end{aligned}$$

where we have used (4.10), (5.2), Theorems 4.2, 4.5, and 3.2, and (5.3). Thus (5.1) is established for $p \geqq 3$.

For $p=2$ we cannot define w by (5.2) since A_Δ^d is not defined on $H^{2,2}$. Instead, $w \in H_\cap^{2j-2,2j}$ is defined by

$$\langle A^d v, w\rangle = \langle v, B^3(u-u_\Delta)\rangle, \qquad v \in H^{2,2}.$$

Again, w is well defined and

$$\|w\|_{H_\cap^{2j-2,2j}} \leqq C\|u-u_\Delta\|_{H^{2,2}}. \tag{5.4}$$

Now

$$\begin{aligned}\|u-u_\Delta\|_{H^{2,2}}^2 &= \langle u-u_\Delta, B^3(u-u_\Delta)\rangle \\ &= \langle A_\Delta^d(u-u_\Delta), w\rangle + \langle (A^d - A_\Delta^d)(u-u_\Delta), w\rangle.\end{aligned} \tag{5.5}$$

As before

$$\begin{aligned}|\langle A_\Delta^d(u-u_\Delta), w\rangle| &= \inf_{v\in \mathcal{M}_d(\Delta)} |\langle A_\Delta^d(u-u_\Delta), w-v\rangle| \\ &\leqq Ch_\Delta^{j-1}\|u-u_\Delta\|_{H_\cap^{j,j+1}}\|u-u_\Delta\|_{H^{2,2}}.\end{aligned} \tag{5.6}$$

To bound the final term of (5.5), note that

$$\begin{aligned}&\langle (A^d - A_\Delta^d)(u-u_\Delta), w\rangle \\ &= (J-J_\Delta)\tilde{L}(u-u_\Delta)\cdot Jw \\ &\quad + (-1)^{j+1}\int_0^1 [\partial_x^{j-1}\otimes (J_y - J_{\Delta_y})]\tilde{L}(u-u_\Delta)\cdot(\partial_x^{j+1}\otimes J_y)w\,dx \\ &\quad + (-1)^{j+1}\int_0^1 [(J_x - J_{\Delta_x})\otimes \partial_y^{j-1}]\tilde{L}(u-u_\Delta)\cdot(J_x\otimes\partial_y^{j+1})w\,dy.\end{aligned} \tag{5.7}$$

Now we distinguish two subcases. First suppose $j=2$ or 3. Then we apply (4.14) and (4.15) to get

$$\begin{aligned}|\langle A^d - A_\Delta^d)(u-u_\Delta), w\rangle| &\leqq Ch_\Delta^{j-1}\|\tilde{L}(u-u_\Delta)\|_{H^{j-1,j-1}}\|w\|_{H_\cap^{j+1,0}} \\ &\leqq Ch_\Delta^{j-1}\|u-u_\Delta\|_{H_\cap^{j-1,j+1}}\|w\|_{H_\cap^{2j,2j-2}} \\ &\leqq Ch_\Delta^{j-1}\|u-u_\Delta\|_{H_\cap^{j-1,j+1}}\|u-u_\Delta\|_{H^{2,2}},\end{aligned} \tag{5.8}$$

where we invoked (5.4) in the last step. In the case $p=2$ and $j=2$ or 3, (5.1) now follows from (5.5), (5.6), and (5.8).

Finally, we consider the case $p=2$, $j\geqq 4$. Integrate by parts in (5.7) to get in place of (5.8)

$$\begin{aligned}|\langle (A^d - A_\Delta^d)(u-u_\Delta), w\rangle| &\leqq |(J-J_\Delta)\tilde{L}(u-u_\Delta)|\,|Jw| \\ &\quad + \|[\partial_x^2\otimes(J_y-J_{\Delta_y})]\tilde{L}(u-u_\Delta)\|_{H^0}\|(\partial_x^{2j-2}\otimes J_y)w\|_{H^0} \\ &\quad + \|[(J_x-J_{\Delta_x})\otimes\partial_y^2]\tilde{L}(u-u_\Delta)\|_{H^0}\|(J_x\otimes\partial_y^{2j-2})w\|_{H^0} \\ &\leqq Ch_\Delta^2\|\tilde{L}(u-u_\Delta)\|_{H^{2,2}}\|w\|_{H_\cap^{0,2j-2}} \\ &\leqq Ch_\Delta^2\|u-u_\Delta\|_{H^{4,4}}\|u-u_\Delta\|_{H^{2,2}}.\end{aligned}$$

Since (5.1) has already been established for $p=4\leqq j$,

$$\|u-u_\Delta\|_{H^{4,4}} \leqq Ch_\Delta^{j-3}\|u-u_\Delta\|_{H_\cap^{j,j+1}},$$

so again we have

$$|\langle (A^d - A_\Delta^d)(u-u_\Delta), w\rangle| \leqq Ch_\Delta^{j-1}\|u-u_\Delta\|_{H_\cap^{j,j+1}}\|u-u_\Delta\|_{H^{2,2}},$$

and (5.1) follows. □

Theorems 5.1 and 5.2 show that for sufficiently smooth u the partial derivatives $\partial_x^k \partial_y^l(u-u_\Delta)$ of the error converge to zero at the optimal rate $O(h^{d+1-\max(k,l)})$ in L^2 for any integers k, l with $\max(k,l)\in[2, j+1]$ and $\min(k,l)<j+1$. The highest rate of convergence, $O(h_\Delta^{d-1})$, is achieved by all derivatives of order at most two in each variable, and is exactly twice the rate of convergence achieved in $H_\cap^{j,j+1}$, the space in which we proved quasioptimality of the collocation method. Note that this estimate is suboptimal by two powers of h_Δ for the L^2 norm of the error itself and by one power for L^2 norm of its gradient. As mentioned in the introduction this suboptimality is a property of the collocation method and not merely due to the method of analysis. It occurs already for ordinary differential equations, as proved in [5].

We conclude by applying a standard argument to show that for quasiuniform meshes optimal order approximation also holds in the Sobolev space H^p, $j+1<p<d+1$. Recall that the mesh Δ is ρ-quasiuniform if $\rho\min(x_m-x_{m-1}, y_m-y_{m-1})\geqq h_\Delta$ for all m.

THEOREM 5.3. *For each $\rho\geqq 1$ there exists a constant C so that if Δ is a ρ-quasiuniform mesh, then for $p\in[j+1,d]$, $q\in[p,d+1]$, $u\in H^q$, there holds*

$$\|u-u_\Delta\|_{H^p}\leqq Ch_\Delta^{q-p}\|u\|_{H_\cap^{j,q}}.$$

Proof. Because of the quasiuniformity the inverse property

$$\|v\|_{H_\omega^m(\mathbb{R})}\leqq Ch_\Delta^{n-m}\|v\|_{H_\omega^n(\mathbb{R})}, \qquad v\in S_d(\Delta_x),\quad 0\leqq n\leqq m\leqq d, \tag{5.9}$$

holds. It follows easily that the $H_\omega^0(\mathbb{R})$ projection $P_{\Delta_x}: H_\omega^0(\mathbb{R})\to S_d(\Delta_x)$ satisfies the optimal order error estimates

$$\|v-P_{\Delta_x}v\|_{H_\omega^m(\mathbb{R})}\leqq Ch_\Delta^{n-m}\|v\|_{H_\omega^n(\mathbb{R})}, \qquad v\in H_\omega^n(\mathbb{R}),$$

for all $m\in[0,d]$, $n\in[m,d+1]$. Letting then $Q_\Delta=P_{\Delta_x}\otimes P_{\Delta_y}: H^0\to\mathcal{M}_d(\Delta)$ it is easy to establish the conclusion of the theorem with u_Δ replaced by $Q_\Delta u$. Also from (5.9) we may infer the two-dimensional inverse property

$$\|w\|_{H^p}\leqq Ch_\Delta^{j+1-p}\|w\|_{H^{j+1}}, \qquad w\in\mathcal{M}_d(\Delta),\quad p\in[j+1,d].$$

Therefore

$$\begin{aligned}\|u-u_\Delta\|_{H^p} &\leqq \|u-Q_\Delta u\|_{H^p}+\|Q_\Delta u-u_\Delta\|_{H^p}\\ &\leqq \|u-Q_\Delta u\|_{H^p}+Ch_\Delta^{j+1-p}[\|u-Q_\Delta u\|_{H^{j+1}}+\|u-u_\Delta\|_{H^{j+1}}]\\ &\leqq Ch_\Delta^{q-p}\|u\|_{H_\Delta^{j,q}},\end{aligned}$$

where we have used Theorem 5.1 to estimate

$$\|u-u_\Delta\|_{H^{j+1}}\leqq\|u-u_\Delta\|_{H_\cap^{j,j+1}}.$$ □

Remark. The rate $O(h_\Delta^{d+1-p})$ in the Sobolev space H^p, which is established in Theorem 5.3 for $u\in H_\cap^{j,d+1}$, is optimal. However if $\max(k,l)\in[j+1,d]$, then we can only infer convergence of $\partial_x^k\partial_y^l(u-u_\Delta)$ to zero in L^2 from the preceding theorems when $k+l\leqq d$, and then only with rate $O(h_\Delta^{d+1-k-l})$. This rate is optimal only if $k=0$ or $l=0$. For the high order mixed derivatives we do not know if optimal order convergence holds.

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