COLLOCATION VERSUS GALERKIN PROCEDURES FOR BOUNDARY INTEGRAL METHODS

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INTRODUCTION

A comparison of the computational complexities of the finite element domain method with the boundary element method for elliptic interior boundary value problems shows that these are roughly the same in two as well as in three dimensions, respectively [16, p. 444] . Hence a decision for one of these methods depends on further, more specific properties of the problem to be solved. If the solution is also required on the boundary curve or boundary surface Γ , respectively, or if an exterior problem is to be solved, then the boundary element method will usually be preferable. Whereas usually the data administration for the boundary element method is simple and, moreover, a few boundary elements often provide rather accurate solutions, one of the disadvantages of the method is seen in the large amount of computing time for the computation of the stiffness matrix. Here we estimate these times for two-dimensional problems for the Galerkin procedure involving double integrations and for the standard collocation procedure which mostly is used in applications. We first compare the time consuming evalutations of the smooth remaining kernel functions of both methods involving Gaussian quadrature formulas against the order of convergence of the L_2-error terms using splines of the same degree.

Then we compare the number of evaluations if the highest orders of convergence, i.e. superapproximation coincide. It turns out that here the Galerkin method needs much more time than collocation whilst using much lower degree splines. But even for collocation the evaluation of the stiffness matrix with Gaussian quadrature is rather costly. Moreover, a mesh refinement requires the new evaluation of the kernel function values at all nodal points since these change with any change of step size.

with the scalar product

$$\langle f,g\rangle_s = \hat{f}_0\overline{\hat{g}_0} + \sum_{0\neq k\in\mathbb{Z}} \hat{f}_k\overline{\hat{g}_k}|2\pi k|^{2s}$$

and $\|f\|_s^2 = \langle f,f\rangle_s$ where $\hat{f}_k, \hat{g}_k$ denote the k-th Fourier coefficients of f and g, respectively. Note that

$$\langle f,g\rangle_0 = \int_0^1 f(t)\overline{g(t)}\,dt .$$

In Equations (1) we further specify A to be a pseudodifferential operator of order 2α on Γ and require strong ellipticity which in turn implies coercivity in form of Gårding's inequality:

There exists a positive constant C and a compact bilinear form $k[u,v]$ on $(H^{j+\alpha})^2$ such that

$$\mathrm{Re}\,\langle Av,v\rangle_\lambda \geq C\|v\|_{\lambda+\alpha}^2 - \mathrm{Re}\,k[v,v] \quad \text{for} \quad v\in H^{\lambda+\alpha} . \qquad (2)$$

$\lambda\in\mathbb{R}$ will be specified later on.

Most boundary integral methods for stationary and time harmonic problems in the applications belong to this class of mathematical problems, for short surveys see [2 , §2.3],[15] . The most frequent cases in applications are $\alpha = -1/2$ for Symm's integral equation and related equations of the first kind with logarithmic principal part with applications in conformal mapping, torsion problems, plane elasticity, Stokes flows and electrostatics; $\alpha = 0$ for singular integral equations involving Cauchy's kernel including Fredholm integral equations of the second kind as a special case and with applications in plane elasticity and thermoelasticity, electromagnetic fields, acoustics, classical potential theory, incompressible flows; $\alpha = 1/2$ for the normal derivative of double layer potentials and for the operator of Prandtl's wing theory with applications in acoustics, ideal flows and plane elasticity; $\alpha = 1$ for integrodifferential operators of second order with many applications involving periodic solutions of second order ordinary differential equations.

For the approximations of Equations (1) we select an increasing sequence of mesh points $\Delta = \{t_i\}$, $i\in\mathbb{Z}$ satisfying $t_{i+N} = t_i + 1$ for fixed N and all $i\in\mathbb{Z}$, and denote by $S_m(\Delta)$ the space of all 1-periodic , $m-1$ times continuously differentiable splines of degree m subordinate to the partition Δ . We also write $S_m(\Delta)$ for $(S_m(\Delta))^p$.

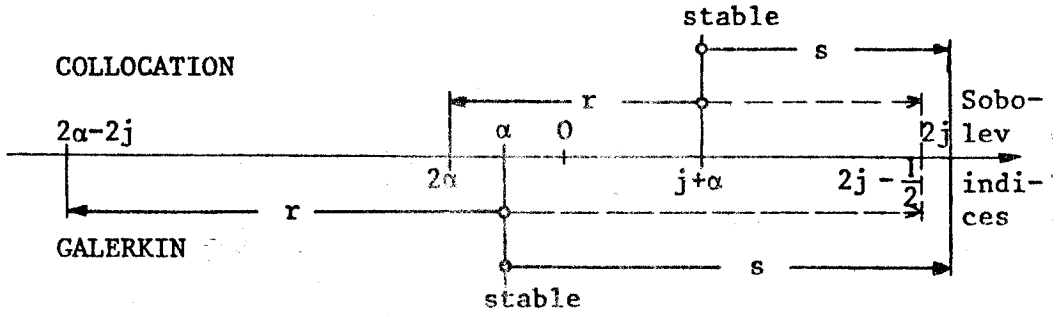
Then the standard Galerkin method for Equations (1) reads as:

Find $u_h \in S_m(\Delta)$, $\omega_h \in \mathbb{R}^q$ such that

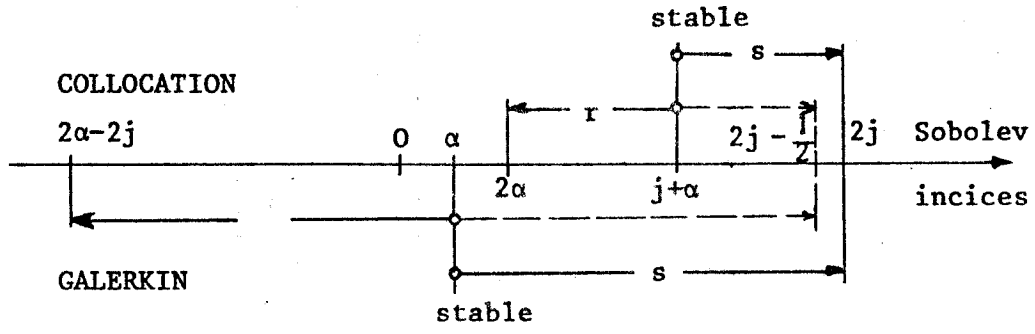
$$\| u-u_\Delta \|_r + |\omega-\omega_\Delta| \le ch^{s-r} \| u \|_s \,. \qquad (6)$$

In case of a quasiuniform family of meshes the Estimate (6) also holds for $j+\alpha \le r < m+1/2$ and $r \le s \le m+1$.

Now we are in the position to compare the two methods. Let us first consider the case of using the same degree splines $S_m(\Delta)$ for both methods. Further let us restrict to quasiuniform families of meshes and smooth solutions. Then the highest rate of convergence achieved by the collocation method is $O(h^{m+1-2\alpha})$ in $H^{2\alpha}$, whilst the Galerkin method converges with rate $O(h^{2m+2-2\alpha})$ in $H^{2\alpha-m-1}$. This situation is summarized in Figure 1 .



The case $\alpha \le 0$ ($j+\alpha > 0$ pictured, $j+\alpha \le 0$ also possible)



The case $\alpha \ge 0$

Figure 1. The indices $r \le s$ for which $\| u-u_\Delta \|_r \le ch_\Delta^{s-r} \| u \|_s$. Dashed lines indicate estimates requiring a quasiuniform mesh family.

$$\| u_h \|_o \le c\{h^{2\alpha'} \| P_h f \|_o + |\beta|\} \ , \ |\omega_h| \le c\{\| P_h f \|_o + |\beta|\} \ ,$$

$$\| P_h A u_h \|_o \le c h^{2(\alpha'-\alpha)} \| u_h \|_o \ ,$$

respectively

$$\| u_\Delta \|_o \le c\{h^{2\alpha'} \| I_h f \|_o + |\beta|\} \ , \ |\omega_\Delta| \le c\{\| I_h f \|_o + |\beta|\} \ ,$$

$$\| I_h A u_\Delta \|_o \le c h^{2(\alpha'-\alpha)} \| u_\Delta \|_o \ ,$$

where $\alpha' = \min\{\alpha, 0\}$.

Besides the above stability estimates we also need consistency which corresponds to the numerical accuracy of the elements in the stiffness matrices. To this end let us denote by

$$g_{\ell k} = \langle A\mu_\ell, \mu_k \rangle_o \ , \ g_k = \langle B^*, \mu_k \rangle_o \ , \ \gamma_\ell = \Lambda \mu_\ell$$

the Galerkin weights and by $\tilde{g}_{\ell k}$, $\tilde{g}_k$, $\tilde{\gamma}_\ell$ their numerically integrated counterparts. The latter define by

$$\langle \tilde{A}_h \mu_\ell, \mu_k \rangle_o = \tilde{g}_{\ell k} \ , \ \langle \tilde{B}_h \omega, \mu_k \rangle_o = \tilde{g}_k \omega \ , \ \tilde{\Lambda}_h \mu_\ell = \tilde{\gamma}_\ell$$

linear mappings $\tilde{A}_h : S_m(\Delta) \to S_m(\Delta)$, $\tilde{B}_h : \mathbb{R}^q \to S_m(\Delta)$ and $\tilde{\Lambda}_h : S_m(\Delta) \to \mathbb{R}^q$ which are approximations to $P_h A P_h, P_h B$ and ΛP_h , respectively.

Correspondingly we denote by

$$c_{\ell k} = A\mu_\ell(t_k) \quad \text{and} \quad c_k = B(t_k) \ , \ \Lambda \mu_\ell = \gamma_\ell$$

the collocation weights, by $\tilde{c}_{\ell k}$, $\tilde{c}_k$, $\tilde{\gamma}_\ell$ their numerically integrated counterparts and by $\tilde{A}_\Delta$, $\tilde{B}_\Delta$, $\tilde{\Lambda}_\Delta$ the corresponding approximations to $I_h A P_h$, $I_h B$ and Λ defined by

$$\tilde{A}_\Delta \mu_\ell(t_k) = \tilde{c}_{\ell k} \quad \text{and} \quad \tilde{B}_\Delta(t_k) = \tilde{c}_k \ , \ \tilde{\Lambda}_\Delta \mu_\ell = \tilde{\gamma}_\ell \ .$$

Theorem 2.2: Let the degree of precision of the numerical integrations of the weights be L [5, p. 49] , i.e. let us assume

$$\begin{aligned} |g_{\ell k} - \tilde{g}_{\ell k}| + h|g_k - \tilde{g}_k| + h|\gamma_\ell - \tilde{\gamma}_\ell| &\le c h^{L+3} \ , \\ |c_{\ell k} - \tilde{c}_{\ell k}| + h|c_k - \tilde{c}_k| + |\gamma_\ell - \tilde{\gamma}_\ell| &\le c h^{L+2} \ . \end{aligned} \tag{8}$$

Then the corresponding operators provide the consistency estimates

provided $-m-1+2\alpha \le r \le s \le m+1$, $r < m + \frac{1}{2}$,

respectively

$$\| \tilde{u}_\Delta - u \|_r + |\tilde{\omega}_\Delta - \omega| \le c(h^{L+1+2\alpha'+(-r)'} \| f \|_{-2\alpha} + h^{s-r} \| f \|_{s-2\alpha}) \tag{12}$$

provided $2\alpha \le r \le s \le m+1$, $(-\tau)' = \min\{0,-r\}$, $r < m + \frac{1}{2}$.

Since the proof is very similar to the proofs of [18, Theorems 6.3, 6.4 and 7.2] we omit the details.

3. COMPARISON INVOLVING GAUSSIAN QUADRATURE

Most numerical implementations of the Galerkin or the collocation method are based on Gaussian quadrature formulae on the patches, i.e. on the subintervalls $[t_{i-1},t_i]$. For singularities of the kernels, however, a special treatment is necessary. Therefore we shall require in the following that the operator A has convolutional principal part, i.e.

$$Au = A_1 u + A_2 u = \int (p_1(t-\tau) + \log|t-\tau| p_2(t-\tau)) u(t) dt + \int K(t,\tau) u(t) dt ,$$

where p_1 and p_2 are homogeneous functions of degree $\beta = -2\alpha - 1$ and where K denotes a smooth remaining kernel [15],[18]. Moreover, let the family of meshes Δ be uniform and the spline spaces $S_m(\Delta)$ be generated by one shape function $\mu(\eta)$ as in [18, Chap. 9.5] . Then the weights of the principal part can be computed in terms of a Toeplitz matrix whose entries we consider to be known exactly. For the Galerkin method these are given by

$$(A_1 \mu_\ell, \mu_k) = h^{1-2\alpha} \{ \iint_{(\text{supp } \mu)^2} p(t'-\tau'+(\ell-k)) \mu(t') \mu(\tau') dt' d\tau' + \log h \iint_{(\text{supp } \mu)^2} p_2(t'-\tau'+(\ell-k)) \mu(t') \mu(\tau') dt' d\tau' \} \tag{13}$$

where $p = p_1 + p_2 \log|\cdot|$, whereas for collocation we have

$$A_1 \mu_\ell(t_k) = h^{-2\alpha} \{ \int_{\text{supp } \mu} p(t'- \frac{m+1}{2} + (\ell-k)) \mu(t') dt' + \log h \int_{\text{supp } \mu} p_2(t'- \frac{m+1}{2} + (\ell-k)) \mu(t') dt' \}. \tag{14}$$

(For special equations see [8, Section 3].)
For all remaining integrals on each patch we use a Gaussian quadrature formula with degree L of precision [5]. This

Now we balance the errors in (2.10) finding

$$L_G \geq 2m_G + 1 - 2(\alpha'+\alpha) \quad \text{in case} \quad 2\alpha \leq m_G + 1 \tag{15}$$

and

$$L_G = m_G \quad \text{in case} \quad m_G + 1 < 2\alpha < 2m_G + 1 \,. \tag{16}$$

Similarly we find for collocation from balancing the errors in (12),

$$L_c \geq m_c - 2\alpha' \,. \tag{17}$$

Note that in general the required degrees of precision L_G and L_c for the Galerkin method, respectively, collocation will be different. In Table 2 we compare again the numbers of evaluations of the kernel function depending on the order of superapproximation for various α. For two-dimensional Gaussian quadrature we refer to [5, p. 424 and 427].

α	$-\frac{1}{2}$				0				$\frac{1}{2}$			1		
order	3	5	7	9	2	4	6	8	3	5	7	2	4	6
m_G	0	1	2	3	0	1	2	3	1	2	3	1	2	3
L_G	3	5	7	9	1	3	5	7	2	4	6	1	3	5
eg	4	9 7	16 12	25 18	1	4	9 7	16 12	4	9 7	16 12	1	4	9 7
m_c	1	3	5	7	1	3	5	7	3	5	7	3	5	7
L_c	3	5	7	9	1	3	5	7	3	5	7	3	5	7
e_c	2	3	4	5	1	2	3	4	2	3	4	2	3	4

Table 2: Evaluations per element of the stiffness matrix for same optimal superapproximation orders.

Note that also in this case the evaluation of the stiffness matrix for Galerkin's method requires almost always significantly more time than for the collocation method. But one should be aware of the fact that the latter deals with more than twice higher degree splines yielding more complicated coding and much higher requirements for the smoothness of Γ and the boundary charges u.

The reason for all the trouble is the use of Gaussians quadrature formulas that implies computational expenses of orders $N^2 \cdot (eg) = h^{-2}(eg)$, respectively, $N^2(ec) = h^{-2}(ec)$ for the evaluation of the stiffness matrix. As we shall see in the following section, the consequent use of the regular nodal

γ	M	m=0		m=1			m=2			
		b_0	b_1	b_0	b_1	b_2	b_0	b_1	b_2	b_3
1	0	1	-	1	-	-	1	-	-	-
	1	$\frac{11}{12}$	$\frac{1}{24}$	$\frac{5}{6}$	$\frac{1}{12}$	-	$\frac{3}{4}$	$\frac{1}{8}$		
$\frac{1}{2}$	2			$\frac{13}{30}$	$\frac{4}{15}$	$\frac{1}{60}$	$\frac{2}{5}$	$\frac{7}{30}$	$\frac{1}{15}$	-
	3						$\frac{358}{945}$	$\frac{157}{630}$	$\frac{19}{315}$	$\frac{1}{945}$

Table 3: Weights of numerical integrations against splines.

If the computation of the values of the kernel functions at the grid points is executed in advance and these values then are stored for further use then the transition of the method to different shape functions can be done in a most efficient way (see [7, Table 4]) .

The computational expense for computing the stiffness matrices now depends only on γ and is proportional to

$(\frac{1}{\gamma_G})^2 N^2 = (\gamma_G \cdot h)^{-2}$ for the Galerkin collocation and to

$(\frac{1}{\gamma_c}) N^2 = \gamma_c^{-1} h^{-2}$ for the collocation with grid point quadrature.

For comparison we consider again the two cases as in Section 3.

Comparison for same orders of L_2-errors

With same splines for both methods we balance

$$2M + 1 + 2\alpha' \geq m = \text{order} - 1 \quad .$$

In Table 4 we collect orders, M and γ for the four cases $\alpha = -\frac{1}{2}, 0, \frac{1}{2}, 1$ and observe that here always $\gamma = 1$, i.e. we need only one function value per stiffness element (and appropriate organization of the code) in contrary to the case using Gaussian quadrature, Table 1.

compete with the above collocation since the corresponding splines are of less than half the degrees. If we compare the Galerkin collocation with Galerkin's method using Gaussian quadrature (Table 2) then we see again that Galerkin collocation is significantly faster and hence superior.

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