REGULAR INVERSION OF THE DIVERGENCE OPERATOR WITH DIRICHLET BOUNDARY CONDITIONS ON A POLYGON*

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Abstract. We consider the existence of regular solutions to the boundary value problem $\operatorname{div}\mathbf{U} = f$ on a plane polygonal domain Ω with the Dirichlet boundary condition $\mathbf{U} = \mathbf{g}$ on $\partial\Omega$. We formulate simultaneously necessary and sufficient conditions on f and $\mathbf{g}$ in order that a solution $\mathbf{U}$ exist in the Sobolev space $W_p^{s+1}(\Omega)$. In addition to the obvious regularity and integral conditions these consist of at most one compatibility condition at each vertex of the polygon. In the special case of homogeneous boundary data, it is necessary and sufficient that f belong to $W_p^s(\Omega)$, have mean value zero, and vanish at each vertex. (The latter condition only applies if s is large enough that the point values make sense.) We construct a solution operator which is independent of s and p. As intermediate results we obtain various new trace theorems for Sobolev spaces on polygons.

Key words. divergence, trace, Sobolev space

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1. Introduction. The constraint of incompressibility arises in many problems of physical interest. In its simplest form this constraint is modelled by the partial differential equation

$$\operatorname{div}\mathbf{U} = 0 \quad \text{in } \Omega,$$

where $\mathbf{U}$ could be, for example, the velocity field in the Navier-Stokes equations or the displacement field in the equations of incompressible elasticity and Ω is the spatial domain. Often in the analysis of such problems the inhomogeneous equation

$$\operatorname{div}\mathbf{U} = f \quad \text{in } \Omega \tag{1.1}$$

is introduced and the question of the existence of regular solutions to this equation arises. If no boundary conditions are imposed, then it is easy to see that solutions to (1.1) may be found which are as regular as the regularity of f permits. That is, if f belongs to the Sobolev space $W_p^s(\Omega)$ for some $s \ge 0$ and $1 < p < \infty$ then there exists a solution $\mathbf{U}$ in $W_p^{s+1}(\Omega)$. To show this it suffices to define $\mathbf{U} = \operatorname{grad} u$ where $u \in W_p^{s+2}(\Omega)$ is a solution of Poisson's equation $\Delta u = f$. Note that there exist such regular solutions of Poisson's

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equation even on a domain whose boundary is assumed no more regular than Lipschitz, since we can always extend f to a smoothly bounded domain and solve a regular boundary value problem for Poisson's equation (such as the Dirichlet problem) on the larger domain.

The existence of regular solutions to (1.1) with specified boundary values

$$\mathbf{U} = \mathbf{g} \quad \text{on } \partial\Omega \tag{1.2}$$

is more subtle. An obvious necessary condition for the existence of such a solution is that

$$\int_\Omega f = \int_\Omega \operatorname{div} \mathbf{U} = \int_{\partial\Omega} \mathbf{U} \cdot \boldsymbol{\nu} = \int_{\partial\Omega} \mathbf{g} \cdot \boldsymbol{\nu}. \tag{1.3}$$

If Ω is a smoothly bounded planar domain, $f \in W_p^s(\Omega)$, $\mathbf{g} \in W_p^{s+1-1/p}(\partial\Omega)$ for $s \geq 0$ with $s - 1/p$ nonintegral, and (1.3) holds, then a simple construction of a solution $\mathbf{U}$ of (1.1), (1.2) in $W_p^{s+1}(\Omega)$ is possible. For example, consider the case $\mathbf{g} \equiv 0$ and suppose Ω is simply connected. First let $u \in W_p^{s+2}(\Omega)$ be a solution to Poisson's equation as above. Then the normal derivative $\partial u/\partial \boldsymbol{\nu}$ and the tangential derivative $\partial u/\partial \boldsymbol{\sigma}$ are in $W_p^{s+1-1/p}(\partial\Omega)$ with $\int_{\partial\Omega} \partial u/\partial \boldsymbol{\nu} = 0$. We can thus find $w \in W_p^{s+2}(\Omega)$ such that

$$\partial w/\partial \boldsymbol{\nu} = \partial u/\partial \boldsymbol{\sigma}, \quad \partial w/\partial \boldsymbol{\sigma} = -\partial u/\partial \boldsymbol{\nu} \quad \text{on } \partial\Omega,$$

or, equivalently,

$$\operatorname{curl} w = -\operatorname{grad} u \quad \text{on } \partial\Omega.$$

(By $\operatorname{curl} w$ we mean the vectorfield $(\partial w/\partial y, -\partial w/\partial x)$). Setting $\mathbf{U} = \operatorname{grad} u + \operatorname{curl} w$ gives the desired solution.

If $\partial\Omega$ is not sufficiently smooth, then this argument fails and the existence of w is far from obvious. In this paper we consider the case of polygonal Ω with sides denoted by Γ_n. Returning to the general case, we show that if f belongs to $W_p^s(\Omega)$, $\mathbf{g}|_{\Gamma_n}$ belongs to $\left[W_p^{s+1-1/p}(\Gamma_n)\right]^2$ for each n, and f and $\mathbf{g}$ satisfy (1.3) and some further necessary compatibility conditions, then (1.1) admits a W_p^{s+1} solution $\mathbf{U}$ satisfying the boundary condition (1.2). Somewhat surprisingly, the compatibility conditions required for $s = 2$, one condition per vertex in addition to (1.3), are sufficient for all higher s. These results have already been applied in [2], [6], and [8].

2. Preliminaries. We will introduce a variety of function spaces allied to the Sobolev spaces. For the convenience of the reader we list here our notation for each and the equation number nearest the definition.

$W_p^s(\Omega)$	(2.1);	$\mathring{W}_p^s(\Omega)$	(2.1);	$\hat{W}_p^s(\Omega)$	(2.1);
$\check{W}_p^s(\Omega)$	(3.1);	X_{sp}^m	(5.1);	$X_{sp}^m(\partial\Omega)$	(6.1);
$Z_p^s(\partial\Omega)$	(6.3);	$\overset{\Delta}{W}{}_p^s(\Omega)$	(6.13);	$V_p^s(\Omega)$	(7.1).

Throughout the letter C is used to denote a generic constant, not necessarily the same from one occurence to the next. For $\Omega \subseteq \mathbb{R}^n$ a domain with Lipschitz boundary (as defined, for example, in [3, Definition 1.2.1.1]), and $f \in C^\infty(\bar{\Omega})$ we define the usual Sobolev norms for $1 < p < \infty$ and $s \geq 0$:

$$\|f\|_{s,p,\Omega}^p = \begin{cases} \|f\|_{L^p(\Omega)}^p, & s = 0, \\ \|f\|_{L^p(\Omega)}^p + \displaystyle\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy, & 0 < s < 1, \\ \displaystyle\sum_{|\alpha| \leq [s]} \|D^\alpha f\|_{s-[s],p,\Omega}^p, & 1 \leq s < \infty. \end{cases} \tag{2.1}$$

(Here $[s]$ denotes the greatest integer not exceeding s.) The spaces $W_p^s(\Omega)$ and $\mathring{W}_p^s(\Omega)$ are defined to be the closures of $C^\infty(\bar{\Omega})$ and $C_0^\infty(\Omega)$, respectively, relative to these norms. There exists a bounded linear extension operator from $W_p^s(\Omega)$ into $W_p^s(\mathbb{R}^n)$ (even if the boundary is only Lipschitz). Cf. [3, Theorem 1.4.3.1]. For $s > 1/p$ the functions in $W_p^s(\Omega)$ have well-defined traces on $\partial\Omega$. If $s \geq 1$, then $W_p^s(\Omega) \cap \mathring{W}_p^1(\Omega) = \{\, v \in W_p^s(\Omega) \mid v = 0 \text{ on } \partial\Omega \,\}$. We denote by $\hat{W}_p^s(\Omega)$ the subspace of $W_p^s(\Omega)$ consisting of elements whose integral is zero. For details and more information regarding Sobolev spaces, we refer the reader to [1], [3], [7], and, for the case $p = 2$, to [5].

We shall also require the Sobolev norms for functions defined on Lipschitz curves in $\mathbb{R}^2$, in particular for an open subset, Γ, of the boundary of a polygon. For a Lipschitz curve, the norms $\|\cdot\|_{s,p,\Gamma}$ may be defined for $0 \leq s \leq 1$, $1 < p < \infty$, via charts. Moreover, for $1/p < s \leq 1$, the trace operator maps $W_p^s(\Omega)$ boundedly onto $W_p^{s-1/p}(\partial\Omega)$. See [3, Theorem 1.5.1.2]. (The norms $\|\cdot\|_{s,p,\Gamma}$, $s > 1$, are not well-defined unless Γ is more regular.)

We recall some properties of these spaces when the domain of definition is a broken line segment. (Cf. [4] or [3, Lemma 1.5.1.8].) Suppose Γ_1 and Γ_2 are line segments in $\mathbb{R}^2$ intersecting at a common endpoint, z, and let f be a function on $\Gamma = \Gamma_1 \cup \Gamma_2$. Then for $1 < p < \infty$ and $0 \leq s < 1/p$, $f \in W_p^s(\Gamma)$ if and only if

$$f|_{\Gamma_1} \in W_p^s(\Gamma_1), \qquad f|_{\Gamma_2} \in W_p^s(\Gamma_2). \tag{2.2}$$

Moreover, the norm

$$\|f\|_{s,p,\Gamma_1} + \|f\|_{s,p,\Gamma_2} \tag{2.3}$$

is equivalent to the $W_p^s(\Gamma)$ norm. For $1/p < s \leq 1$, $f \in W_p^s(\Gamma)$ if and only if (2.2) holds and f is continuous at z. (Note that (2.2) implies the continuity of f everywhere else in view of the Sobolev imbedding theorem). In this case too we have equivalence of norms. The case $s = 1/p$ is more involved. Let $\boldsymbol{\sigma}_1$ denote the unit direction along Γ_1 pointing toward

z, and let $\boldsymbol{\sigma}_2$ denote the unit direction along Γ_2 pointing away from z. Then $f \in W_p^{1/p}(\Gamma)$ if and only if (2.2) holds and

$$I_\Gamma^p(f) = \int_0^\epsilon t^{-1}|f(z - t\boldsymbol{\sigma}_1) - f(z + t\boldsymbol{\sigma}_2)|^p \, dt < \infty$$

where ϵ is a positive number not exceeding the lengths of Γ_1 or Γ_2. In this case

$$\Big(\|f\|_{1/p,p,\Gamma_1}^p + \|f\|_{1/p,p,\Gamma_2}^p + I_\Gamma^p(f)\Big)^{1/p} \tag{2.4}$$

defines a norm equivalent to $\|f\|_{1/p,p,\Gamma}$.

If Γ_1 and Γ_2 are collinear, so Γ is a line segment, one can easily extend these results to determine when f belongs to $W_p^s(\Gamma)$ for $s > 1$. Namely, if $s - 1/p$ is not an integer, then $f \in W_p^s(\Gamma)$ if and only if (2.2) holds and the tangential derivatives $f^{(k)}$ are continuous at z for $0 \le k < s - 1/p$. If this case, (2.3) is an equivalent norm on $W_p^s(\Gamma)$. If $s - 1/p$ is an integer, it is required in addition that $I_\Gamma^p(f^{(s-1/p)}) < \infty$ and then $[\|f\|_{s,p,\Gamma_1}^p + \|f\|_{s,p,\Gamma_2}^p + I_\Gamma^p(f^{(s-1/p)})]^{1/p}$ is an equivalent norm on $W_p^s(\Gamma)$.

We close this section with two lemmas concerning the space $W_p^{2/p}$ on a sector. Let $S_\alpha = \{\, (r\cos\theta, r\sin\theta) \mid 0 < r < 1,\ 0 < \theta < \alpha \,\}$, $\Gamma_1^\alpha = \{\, (r\cos\alpha, r\sin\alpha) \mid 0 < r < 1 \,\}$, and $\Gamma_2^\alpha = \{\, (r, 0) \mid 0 < r < 1 \,\}$. Then $\Gamma^\alpha := \Gamma_1^\alpha \cup \Gamma_2^\alpha \cup (0,0)$ is the linear part of the boundary of S_α. By γu we denote the trace of the function u on Γ^α.

Lemma 2.1. *Let $\alpha \in (0, 2\pi)$. Then the trace operator γ maps $W_p^{2/p}(S_\alpha)$ continuously into $W_p^{1/p}(\Gamma^\alpha)$.*

Proof. For $p \ge 2$ this lemma follows immediately from the trace theorem quoted earlier ([3, Theorem 1.5.1.2]) since then $s = 2/p \le 1$. For $1 < p < 2$ we first show that γ maps $W_p^t(S_\alpha)$ continuously into $W_p^{t-1/p}(\Gamma^\alpha)$, $2/p < t < 1 + 1/p$. Indeed, if $u \in W_p^t(S_\alpha)$, then $\gamma u|_{\Gamma_i^\alpha} \in W_p^{t-1/p}(\Gamma_i^\alpha)$, and since $t > 2/p$, γu is also continuous at the origin. As $t - 1/p < 1$, this implies that $\gamma u \in W_p^{t-1/p}(\Gamma^\alpha)$. Furthermore, we know that γ maps $W_p^1(S_\alpha)$ continuously into $W_p^{1-1/p}(\Gamma^\alpha)$. The conclusion of the lemma now follows by interpolation.[1] □

Remarks. (1) If α varies in a compact subinterval of $(0, 2\pi)$, then the distance between points $x_1 \in \Gamma_1^\alpha$ and $x_2 \in \Gamma_2^\alpha$ is equivalent to the sum of the distances of x_1 and x_2 to

[1] The fractional order spaces $W_p^t(S_\alpha)$ may also be defined by real interpolation between two consecutive integer orders. This is the approach used in [1, Ch. 7], where the specific interpolation process is defined. The equivalence with the definition given here is shown in [1, Theorem 7.48]. Similarly the spaces $W_p^s(I)$ for an interval I may be defined by real interpolation between integer orders. The same holds for the spaces $W_p^s(\Gamma^\alpha)$ for $0 < s < 1$, because these are defined by pulling back the spaces $W_p^s(I)$ via Lipschitz charts. Since $1 < t < 2$ it is possible to interpolate between $W_p^1(S_\alpha)$ and $W_p^t(S_\alpha)$ and since $0 < 1-1/p < t-1/p < 1$ it is possible to interpolate between $W_p^{1-1/p}(\Gamma^\alpha)$ and $W_p^{t-1/p}(\Gamma^\alpha)$. Choosing the interpolation index appropriately we get that γ maps $W_p^{2/p}(S_\alpha)$ continuously into $W_p^{1/p}(\Gamma^\alpha)$.

the origin, uniformly in α. Consequently, the equivalence of the norm in (2.4) and the $W_p^{1/p}(\Gamma^\alpha)$ norm is uniform in α. It is then easy to see that there is a single constant which bounds the norm of the trace operator between the spaces $W_p^{2/p}(S_\alpha)$ and $W_p^{1/p}(\Gamma^\alpha)$.
(2) Using a partition of unity, we can easily extend Lemma 2.1 to show that trace operator maps $W_p^{2/p}(\Omega)$ continuously into $W_p^{1/p}(\partial\Omega)$ for any polygonal domain Ω.

The next lemma relates the decay of the trace of a $W_p^{2/p}$ function near the vertex to the decay of the function itself.

LEMMA 2.2. *Let $\alpha \in (0, 2\pi)$. Then there exists a constant C such that*

$$\int_0^1 \frac{|u(x,0)|^p}{x}\,dx \le C\left(\|u\|^p_{2/p,p,S_\alpha} + \iint\limits_{S_a} \frac{|u(x,y)|^p}{x^2+y^2}\,dx\,dy\right) \tag{2.5}$$

and

$$\iint\limits_{S_a} \frac{|u(x,y)|^p}{x^2+y^2}\,dx\,dy \le C\left(\|u\|^p_{2/p,p,S_\alpha} + \int_0^1 \frac{|u(x,0)|^p}{x}\,dx\right) \tag{2.6}$$

for all $u \in W_p^{2/p}(S_\alpha)$.[2]

Proof. Suppose θ lies between $\alpha/2$ and α. Then θ is bounded away from 0 and 2π, and so we may find a constant C depending on α but not θ for which

$$\int_0^1 r^{-1}|u(r\cos\theta, r\sin\theta) - u(r,0)|^p\,dr \le C\|u\|^p_{1/p,p,\Gamma^\theta}.$$

Moreover by Lemma 2.1, there is a single constant C such that

$$\|u\|^p_{1/p,p,\Gamma^\theta} \le C\|u\|^p_{2/p,p,S_\theta} \le C\|u\|^p_{2/p,p,S_\alpha}$$

holds for all such θ. Thus

$$\int_0^1 r^{-1}|u(r,0)|^p\,dr \le C\left(\int_0^1 r^{-1}|u(r\cos\theta, r\sin\theta)|^p\,dy + \|u\|^p_{2/p,p,S_\alpha}\right).$$

Integration over θ in $(\alpha/2, \alpha)$ gives (2.5).

To prove (2.6) it suffices to show that

$$\iint\limits_{D} \frac{|u(x,y)|^p}{x^2+y^2}\,dx\,dy \le C\left(\|u\|^p_{2/p,p,D} + \int_0^1 \frac{|u(x,0)|^p}{x}\,dx\right) \tag{2.7}$$

[2]Either of the two integrals entering in (2.5) and (2.6) may be infinite. In that case the inequalities imply that they both are infinite.

where D is the unit disc, since we can always extend a function in $W_p^{2/p}(S_\alpha)$ to one in $W_p^{2/p}(D)$. Considering the trace of u on Γ^π and applying Lemma 2.1, we see that

$$\int_0^1 \frac{|u(-x,0)|^p}{x}\,dx \le C\left(\int_0^1 \frac{|u(x,0)|^p}{x}\,dx + \|u\|_{2/p,p,D}^p\right). \tag{2.8}$$

Let $\theta \in [\pi/2, 3\pi/2]$. Applying the lemma again, we get

$$\int_0^1 r^{-1}|u(r\cos\theta, r\sin\theta) - u(r,0)|^p\,dr \le C\|u\|_{1/p,p,\Gamma^\theta}^p \le C\|u\|_{2/p,p,D}^p,$$

where we may choose C independent of θ. Consequently,

$$\int_0^1 r^{-1}|u(r\cos\theta, r\sin\theta)|^p\,dr \le C\left(\|u\|_{2/p,p,D}^p + \int_0^1 r^{-1}|u(r,0)|^p\,dr\right). \tag{2.9}$$

Similarly for $\theta \in [0,\pi/2]\cup[3\pi/2,2\pi]$, we may consider the trace of u on the boundary of the sector formed by the negative x-axis and the ray emanating from the origin with angle θ and use (2.8) to conclude (2.9). Integration of (2.9) with respect to $\theta \in (0,2\pi)$ gives (2.7). $\square$

3. The main theorem. Homogeneous Boundary Conditions. Let Ω denote a (bounded and simply-connected) polygonal domain in the plane. Denote by z_j, $j = 1,2,\dots N$, the vertices of Ω listed in order as $\partial\Omega$ is traversed counterclockwise, and by Γ_j the open line segment connecting z_{j-1} to z_j (we interpret the subscripts on z and Γ modulo N). We explicitly assume that the magnitude of the angle formed by Γ_j and Γ_{j+1} at z_j lies strictly between 0 and 2π, i.e., we exclude domains with slits. We also denote by $\boldsymbol{\nu}$ the outward-pointing unit vector normal to Ω which is defined on $\partial\Omega\setminus\{z_1,\dots,z_N\}$ and by $\boldsymbol{\nu}_j$ its constant value on Γ_j. Similarly $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_j$ refer to the counterclockwise tangent vector.

Suppose that $\mathbf{U} \in W_p^{s+1}(\Omega)$ satisfies (1.1) and vanishes on $\partial\Omega$. Then, of course, $\int_\Omega f = 0$. In addition, $\partial\mathbf{U}/\partial\boldsymbol{\sigma}_j = 0$ on Γ_j and $\partial\mathbf{U}/\partial\boldsymbol{\sigma}_{j+1} = 0$ on Γ_{j+1}. For $s > 2/p$ the Sobolev imbedding theorem implies that both the directional derivatives $\partial\mathbf{U}/\partial\boldsymbol{\sigma}_j$ and $\partial\mathbf{U}/\partial\boldsymbol{\sigma}_{j+1}$ are continuous on $\bar\Omega$, so they both vanish at z_j. Since the vectors $\boldsymbol{\sigma}_j$ and $\boldsymbol{\sigma}_{j+1}$ are linearly independent, it follows that $f = \operatorname{div}\mathbf{U}$ vanishes at z_j. Define

$$\check W_p^s(\Omega) = \begin{cases} \hat W_p^s(\Omega), & 0 \le s < 2/p,\\ \left\{ f \in \hat W_p^s(\Omega) \mid \int_\Omega \frac{|f(z)|^p}{|z-z_j|^2}\,dz < \infty,\ j = 1,\dots,N \right\}, & s = 2/p,\\ \{\, f \in \hat W_p^s(\Omega) \mid f(z_j) = 0,\ j = 1,\dots,N\,\}, & s > 2/p. \end{cases} \tag{3.1}$$

This space is normed by the restriction of the W_p^s norm except if $s = 2/p$ in which case we take the norm to be

$$\|f\|_{\check W_p^s(\Omega)}^p = \|f\|_{s,p,\Omega}^p + \sum_{j=1}^N \int_\Omega \frac{|f(z)|^p}{|z-z_j|^2}\,dz.$$

We have just seen that for $s \ge 0$, $s \ne 2/p$,

$$\operatorname{div}([W_p^{s+1}(\Omega) \cap \mathring{W}_p^1(\Omega)]^2) \subset \check{W}_p^s(\Omega). \tag{3.2}$$

This is also true if $s = 2/p$, as we now demonstrate. For any $\phi \in W_p^{2/p+1}(\Omega) \cap \mathring{W}_p^1(\Omega)$ the tangential derivative $\partial\phi/\partial\boldsymbol{\sigma}$ vanishes on every edge. In light of (2.6) it follows that

$$\int_\Omega \frac{|\partial\phi(z)/\partial\boldsymbol{\sigma}_j|^p}{|z-z_j|^2}\,dz + \int_\Omega \frac{|\partial\phi(z)/\partial\boldsymbol{\sigma}_{j+1}|^p}{|z-z_j|^2}\,dz < \infty.$$

Since $\boldsymbol{\sigma}_j$ and $\boldsymbol{\sigma}_{j+1}$ are linearly independent,

$$\int_\Omega \frac{|\operatorname{grad}\phi(z)|^p}{|z-z_j|^2}\,dz < \infty.$$

The inclusion (3.2) follows by application of this result componentwise to $\mathbf{U}$.

The next theorem is our main result for homogeneous boundary conditions. It shows that in fact equality holds in (3.2) and that the divergence operator admits a bounded right inverse which does not depend on s or p.

THEOREM 3.1. *Let $1 < p < \infty$, $s \ge 0$. If $p \ne 2$ suppose that $s - 1/p \notin \mathbb{Z}$. Then there exists a bounded linear map*

$$\mathcal{L} : \check{W}_p^s(\Omega) \to [W_p^{s+1}(\Omega) \cap \mathring{W}_p^1(\Omega)]^2$$

such that $\operatorname{div}\mathcal{L}(f) = f$ *for all* $f \in \check{W}_p^s(\Omega)$. *The operator $\mathcal{L}$ may be chosen independently of s and p.*[3]

Remark. We conjecture that the hypothesis that $s - 1/p \notin \mathbb{Z}$ is unnecessary for all p, not just $p = 2$. However this hypothesis is necessary for certain trace results we use in our proof. See the remark to Theorem 4.1.

Several ingredients of the proof of this theorem will be developed in the next three sections. These ingredients assembled, the proof of Theorem 3.1 becomes very short. It is given in the last section of this paper. The analogue of Theorem 3.1 for the case of inhomogeneous boundary conditions is also true. This result is stated as Theorem 7.1.

[3]More precisely, there is a linear operator

$$\mathcal{L} : \bigcup_{s,p} \check{W}_p^s(\Omega) \to [W_1^1(\Omega)]^2$$

such that for each s and p, $\mathcal{L}$ maps $\check{W}_p^s(\Omega)$ boundedly into $[W_p^{s+1}(\Omega) \cap \mathring{W}_p^1(\Omega)]^2$.

4. A result concerning traces on a line. In the case of a smooth boundary, the construction of a solution of (1.1) vanishing on $\partial\Omega$ as sketched in Section 1 was based on the existence of a function w whose curl coincides with the negative gradient of another function u on the boundary. In extending this construction to the case of a polygon, we need to know what vectorfields may arise on a polygonal boundary as the traces of curls or gradients. The main results of section 6, Theorem 6.2 and Corollary 6.3, provide the answer. In deriving them, we need a collection of other trace theorems, starting with those of this section regarding traces of a function of two variables on a line. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space, i.e., the space of C^∞ functions of one variable, all of whose derivatives decrease faster than any polynomial at infinity, and let $\mathcal{S}'(\mathbb{R})$ denote its dual. Let $\bigoplus_{j=0}^\infty \mathcal{S}'(\mathbb{R})$ denote the subset of $\prod_{j=0}^\infty \mathcal{S}'(\mathbb{R})$ consisting of vectors having at most a finite number of non-zero entries. Given subspaces $V_j \subset \mathcal{S}'(\mathbb{R})$, $0 \le j \le m$, we view the Cartesian product $\prod_{j=0}^m V_j$ naturally as a subspace of $\bigoplus_{j=0}^\infty \mathcal{S}'(\mathbb{R})$ by setting all entries with index greater than m to zero.

Theorem 4.1. *Let s and p denote real numbers such that $1 < p < \infty$, $s > 1/p$. If $p \neq 2$ assume that $s - 1/p \notin \mathbb{Z}$. Then the trace map*

$$u \mapsto \gamma_x^j u := \frac{\partial^j u}{\partial y^j}(\,\cdot\,, 0)$$

maps $W_p^s(\mathbb{R}^2)$ boundedly onto $W_p^{s-j-1/p}(\mathbb{R})$ provided $j < s - 1/p$. Moreover, there is a map

$$\mathcal{E}_x : \bigoplus_{j=0}^\infty \mathcal{S}'(\mathbb{R}) \to C^\infty(\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}))$$

such that, if $\mathbf{f} = (f_j) \in \prod_{j=0}^m W_p^{s-j-1/p}(\mathbb{R})$ with $m < s - 1/p$, then

$$\|\mathcal{E}_x \mathbf{f}\|_{s,p,\mathbb{R}^2} \le C_{s,p} \sum_{j=0}^m \|f_j\|_{s-j-1/p,p,\mathbb{R}}$$

and

$$\gamma_x^j \mathcal{E}_x \mathbf{f} = f_j, \qquad j = 0, 1, \ldots, m.$$

Remarks. (1) Henceforth we avoid the case $s - 1/p$ integral if $p \neq 2$, since in this case, the range of the trace operator is a Besov space which does not coincide with any Sobolev space. Cf. [1, Chapter 7].

(2) It follows from the theorem that, if $v = \mathcal{E}_x(f_0, ..., f_m, 0, 0, ...)$ with $f_j \in W_p^{s-j-1/p}(\mathbb{R})$ then the trace

$$\frac{\partial^k v}{\partial y^k}(\,\cdot\,, 0) = 0$$

as a function in $W_p^{s-k-1/p}(\mathbb{R})$, provided $m < k < s - 1/p$. By the Sobolev imbedding theorem,

$$\frac{\partial^{j+k} v}{\partial x^j \partial y^k}(x,0) = 0, \quad x \in \mathbb{R}, \tag{4.1}$$

(in the pointwise sense) provided $m < k$ and $j + k < s - 2/p$.

Proof. The asserted properties of the trace operators γ_x^j are standard, cf. [3] and [7]. Since we require a *fixed* extension operator $\mathcal{E}_x$, independent of s, we include a detailed derivation of this part of the result. Our construction extends that given in [7]. Let S be a function in $\mathcal{S}(\mathbb{R})$ which satisfies

$$\int_{\mathbb{R}} S(t)\,dt = 1, \qquad \int_{\mathbb{R}} t^j S(t)\,dt = 0, \quad j = 1, 2, \ldots.$$

Such a function exists, as is easily seen using the Fourier transform, $\mathcal{F}$. These conditions translate to choosing $(\mathcal{F}S)(0) = 1$ and $(\mathcal{F}S)^{(j)}(0) = 0$ for $j = 1, 2, \ldots$. Since $\mathcal{F}$ is an isomorphism on $\mathcal{S}(\mathbb{R})$, we can choose $S \in \mathcal{S}(\mathbb{R})$ with these properties as follows: let $\chi \in C_0^\infty(\mathbb{R})$ be any function that is identically one in a neighborhood of the origin, and set $S = \mathcal{F}^{-1}\chi$. Now we define

$$\begin{aligned} v(x,y) &= \mathcal{E}_x(f_0, f_1, \ldots)(x,y) \\ &:= \chi(y) \sum_{j=0}^{\infty} \frac{y^j}{j!} \int_{\mathbb{R}} S(t) f_j(x+yt)\,dt, \quad (x,y) \in \mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}), \end{aligned} \tag{4.2}$$

where again, $\chi \in C_0^\infty(\mathbb{R})$ is any function that is identically one in a neighborhood of the origin. Note that only a finite number of terms in (4.2) are non-zero because we are assuming that $(f_0, f_1, \ldots) \in \bigoplus_{j=0}^\infty \mathcal{S}'(\mathbb{R})$. It is easy to verify that v has the asserted traces. The boundedness of $\mathcal{E}_x$ from $\prod_{j=0}^m W_p^{s-j-1/p}(\mathbb{R})$ to $W_p^s(\mathbb{R}^2)$ follows from the following result. ∎

Lemma 4.2. *Let j be a nonnegative integer, $S \in \mathcal{S}(\mathbb{R})$, $\chi \in C_0^\infty(\mathbb{R})$, and $f \in C_0^\infty(\mathbb{R})$. Define*

$$v(x,y) = \chi(y) y^j \int_{\mathbb{R}} S(t) f(x+yt)\,dt, \quad (x,y) \in \mathbb{R}^2.$$

Then, for any real s, p with $1 < p < \infty$ and $s > j + 1/p$ with $s - 1/p \notin \mathbb{Z}$ if $p \neq 2$, there holds

$$\|v\|_{s,p,\mathbb{R}^2} \le C \|f\|_{s-j-1/p,p,\mathbb{R}}.$$

The constant C depends on χ, S, s, and p but not on f.

Proof. Let

$$w(x,y) = y^j \int_{\mathbb{R}} S(t) f(x+yt)\,dt.$$

For any positive integers $k \geq j$, there are coefficients $c_0, \ldots, c_j$ and a function $S_{kj} \in \mathcal{S}(\mathbb{R})$ such that

$$\begin{aligned}
\frac{\partial^k w}{\partial y^k}(x,y) &= \sum_{l=0}^{j} c_l y^{j-l} \int_{\mathbb{R}} S(t) f^{(k-l)}(x+yt) t^{k-l}\, dt \\
&= \sum_{l=0}^{j} c_l \int_{\mathbb{R}} S(t) t^{k-l} \left(\frac{\partial}{\partial t}\right)^{j-l} f^{(k-j)}(x+yt)\, dt \\
&= \int_{\mathbb{R}} \left[\sum_{l=0}^{j} c_l \left(-\frac{\partial}{\partial t}\right)^{j-l} [S(t)t^{k-l}]\right] f^{(k-j)}(x+yt)\, dt \\
&=: \int_{\mathbb{R}} S_{kj}(t) f^{(k-j)}(x+yt)\, dt.
\end{aligned}$$

Similarly, for positive integers $k < j$, there exists a function $S_{kj} \in \mathcal{S}(\mathbb{R})$ such that

$$\frac{\partial^k w}{\partial y^k}(x,y) = y^{j-k} \int_{\mathbb{R}} S_{kj}(t) f(x+yt)\, dt.$$

That is, for any integers k and j,

$$\frac{\partial^k w}{\partial y^k}(x,y) = y^{\max\{j-k,0\}} \int_{\mathbb{R}} S_{kj}(t) f^{(\max\{k-j,0\})}(x+yt)\, dt.$$

For differentiation with respect to x, we get

$$\frac{\partial^k w}{\partial x^k}(x,y) = y^{\max\{j-k,0\}} \int_{\mathbb{R}} \tilde{S}_{kj}(t) f^{(\max\{k-j,0\})}(x+yt)\, dt$$

with some function $\tilde{S}_{kj} \in \mathcal{S}(\mathbb{R})$. Combining these formulas, we have for any multi-index α that

$$D^\alpha w(x,y) = y^{\max\{j-|\alpha|,0\}} \int_{\mathbb{R}} S_\alpha(t) f^{(\max\{|\alpha|-j,0\})}(x+yt)\, dt$$

where $S_\alpha \in \mathcal{S}(\mathbb{R})$. Leibniz' rule then gives

$$\begin{aligned}
D^\alpha v(x,y) &= \sum_{\beta \leq \alpha} c_\beta D^{\alpha-\beta} \chi(y) D^\beta w(x,y) \\
&= \sum_{\beta \leq \alpha} \chi_{\alpha\beta}(y) \int_{\mathbb{R}} S_\beta(t) f^{(\max\{|\beta|-j,0\})}(x+yt)\, dt,
\end{aligned} \tag{4.3}$$

for some $\chi_{\alpha\beta} \in C_0^\infty(\mathbb{R})$.

Now for $S \in \mathcal{S}(\mathbb{R})$, $f \in C_0^\infty(\mathbb{R})$, and any real y, Young's inequality gives

$$\int_{\mathbb{R}} \left|\int_{\mathbb{R}} S(t) f(x+yt)\, dt\right|^p dx \leq \left(\int_{\mathbb{R}} |S(t)|\, dt\right)^p \int_{\mathbb{R}} |f(x)|^p\, dx. \tag{4.4}$$

In view of (4.3) and (4.4) the case of integer s reduces to proving that if $S \in \mathcal{S}(\mathbb{R})$, $f \in C_0^\infty(\mathbb{R})$, then

$$\iint_{\mathbb{R}^2} \left| \int_{\mathbb{R}} S(t) f'(x+yt)\, dt \right|^p dx\, dy \le C_S \|f\|^p_{1-1/p,p,\mathbb{R}}. \tag{4.5}$$

To show this, set

$$\begin{aligned} g(x,y) &= \int_{\mathbb{R}} S(t) f'(x+yt)\, dt = \int_{\mathbb{R}} (-1/y) S'(t) f(x+yt)\, dt \\ &= \int_{\mathbb{R}} (1/y) S'(t)[f(x) - f(x+yt)]\, dt = \int_{\mathbb{R}} t S'(t) \frac{f(x) - f(x+yt)}{yt}\, dt. \end{aligned}$$

From Hölder's inequality, we find

$$|g(x,y)|^p \le C_S \int_{\mathbb{R}} |tS'(t)| \frac{|f(x) - f(x+yt)|^p}{|yt|^p}\, dt.$$

Fubini's Theorem thus yields

$$\begin{aligned} \iint_{\mathbb{R}^2} |g(x,y)|^p\, dx\, dy &\le C_S \iiint_{\mathbb{R}^3} |tS'(t)| \frac{|f(x) - f(x+yt)|^p}{|yt|^p}\, dt\, dx\, dy \\ &= C_S \iiint_{\mathbb{R}^3} |S'(t)| \frac{|f(x) - f(x+z)|^p}{|z|^p}\, dt\, dx\, dz \\ &= C'_S \iint_{\mathbb{R}^2} \frac{|f(x) - f(x+z)|^p}{|z|^p}\, dx\, dz \end{aligned}$$

which is the desired estimate (4.5). This establishes the theorem for integer s. Interpolation between consecutive integers then gives the theorem for all real $s \ge j+1$.

To complete the proof, we now consider the case $j + 1/p < s < j+1$. In view of (4.3) and (4.4) , it suffices to show that for $|\alpha| = j$ we have

$$\begin{aligned} |D^\alpha v|_{s-j,p,\mathbb{R}^2} &:= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|D^\alpha v(x,y) - D^\alpha v(\tilde{x},\tilde{y})|^p}{|(x,y) - (\tilde{x},\tilde{y})|^{2+(s-j)p}}\, dx\, dy\, d\tilde{x}\, d\tilde{y} \\ &\le C \|f\|_{s-j-1/p,p,\mathbb{R}}. \end{aligned}$$

Because of (4.3) this reduces to showing that for $S \in \mathcal{S}(\mathbb{R})$, $f \in C_0^\infty(\mathbb{R})$, and g defined by

$$g(x,y) = \int_{\mathbb{R}} S(t) f(x+yt)\, dt$$

we have

$$|g|_{s,p,\mathbb{R}^2} \le C_S \|f\|_{s-1/p,p,\mathbb{R}} \tag{4.6}$$

provided that $1/p < s < 1$. To show this, we estimate:

$$\begin{aligned}|g(x,y) - g(\tilde{x},\tilde{y})|^p &\le \left|\int_{\mathbb{R}} S(t)[f(x+yt) - f(\tilde{x}+\tilde{y}t)]\,dt\right|^p \\ &\le C_S \int_{\mathbb{R}} |S(t)|\,|f(x+yt) - f(\tilde{x}+\tilde{y}t)|^p\ dt.\end{aligned}$$

Thus

$$|g|^p_{s,p,\mathbb{R}^2} \le C_S \iiiint\!\!\int_{\mathbb{R}^5} |S(t)| \frac{|f(x+yt) - f(\tilde{x}+\tilde{y}t)|^p}{|(x-\tilde{x})^2 + (y-\tilde{y})^2|^{1+sp/2}}\,dt\,dx\,dy\,d\tilde{x}\,d\tilde{y}.$$

A simple change of variables yields

$$\begin{aligned}&\iiint_{\mathbb{R}^3} |S(t)| \frac{|f(x+yt) - f(\tilde{x}+\tilde{y}t)|^p}{|(x-\tilde{x})^2 + (y-\tilde{y})^2|^{1+sp/2}}\,dt\,dx\,d\tilde{x} \\ &\quad = \iiint_{\mathbb{R}^3} |S(t)| \frac{|f(x+yt) - f(\tilde{x}+\tilde{y}t)|^p}{|(x+yt) - (\tilde{x}+\tilde{y}t)|^{sp}} \frac{|(x+yt) - (\tilde{x}+\tilde{y}t)|^{sp}}{|(x-\tilde{x})^2 + (y-\tilde{y})^2|^{1+sp/2}}\,dt\,dx\,d\tilde{x} \\ &\quad = \iiint_{\mathbb{R}^3} \frac{|f(\xi) - f(\eta)|^p}{|\xi-\eta|^{sp}} \frac{|S(t)|\,|\xi-\eta|^{sp}}{|[(\xi - yt) - (\eta - \tilde{y}t)]^2 + (y-\tilde{y})^2|^{1+sp/2}}\,dt\,d\xi\,d\eta.\end{aligned}$$

Thus to verify (4.6) it is sufficient to show that

$$\iiint_{\mathbb{R}^3} \frac{|S(t)|\,dt\,dy\,d\tilde{y}}{|[a - (y-\tilde{y})t]^2 + (y-\tilde{y})^2|^{1+sp/2}} \le \frac{C_S}{|a|^{sp}}, \quad a \in \mathbb{R}.$$

A simple computation gives

$$\int_{\mathbb{R}} \frac{|S(t)|\,dt}{|(a-bt)^2 + b^2|^{1+sp/2}} \le \frac{2^{1+sp/2}}{|a^2+b^2|^{1+sp/2}} \int_{\mathbb{R}} |S(t)|(1+t^2)^{1+sp/2}\,dt = \frac{C_S}{|a^2+b^2|^{1+sp/2}}$$

because

$$[(a-bt)^2 + b^2](1+t^2) \ge (a^2+b^2)/2.$$

Therefore

$$\iiint_{\mathbb{R}^3} \frac{|S(t)|\,dt\,dy\,d\tilde{y}}{|[a - (y-\tilde{y})t]^2 + (y-\tilde{y})^2|^{1+sp/2}} \le C_S \iint_{\mathbb{R}^2} \frac{dy\,d\tilde{y}}{[a^2 + (y-\tilde{y})^2]^{1+sp/2}} = \frac{C'_S}{|a|^{sp}}.$$

This establishes the lemma, and thus the theorem, under the assumption that $f_j \in C_0^\infty(\mathbb{R})$. The theorem now follows for all f_j by a density argument. □

Remark. For $(f_j) \in \bigoplus_{j=0}^\infty \mathcal{S}'(\mathbb{R})$,

$$\frac{\partial^k \mathcal{E}_x(f_j)}{\partial x^k} = \mathcal{E}_x(f_j^{(k)}), \quad k = 1, 2, \ldots.$$

5. Traces on intersecting half-lines. Theorem 4.1 concerns the traces of a W_p^s function on a line. Using a simple argument involving a partition of unity and local coordinates it is easy to extend it to cover traces on smooth curves. Since we are interested in polygons, however, we will need a result analogous to Theorem 4.1 for traces on two intersecting line segments. To state this result, we require some notation. By $\mathbb{R}_+$ we denote the set of positive real numbers. Let

$$I^p(f) = \int_0^1 t^{-1}|f(t)|^p\,dt.$$

For m a nonnegative integer and $s > m + 1/p$ define X_{sp}^m to be the space of pairs of $(m+1)$-tuples

$$\big((f_j)_{j=0}^m, (g_j)_{j=0}^m\big) \in \prod_{j=0}^m W_p^{s-j-1/p}(\mathbb{R}_+) \times \prod_{j=0}^m W_p^{s-j-1/p}(\mathbb{R}_+) \tag{5.1}$$

satisfying

$$f_j^{(k)}(0) = g_k^{(j)}(0), \qquad 0 \le j,k \le m,\ j+k < s - 2/p, \tag{5.2}$$

$$I^p(f_j^{(k)} - g_k^{(j)}) < \infty, \qquad 0 \le j,k \le m,\ j+k = s - 2/p. \tag{5.3}$$

Note that (5.3) only applies if $s - 2/p$ is integral. The space X_{sp}^m is a Banach space with the norm $\|\big((f_j),(g_j)\big)\|_{X_{sp}^m}$ given by

$$\left(\sum_{j=0}^m \Big(\|f_j\|^p_{s-j-1/p,p,\mathbb{R}_+} + \|g_j\|^p_{s-j-1/p,p,\mathbb{R}_+}\Big)\right)^{1/p}$$

if $s - 2/p$ is not an integer, and

$$\left(\sum_{j=0}^m \Big(\|f_j\|^p_{s-j-1/p,p,\mathbb{R}_+} + \|g_j\|^p_{s-j-1/p,p,\mathbb{R}_+}\Big) + \sum_{\substack{0\le j,k\le m\\ j+k=r}} I^p(f_j^{(k)} - g_k^{(j)}) \right)^{1/p}$$

if $r = s - 2/p$ is an integer.

THEOREM 5.1. *Let m be a nonnegative integer and s and p real numbers with $1 < p < \infty$ and $s > m + 1/p$. If $p \ne 2$ suppose that $s - 1/p \ne \mathbb{Z}$. Then the trace map*

$$u \mapsto \left(\left(\frac{\partial^j u}{\partial y^j}(\,\cdot\,,0)\right)_{j=0}^m, \left(\frac{\partial^j u}{\partial x^j}(0,\,\cdot\,)\right)_{j=0}^m\right) \tag{5.4}$$

maps $W^s_p(\mathbb{R}^2)$ boundedly onto X^m_{sp} and admits a bounded right inverse

$$\mathcal{E}^m_{\llcorner} : X^m_{sp} \to W^s_p(\mathbb{R}^2).$$

The operator $\mathcal{E}^m_{\llcorner}$ is independent of s and p.

Proof. Our proof is similar to that of [3, Theorem 1.5.2.4]. First of all, we show that the operator defined in (5.4) maps $W^s_p(\mathbb{R}^2)$ into X^m_{sp}. Let $u \in W^s_p(\mathbb{R}^2)$, and set

$$f_j = \left(\frac{\partial^j u}{\partial y^j}(\,\cdot\,,0)\right), \quad g_j = \left(\frac{\partial^j u}{\partial x^j}(0,\,\cdot\,)\right), \qquad j = 0,\ldots,m.$$

By the Sobolev imbedding theorem $\partial^{k+j}u/\partial x^k\partial y^j$ is continuous on $\mathbb{R}^2$ for $k+j < s-2/p$, so (5.2) holds. If $s = m+2/p$ and $k+j = m$, then we apply Lemma 2.1 to $\partial^m u/\partial x^k \partial y^j$ to infer (5.3). Now we define, in several steps, the operator $\mathcal{E}^m_{\llcorner}$. Let $\mathcal{E}_x$ denote the extension operator in Theorem 4.1. Let $\mathcal{R}^m_x$ denote the vector of $m+1$ restrictions

$$\mathcal{R}^m_x u := \left(u\,(\,\cdot\,,0)\,, \frac{\partial u}{\partial y}(\,\cdot\,,0)\,, \ldots, \frac{\partial^m u}{\partial y^m}(\,\cdot\,,0)\right).$$

Let $\mathcal{E}_y$ and $\mathcal{R}^m_y$ denote the corresponding operators with the x and y variables reversed, e.g.,

$$\mathcal{R}^m_y u := \left(u\,(0,\,\cdot\,)\,, \frac{\partial u}{\partial x}(0,\,\cdot\,)\,, \ldots, \frac{\partial^m u}{\partial x^m}(0,\,\cdot\,)\right).$$

First extend each f_j and g_j as functions in $W^{s-j-1/p}_p(\mathbb{R})$ with an extension that is independent of s and p. (Cf. [9, Chapter 17] or [10, Chapter 6].) Explicitly we take the extension of f_j given by

$$f_j(x) = \int_0^\infty \lambda(s) f_j(-sx)\,ds, \qquad x < 0,$$

and similarly for g_j. Here λ is chosen to be bounded, smooth for $t > 0$, rapidly decreasing at infinity, and satisfying

$$\int_0^\infty (-t)^k \lambda(t)\,dt = 1, \qquad k = 0,1,2,\ldots. \tag{5.5}$$

For example, we can take $\lambda(t) = R(t^{1/2})$ where $R \in \mathcal{S}(\mathbb{R})$ is an odd function that satisfies

$$\int_{\mathbb{R}} R(t) t^{2k+1}\,dt = (-1)^k, \qquad k = 0,1,2,\ldots.$$

In particular, we can take $\mathcal{F}R(\xi) = \mathrm{i}\,(\sinh \xi)\chi(\xi)$ where χ is any even function in $C^\infty_0(\mathbb{R})$ that is identically one in a neighborhood of the origin and $\mathcal{F}$ is the Fourier transform.

Let $\mathbf{f}$ (resp. $\mathbf{g}$) denote the vector of extensions of the boundary data (f_j) on the x-axis (resp. (g_j) on the y-axis). Define

$$v = \mathcal{E}_x(\mathbf{f} - \mathcal{R}_x^m \mathcal{E}_y \mathbf{g}) + \mathcal{E}_y \mathbf{g}.$$

Note that $\mathcal{R}_x^m v = \mathbf{f}$ because $\mathcal{R}_x^m \mathcal{E}_x$ is the identity operator. Also

$$\mathcal{R}_y^m v = \mathcal{R}_y^m \mathcal{E}_x(\mathbf{f} - \mathcal{R}_x^m \mathcal{E}_y \mathbf{g}) + \mathbf{g}.$$

Define

$$\mathbf{h} = \mathbf{g} - \mathcal{R}_y^m v = -\mathcal{R}_y^m \mathcal{E}_x(\mathbf{f} - \mathcal{R}_x^m \mathcal{E}_y \mathbf{g}).$$

If $m < k$ and $k + j < s - 2/p$, then $h_j^{(k)}(0) = 0$ since (4.1) implies

$$\frac{\partial^{j+k}}{\partial x^j \partial y^k} \mathcal{E}_x(\mathbf{f} - \mathcal{R}_x^m \mathcal{E}_y \mathbf{g})(0,0) = 0.$$

If $k \le m$ and $k + j < s - 2/p$ then

$$h_j^{(k)}(0) = g_j^{(k)}(0) - \frac{\partial^{j+k} v}{\partial x^j \partial y^k}(0,0) = g_j^{(k)}(0) - f_k^{(j)}(0)$$

which is zero in view of (5.2). Also, if $k = s - j - 2/p$, then

$$\begin{aligned}
I^p(h_j^{(k)}) &= \int_0^1 t^{-1} \left| g_j^{(k)}(t) - \frac{\partial^{j+k} v}{\partial x^j \partial y^k}(0,t) \right|^p dt \\
&\le 2 \int_0^1 t^{-1} \left| g_j^{(k)}(t) - \frac{\partial^{j+k} v}{\partial x^j \partial y^k}(t,0) \right|^p dt + 2 \int_0^1 t^{-1} \left| \frac{\partial^{j+k} v}{\partial x^j \partial y^k}(t,0) - \frac{\partial^{j+k} v}{\partial x^j \partial y^k}(0,t) \right|^p dt \\
&\le 2 I^p(f_k^{(j)} - g_j^{(k)}) + C \| \frac{\partial^{j+k} v}{\partial x^j \partial y^k} \|^p_{1/p,p,\Gamma^{\pi/2}} \\
&\le 2 I^p(f_k^{(j)} - g_j^{(k)}) + C \|v\|^p_{s,p,\mathbb{R}^2} \\
&\le C \| \left((f_j)_{j=0}^m, (g_j)_{j=0}^m \right) \|_{X_{sp}^m} < \infty,
\end{aligned}$$

where we have used Lemma 2.1 in the third inequality. Thus, the function $\tilde{h}_j$ which agrees with h_j on $\mathbb{R}_+$ and vanishes for $y < 0$ belongs to $W_p^{s-j-1/p}(\mathbb{R})$ and satisfies

$$\|\tilde{h}_j\|_{s-j-1/p,p,\mathbb{R}} \le C \| \left((f_j)_{j=0}^m, (g_j)_{j=0}^m \right) \|_{X_{sp}^m}.$$

The function $w = \mathcal{E}_y(\tilde{h}_0, \ldots, \tilde{h}_m)$ satisfies

$$\frac{\partial^j w}{\partial x^j}(0,y) = \begin{cases} g_j(y) - \dfrac{\partial^j v}{\partial x^j}(0,y), & 0 < y < \infty, \\ 0, & -\infty < y < 0, \end{cases}$$

for $j = 0, \ldots, m$. Finally we set $\mathcal{E}_{\llcorner}^m\left((f_j),(g_j)\right) = u$ with (see (5.5))

$$u(x,y) = v(x,y) + w(x,y) - \int_0^\infty \lambda(s) w(x,-sy)\, ds, \qquad (x,y) \in \mathbb{R}^2.$$

Note that the last term insures that $\dfrac{\partial^j u}{\partial y^j}(x,0) = \dfrac{\partial^j v}{\partial y^j}(x,0) = f_j(x)$ for $j = 0, \ldots, m$. □

6. Extension to general domains. Traces of curls and gradients. We now extend Theorem 5.1 from the case of a single corner to a polygonal domain. For simplicity of notation, and since this is all we need to prove our main theorem, we consider only the cases $m = 0$ and $m = 1$. We use the notation of section 3 for a polygonal domain. If f and g are functions defined on Γ_n and Γ_{n+1} respectively, we set

$$I_n^p(f, g) = \int_0^\epsilon t^{-1} |f(z_n - t\boldsymbol{\sigma}_n) - g(z_n + t\boldsymbol{\sigma}_{n+1})|^p \, dt$$

where $\epsilon = \min_{j=1,\dots,N} |z_j - z_{j-1}|$.

For $s > 1/p$ we define $X_{sp}^0(\partial\Omega)$ to be the space of N-tuples

$$(\phi_n)_{n=1}^N \in \prod_{n=1}^N W_p^{s-1/p}(\Gamma_n) \tag{6.1}$$

satisfying

$$\begin{aligned} I_n^p(\phi_n, \phi_{n+1}) < \infty, &\quad \text{if } s = 2/p \\ \phi_n(z_n) = \phi_{n+1}(z_n), &\quad \text{if } s > 2/p, \end{aligned} \tag{6.2}$$

for $n = 1, \dots, N$. The space $X_{sp}^0(\partial\Omega)$ is a Banach space with the norm $\|(\phi_n)_{n=1}^N\|_{X_{sp}^0(\partial\Omega)}$ given by $(A_1 + A_2)^{1/p}$ where

$$A_1 = \sum_{n=1}^N \|\phi_n\|_{s-1/p,p,\Gamma_n}^p,$$

$$A_2 = \begin{cases} \sum_{n=1}^N I_n^p(\phi_n, \phi_{n+1}), & s = 2/p, \\ 0, & \text{otherwise.} \end{cases}$$

For $s > 1 + 1/p$ we define $X_{sp}^1(\partial\Omega)$ to be the space of N-tuples of pairs

$$(\phi_n, \psi_n)_{n=1}^N \in \prod_{n=1}^N W_p^{s-1/p}(\Gamma_n) \times W_p^{s-1-1/p}(\Gamma_n)$$

satisfying

$$\phi_n(z_n) = \phi_{n+1}(z_n),$$

$$I_n^p\left(\frac{\partial\phi_n}{\partial\boldsymbol{\sigma}_n}\boldsymbol{\sigma}_n + \psi_n\boldsymbol{\nu}_n, \frac{\partial\phi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\boldsymbol{\sigma}_{n+1} + \psi_{n+1}\boldsymbol{\nu}_{n+1}\right) < \infty, \qquad \text{if } s = 1 + 2/p,$$

$$\left(\frac{\partial\phi_n}{\partial\boldsymbol{\sigma}_n}\boldsymbol{\sigma}_n + \psi_n\boldsymbol{\nu}_n\right)(z_n) = \left(\frac{\partial\phi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\boldsymbol{\sigma}_{n+1} + \psi_{n+1}\boldsymbol{\nu}_{n+1}\right)(z_n), \text{ if } s > 1 + 2/p,$$

$$I_n^p\Big(\frac{\partial^2\phi_n}{\partial\boldsymbol{\sigma}_n^2}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_{n+1}+\frac{\partial\psi_n}{\partial\boldsymbol{\sigma}_n}\boldsymbol{\nu}_n\cdot\boldsymbol{\sigma}_{n+1},\frac{\partial^2\phi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}^2}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_{n+1}+\frac{\partial\psi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\boldsymbol{\nu}_{n+1}\cdot\boldsymbol{\sigma}_n\Big)<\infty, \qquad \text{if } s=2+2/p,$$

$$\Big(\frac{\partial^2\phi_n}{\partial\boldsymbol{\sigma}_n^2}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_{n+1}+\frac{\partial\psi_n}{\partial\boldsymbol{\sigma}_n}\boldsymbol{\nu}_n\cdot\boldsymbol{\sigma}_{n+1}\Big)(z_n)=\Big(\frac{\partial^2\phi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}^2}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_{n+1}+\frac{\partial\psi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\boldsymbol{\nu}_{n+1}\cdot\boldsymbol{\sigma}_n\Big)(z_n), \qquad \text{if } s>2+2/p,$$

for $n=1,\ldots,N$. The space $X^1_{sp}(\partial\Omega)$ is a Banach space with the norm

$$\|\,(\phi_n,\psi_n)_{n=1}^N\,\|_{X^1_{sp}(\partial\Omega)}$$

given by $(B_1+B_2)^{1/p}$ where

$$B_1=\sum_{n=1}^N\Big(\|\phi_n\|^p_{s-1/p,p,\Gamma_n}+\|\psi_n\|^p_{s-1-1/p,p,\Gamma_n}\Big)$$

and

$$B_2=\begin{cases}\sum_{n=1}^N I_n^p\big(\frac{\partial\phi_n}{\partial\boldsymbol{\sigma}_n}\boldsymbol{\sigma}_n+\psi_n\boldsymbol{\nu}_n,\frac{\partial\phi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\boldsymbol{\sigma}_{n+1}+\psi_{n+1}\boldsymbol{\nu}_{n+1}\big), & s=1+2/p,\\ \sum_{n=1}^N I_n^p\big(\frac{\partial^2\phi_n}{\partial\boldsymbol{\sigma}_n^2}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_{n+1}+\frac{\partial\psi_n}{\partial\boldsymbol{\sigma}_n}\boldsymbol{\nu}_n\cdot\boldsymbol{\sigma}_{n+1}, & \\ \qquad\frac{\partial^2\phi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}^2}\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_{n+1}+\frac{\partial\psi_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\boldsymbol{\nu}_{n+1}\cdot\boldsymbol{\sigma}_n\big), & s=2+2/p,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 6.1. *Let s and p denote real numbers such that $1<p<\infty$, $s>1/p$. If $p\neq 2$ assume that $s-1/p\notin\mathbb{Z}$. Then the trace map*

$$u\mapsto(u|_{\Gamma_n})_{n=1}^N$$

maps $W_p^s(\Omega)$ boundedly onto $X^0_{sp}(\partial\Omega)$ and admits a bounded right inverse

$$\mathcal{E}^0_\pi: X^0_{sp}(\partial\Omega)\to W^s_p(\Omega).$$

If in addition $s>1+1/p$, then the trace map

$$u\mapsto\left(u|_{\Gamma_n},\left.\frac{\partial u}{\partial\boldsymbol{\nu}_n}\right|_{\Gamma_n}\right)_{n=1}^N$$

maps $W_p^s(\Omega)$ boundedly onto $X^1_{sp}(\partial\Omega)$, and admits a bounded right inverse

$$\mathcal{E}^1_\pi: X^1_{sp}(\partial\Omega)\to W^s_p(\Omega).$$

Both the operators $\mathcal{E}_\pi^0$ and $\mathcal{E}_\pi^1$ are independent of s and p.

Proof. We can cover $\partial\Omega$ with a finite collection of open sets such that each set in this collection may be mapped by an invertible affine map onto the square $(-1,1)\times(-1,1)$ in such a way that the intersection of the set with Ω is mapped onto one of the following sets: $(-1,1)\times(0,1)$, $(0,1)\times(0,1)$, or $(-1,1)\times(-1,1)\setminus[0,1]\times[0,1]$. In light of Theorem 5.1 and the extension theorem for Sobolev spaces, a simple partition of unity argument gives the present result. ∎

We can now characterize the traces of curls of W_p^s functions. Let $Z_p^s(\partial\Omega)$ $(s>1+1/p)$ denote the subspace of N-tuples of vector-valued functions

$$(\boldsymbol{\psi}_n)_{n=1}^N \in \prod_{n=1}^N W_p^{s-1-1/p}(\Gamma_n)\times W_p^{s-1-1/p}(\Gamma_n) \tag{6.3}$$

satisfying

$$\sum_{n=1}^N \int_{\Gamma_n} \boldsymbol{\psi}_n\cdot\boldsymbol{\nu}_n = 0, \tag{6.7}$$

$$I_n^p(\boldsymbol{\psi}_n,\boldsymbol{\psi}_{n+1})<\infty, \qquad \text{if } s=1+2/p, \tag{6.8}$$

$$\boldsymbol{\psi}_n(z_n)=\boldsymbol{\psi}_{n+1}(z_n), \qquad \text{if } s>1+2/p, \tag{6.9}$$

$$I_n^p\Big(\frac{\partial\boldsymbol{\psi}_n\cdot\boldsymbol{\nu}_{n+1}}{\partial\boldsymbol{\sigma}_n},\frac{\partial\boldsymbol{\psi}_{n+1}\cdot\boldsymbol{\nu}_n}{\partial\boldsymbol{\sigma}_{n+1}}\Big)<\infty, \qquad \text{if } s=2+2/p, \tag{6.10}$$

$$\frac{\partial\boldsymbol{\psi}_n\cdot\boldsymbol{\nu}_{n+1}}{\partial\boldsymbol{\sigma}_n}(z_n)=\frac{\partial\boldsymbol{\psi}_{n+1}\cdot\boldsymbol{\nu}_n}{\partial\boldsymbol{\sigma}_{n+1}}(z_n), \qquad \text{if } s>2+2/p. \tag{6.11}$$

This space is normed in the usual way.

Theorem 6.2. *Let s and p denote real numbers such that $1<p<\infty$, $s>1+1/p$. If $p\neq 2$ assume that $s-1/p\notin\mathbb{Z}$. Then the operator*

$$u\mapsto(\operatorname{curl} u|_{\Gamma_n})_{n=1}^N \tag{6.12}$$

maps $W_p^s(\Omega)$ boundedly onto $Z_p^s(\partial\Omega)$, and admits a bounded right inverse $\mathcal{E}_{\mathrm{curl}}$ which is independent of s and p.

Proof. That the operator in (6.12) maps u into $Z_p^s(\partial\Omega)$ follows from the continuity and trace properties of $\operatorname{curl} u$ and $\partial^2 u/\partial\boldsymbol{\sigma}_n\partial\boldsymbol{\sigma}_{n+1}$. Given $(\boldsymbol{\psi}_n)$ we define $\mathcal{E}_{\mathrm{curl}}\big((\boldsymbol{\psi}_n)\big)$ as follows. For $z\in\partial\Omega$ let

$$\phi(z)=\oint_{z_1}^{z}\boldsymbol{\psi}\cdot\boldsymbol{\nu}_\Gamma\,ds_\Gamma,$$

(the integral taken counterclockwise along Γ) and set

$$\phi_n = \phi|_{\Gamma_n}, \quad \psi_n = -\boldsymbol{\psi}_n \cdot \boldsymbol{\sigma}_n, \qquad n = 1, \ldots, N.$$

It is easy to verify that $(\boldsymbol{\psi}_n) \in Z_p^s(\partial\Omega)$ implies $(\phi_n, \psi_n) \in X_{sp}^1(\partial\Omega)$. We set

$$\mathcal{E}_{\mathrm{curl}}\big((\boldsymbol{\psi}_n)\big) = \mathcal{E}_\pi^1\big((\phi_n, \psi_n)\big). \quad \square$$

We conclude this section by showing that the trace of the gradient of a $W_p^s(\Omega)$ function is also in $Z_p^s(\partial\Omega)$ as long as the function satisfies some necessary compatibility conditions. Namely we define the subspace $\overset{\Delta}{W}{}_p^s(\Omega)$ of $W_p^s(\Omega)$ for $s \geq 2$ as follows. For $s < 2 + 2/p$,

$$\overset{\Delta}{W}{}_p^s(\Omega) = \left\{ u \in W_p^s(\Omega) \mid \int_\Omega \Delta u = 0 \right\}; \tag{6.13}$$

for $s > 2 + 2/p$,

$$\overset{\Delta}{W}{}_p^s(\Omega) = \left\{ u \in W_p^s(\Omega) \mid \int_\Omega \Delta u = 0,\ \Delta u(z_j) = 0,\ j = 1, \ldots, N \right\};$$

and for $s = 2 + 2/p$,

$$\overset{\Delta}{W}{}_p^s(\Omega) = \left\{ u \in W_p^s(\Omega) \mid \int_\Omega \Delta u = 0,\ \int_\Omega \frac{|\Delta u(z)|^p}{|z - z_j|^2}\, dz < \infty,\ j = 1, \ldots, N \right\}.$$

The Banach norm on $\overset{\Delta}{W}{}_p^s(\Omega)$ is the restriction of the W_p^s norm except if $s = 2 + 2/p$, in which case

$$\|u\|^p_{\overset{\Delta}{W}{}_p^s(\Omega)} = \|u\|^p_{s,p,\Omega} + \sum_{j=1}^N \int_\Omega \frac{|\Delta u(z)|^p}{|z - z_j|^2}\, dz.$$

Corollary 6.3. *If $s \geq 2$, then the operator*

$$u \mapsto (\operatorname{grad} u|_{\Gamma_n})_{n=1}^N$$

maps $\overset{\Delta}{W}{}_p^s(\Omega)$ boundedly into $Z_p^s(\partial\Omega)$.

Proof. It suffices to verify conditions (6.7)-(6.11) for $\boldsymbol{\psi} = \operatorname{grad} u$, $u \in \overset{\Delta}{W}{}_p^s(\Omega)$. The first condition holds since $\int \Delta u = 0$ for $u \in \overset{\Delta}{W}{}_p^s(\Omega)$:

$$\sum_{n=1}^N \int_{\Gamma_n} (\operatorname{grad} u) \cdot \boldsymbol{\nu}_n = \int_\Gamma \frac{\partial u}{\partial \boldsymbol{\nu}} = \int_\Omega \Delta u = 0.$$

Condition (6.9) follows from the continuity of $\operatorname{grad} u$ on $\bar{\Omega}$ if $s > 1 + 2/p$, and similarly (6.8) follows from the inclusion of $(\operatorname{grad} u)|_\Gamma$ in $W_p^{1/p}(\Gamma)$ if $s = 1 + 2/p$. To verify (6.10) and (6.11), we note that $(\operatorname{grad} u) \cdot \boldsymbol{\nu} = \partial u / \partial \boldsymbol{\nu}$, so

$$\frac{\partial(\operatorname{grad} u \cdot \boldsymbol{\nu}_{n+1})}{\partial \boldsymbol{\sigma}_n} - \frac{\partial(\operatorname{grad} u \cdot \boldsymbol{\nu}_n)}{\partial \boldsymbol{\sigma}_{n+1}} = \frac{\partial^2 u}{\partial \boldsymbol{\sigma}_n \partial \boldsymbol{\nu}_{n+1}} - \frac{\partial^2 u}{\partial \boldsymbol{\sigma}_{n+1} \partial \boldsymbol{\nu}_n} = -(\boldsymbol{\sigma}_{n+1} \cdot \boldsymbol{\nu}_n)\Delta u.$$

If $s > 2 + 2/p$, then $\Delta u(z_j) = 0$ by definition, so (6.11) holds. If $s = 2 + 2/p$, then

$$\begin{aligned} I_n^p & \left(\frac{\partial(\operatorname{grad} u \cdot \boldsymbol{\nu}_{n+1})}{\partial \boldsymbol{\sigma}_n}, \frac{\partial(\operatorname{grad} u \cdot \boldsymbol{\nu}_n)}{\partial \boldsymbol{\sigma}_{n+1}} \right) \\ &= \int_0^\epsilon t^{-1} \left| \frac{\partial^2 u}{\partial \boldsymbol{\sigma}_n \partial \boldsymbol{\nu}_{n+1}}(z_n - t\boldsymbol{\sigma}_n) - \frac{\partial^2 u}{\partial \boldsymbol{\sigma}_{n+1} \partial \boldsymbol{\nu}_n}(z_n + t\boldsymbol{\sigma}_{n+1}) \right|^p dt \\ &\le C \left(\int_0^\epsilon t^{-1} \left| \frac{\partial^2 u}{\partial \boldsymbol{\sigma}_n \partial \boldsymbol{\nu}_{n+1}}(z_n - t\boldsymbol{\sigma}_n) - \frac{\partial^2 u}{\partial \boldsymbol{\sigma}_n \partial \boldsymbol{\nu}_{n+1}}(z_n + t\boldsymbol{\sigma}_{n+1}) \right|^p dt \right. \\ &\qquad \left. + \int_0^\epsilon t^{-1} |\Delta u(z_n + t\boldsymbol{\sigma}_{n+1})|^p \, dt \right) \\ &\le C \left(\|u\|_{2+2/p,p,\Omega}^p + \int_0^\epsilon t^{-1} |\Delta u(z_n + t\boldsymbol{\sigma}_{n+1})|^p \, dt \right). \end{aligned}$$

For the last inequality we have used Lemma 2.1. The estimate (6.10) now follows immediately in light of Lemma 2.2. ∎

7. Proof of the main result. Inhomogeneous boundary conditions. We now prove Theorem 3.1. Given $f \in \check{W}_p^s(\Omega)$ for some $1 < p < \infty$ and $s \ge 0$ with $s - 1/p$ nonintegral in the case $p \ne 2$, extend f boundedly to an open disc containing $\bar{\Omega}$ and define u as the solution to $\Delta u = f$ which vanishes on the boundary of the disk. Then $u \in \overset{\Delta}{W}{}_p^{s+2}(\Omega)$ and $\|u\|_{\overset{\Delta}{W}{}_p^{s+2}(\Omega)} \le C \|f\|_{\check{W}_p^s(\Omega)}$. In light of Corollary 6.3 and Theorem 6.2, we may define $w = -\mathcal{E}_{\mathrm{curl}} \left((\operatorname{grad} u|_{\Gamma_n})_{n=1}^N \right)$, and thus we have $\operatorname{curl} w = -\operatorname{grad} u$ on $\partial\Omega$ and $\|w\|_{s+2,p,\Omega} \le C \|u\|_{\overset{\Delta}{W}{}_p^{s+2}(\Omega)}$. Setting $\mathcal{L}(f) = \operatorname{grad} u + \operatorname{curl} w$ gives the desired operator. This completes the proof.

Let us consider the case of inhomogeneous boundary data. For $s > 1/p$ we set

$$\mathbf{X}_{sp}^0(\partial\Omega) = \left[X_{sp}^0(\partial\Omega) \right]^2,$$

so $\mathbf{g} \in \mathbf{X}_{sp}^0(\partial\Omega)$ means that $\mathbf{g} = (\mathbf{g}_1, \cdots, \mathbf{g}_N)$ with $\mathbf{g}_n \in \left[W_p^{s-1/p}(\Gamma_n) \right]^2$ satisfying the conditions given by (6.2). For $s \ge 0$ we define $V_p^s(\Omega)$ to be the space consisting of those pairs

$$(f, \mathbf{g}) \in W_p^s(\Omega) \times \mathbf{X}_{s+1,p}^0(\partial\Omega) \tag{7.1}$$

for which

$$\int_{\Omega} f = \int_{\partial\Omega} \mathbf{g}\cdot\boldsymbol{\nu}, \tag{7.2}$$

$$I_n^p\Big(\frac{\partial \mathbf{g}_{n+1}\cdot\boldsymbol{\nu}_n}{\partial\boldsymbol{\sigma}_{n+1}} - \frac{\partial \mathbf{g}_{n}\cdot\boldsymbol{\nu}_{n+1}}{\partial\boldsymbol{\sigma}_{n}}, \boldsymbol{\nu}_n\cdot\boldsymbol{\sigma}_{n+1} f\Big) < \infty, \quad \text{if } s = 2/p, \tag{7.3}$$

$$\left(\frac{\partial \mathbf{g}_{n+1}\cdot\boldsymbol{\nu}_n}{\partial\boldsymbol{\sigma}_{n+1}} - \frac{\partial \mathbf{g}_{n}\cdot\boldsymbol{\nu}_{n+1}}{\partial\boldsymbol{\sigma}_{n}}\right)(z_n) = \boldsymbol{\nu}_n\cdot\boldsymbol{\sigma}_{n+1} f(z_n), \quad \text{if } s > 2/p. \tag{7.4}$$

It is not difficult to see that a necessary condition for the existence of $\mathbf{U} \in \left[W_p^{s+1}(\Omega)\right]^2$ satisfying

$$\begin{aligned} \operatorname{div}\mathbf{U} &= f \quad \text{in } \Omega, \\ \mathbf{U} &= \mathbf{g} \quad \text{on } \partial\Omega, \end{aligned} \tag{7.5}$$

is that $(f,\mathbf{g}) \in V_p^s(\Omega)$. The following theorem shows that this condition is also sufficient and that the problem (7.5) admits a bounded right inverse which does not depend on s or p.

Theorem 7.1. *Let $1 < p < \infty$, $s \geq 0$. If $p \neq 2$ suppose that $s - 1/p \notin \mathbb{Z}$. Then there exists a bounded linear map*

$$\mathcal{K} : V_p^s(\Omega) \to \left[W_p^{s+1}(\Omega)\right]^2$$

such that for any $(f,\mathbf{g}) \in V_p^s(\Omega)$, the function $\mathbf{U} := \mathcal{K}(f,\mathbf{g})$ solves (7.5). The operator $\mathcal{K}$ may be chosen independent of s and p.

Proof. For $(f,\mathbf{g}) \in V_p^s(\Omega)$, the function $\mathbf{V} = \mathcal{E}_\pi^0\mathbf{g} \in \left[W_p^{s+1}(\Omega)\right]^2$ satisfies $\mathbf{V} = \mathbf{g}$ on $\partial\Omega$. Indeed, this follows from Theorem 6.1 since $\mathbf{g} \in \mathbf{X}_{s+1,p}^0(\partial\Omega)$. The conditions (7.2)–(7.4) have been chosen precisely to guarantee that

$$f - \operatorname{div}\mathbf{V} \in \check{W}_p^s(\Omega).$$

Therefore the operator

$$\mathcal{K}(f,\mathbf{g}) := \mathcal{L}(f - \operatorname{div}\mathbf{V}) + \mathbf{V} = \mathcal{L}(f - \operatorname{div}\mathcal{E}_\pi^0\mathbf{g}) + \mathcal{E}_\pi^0\mathbf{g}$$

has the desired properties. ∎

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