The Delta-Trigonometric Method using the Single-Layer Potential Representation

Raymond Sheng-Chieh Cheng
Global Associates, Ltd.
1423 Powhatan Street
Alexandria, VA 22314

Douglas N. Arnold*
Department of Mathematics
University of Maryland
College Park, MD 20742

Abstract

The Dirichlet problem for Laplace's equation is often solved by means of the single layer potential representation, leading to a Fredholm integral equation of the first kind with logarithmic kernel. We propose to solve this integral equation using a Petrov-Galerkin method with trigonometric polynomials as test functions and, as trial functions, a span of delta distributions centered at boundary points. The approximate solution to the boundary value problem thus computed converges exponentially away from the boundary and algebraically up to the boundary. We show that these convergence results hold even when the discretization matrices are computed via numerical quadratures. Finally, we discuss our implementation of this method using the fast Fourier transform to compute the discretization matrices, and present numerical experiments in order to confirm our theory and to examine the behavior of the method in cases where the theory doesn't apply due to lack of smoothness.

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1. Introduction

We study numerical methods for solving the Dirichlet problem,

$$\Delta u = 0 \quad \text{on} \quad \mathbb{R}^2 \backslash \Gamma, \qquad u = G \quad \text{on} \quad \Gamma,$$

based on a single-layer potential representation where Γ is a simple closed analytic curve, G is an analytic function, and u is bounded at infinity. The single-layer potential representation is:

$$u(z) = \int_\Gamma \Phi(y) \log|z-y|\, d\sigma_y \quad \text{for} \quad z \in \mathbb{R}^2, \tag{1.1}$$

where Φ is the density. For any harmonic u, there exists a unique Φ satisfying the representation (1.1) if the conformal radius of Γ does not equal 1. The density Φ solves the boundary integral equation,

$$G(z) = \int_\Gamma \Phi(y) \log|z-y|\, d\sigma_y \quad \forall \;\; z \in \Gamma. \tag{1.2}$$

REMARK: There are two ways to handle the uniqueness problem when the conformal radius equals 1 [8]. One approach is to add an unknown constant to the right side of (1.1) and (1.2), c.f. [2]. The other approach is to scale the domain so that the conformal radius does not equal 1. For more details, see [7, appendix]. For simplicity, we assume that the conformal radius does not equal 1.

In this paper, we use Petrov-Galerkin methods to approximate Φ in (1.2). Then we approximate the potential u by using the approximate density instead of Φ in equation (1.1). A Petrov-Galerkin method is specified by choosing the space of trial functions and the space of test functions. These methods usually require integrations over Γ and therefore we study the effects of numerical integration.

Two common choices of trial spaces are spline spaces and spaces of trigonometric polynomials. Another possibility is to use the span of a finite set of delta functions. We call a linear combination of delta functions a spline of degree -1. In this case, the approximate potential has the form:

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log|z-y_j| \quad \text{for} \quad z \in \mathbb{R}^2, \tag{1.3}$$

where the y_j's are given points on the boundary and the α_j's are the unknown coefficients. An advantage of using a sum of delta functions instead of a spline function is that no numerical integration is needed to compute the action of the integral operator on the trial function. Furthermore, the computation of the approximate potential in equation (1.3) does not require any further quadrature after the trial function is found.

Common Petrov-Galerkin methods are collocation methods, least square methods, and methods involving spline or trigonometric trial and test spaces. Spline-collocation methods (splines as trial functions and collocation of the boundary integral equation (1.2)) are known to give the optimal asymptotic convergence rates in certain Sobolev spaces.

The optimal asymptotic convergence rates are also achieved for elliptic equations of other orders. For more details, see Arnold and Wendland [3–5], Saranen and Wendland [23], Prössdorf and Schmidt [19, 20], Prössdorf and Rathsfeld [17, 18], and Schmidt [24].

Spline-spline Galerkin methods obtain the optimal convergence rates in a wider range of spaces than spline-collocation methods. However, they are more costly to implement. For more details, see Arnold and Wendland [3, 4], Hsiao, Kopp, and Wendland [11, 12] and Ruotsalainen and Saranen [21, pg. 5].

Ruotsalainen and Saranen [21] proved that the delta-spline Petrov-Galerkin method (splines of degree -1 as trial functions and ordinary splines as test functions) achieves optimal asymptotic convergence rates. The advantages of their method compared to spline-spline and spline-collocation methods are that fewer numerical integrations are needed and a lesser regularity is required of the boundary data. Numerical results were presented by Lusikka, Ruotsalainen, and Saranen [15].

Arnold [2] showed that the approximate potentials produced by the spline-trigonometric method (splines as trial functions and trigonometric polynomials as test functions) converge exponentially (in the L^∞ norm) on compact sets disjoint from Γ and algebraically up to the boundary. McLean [16] showed that the approximate potentials produced by the trigonometric-trigonometric Galerkin method converge exponentially on all of $\mathbb{R}^2$. Neither Arnold nor McLean took into account the effect of quadrature errors.

In this paper, we consider delta-trigonometric Petrov-Galerkin method. That is, we take the approximate potential to be of the form (1.3) and determine the unknown coefficients α_j by restricting (1.3) to Γ and using orthogonality to trigonometric polynomials. We consider also the fully discrete case, in which a quadrature rule is applied in computing the orthogonalities. We show that for both the semidiscrete and fully discrete methods the approximate potentials converge exponentially quickly on compact sets disjoint from Γ. The potential converges at an algebraic rate up to the boundary.

The paper is organized as follows. In section 2, we present the delta-trigonometric Petrov-Galerkin method and define the corresponding matrices with and without numerical quadrature. In section 3, we show that the approximate potentials produced by the delta-trigonometric Petrov-Galerkin method converge exponentially (in the L^∞ norm) on compact sets disjoint from the boundary and algebraically in a global weighted Sobolev norm. We also show that the condition numbers of the corresponding matrices are bounded proportionally to the numbers of subintervals. In section 4, we show that the convergence rates do not change when we use appropriate quadrature rules. This is significant since now we have a fully discrete method using the single-layer potential representation (1.1) which approximates the potential exponentially. In section 5, we discuss our implementation of this method using the fast Fourier transform and present computer results which confirm our theoretical analyses.

We conclude this section by collecting some notation to be used below. Let $\mathbb{Z}^+$ denote the set of positive integers and $\mathbb{Z}^*$ the set of nonzero integers. We define the space of trigonometric polynomials with complex coefficients,

$$T := \operatorname{span}\{\exp(2\pi i k t) \,|\, k \in \mathbb{Z}\}.$$

Any function f in this space can be represented as

$$f(t) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\exp(2\pi ikt)$$

where

$$\widehat{f}(k) := \int_0^1 f(t)\exp(-2\pi ikt)\ dt$$

are arbitrary complex numbers, all but finitely many zero.

For $f \in T$, $s \in \mathbb{R}$, and $\epsilon > 0$, we define the Fourier norm [2, section 3]

$$\|f\|_{s,\epsilon} := \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s}$$

where

$$\underline{k} := \begin{cases} 1, & \text{if } k = 0, \\ 2\pi|k|, & \text{if } k \neq 0. \end{cases}$$

We denote by $X_{s,\epsilon}$ the completion of T with respect to this norm. The L^2 innerproduct,

$$(f, g) := \int_0^1 f\overline{g} = \sum \widehat{f}(k)\overline{\widehat{g}(k)},$$

extends to a bounded binear form on $X_{s,\epsilon} \times X_{-s,\epsilon^{-1}}$ for all $s \in \mathbb{R}$, $\epsilon > 0$, and allows us to identify $X_{-s,\epsilon^{-1}}$ with the dual space of $X_{s,\epsilon}$. In case $\epsilon = 1$, $X_{s,\epsilon}$ is the usual periodic Sobolev space of of order s, and we use the more common notations $H^s = X_{s,1}$ and $\|\cdot\|_t = \|\cdot\|_{t,1}$. See [2, section 3] for a more complete discussion of these spaces.

We denote by $L(X, Y)$ the set of bounded linear functions from X to Y. The standard Euclidean norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$ as is the associated matrix norm. We use C and ϵ to denote generic positive constants, not necessarily the same in each occurence.

2. The Delta-Trigonometric Method

Let $x : \mathbb{R} \to \Gamma$ be a 1-periodic analytic function which parametrizes Γ and has non-vanishing derivatives, and define

$$\phi(t) = \Phi(x(t))\left|\frac{dx}{dt}(t)\right|, \quad g(t) = G(x(t))\left|\frac{dx}{dt}(t)\right|.$$

Next, we define three integral operators in $L(X_{s,\epsilon}, X_{s+1,\epsilon})$. Let

$$A\phi(s) := \int_0^1 \phi(t)\log|x(s) - x(t)|\,dt, \tag{2.1}$$

$$V\phi(s) := \int_0^1 \phi(t)\log|2\sin(\pi(s - t))|\,dt, \tag{2.2}$$

and

$$B\phi(s) := A\phi(s) - V\phi(s) = \int_0^1 \phi(t)K(s,t)\,dt,$$

where $K : \mathbb{R}^2 \to \mathbb{R}$ is a smooth kernel defined by

$$K(s,t) := \begin{cases} \log\left|\dfrac{x(s)-x(t)}{2\sin\pi(s-t)}\right|, & \text{if } s-t \notin \mathbb{Z}, \\ \log\left|\dfrac{x'(s)}{2\pi}\right|, & \text{if } s-t \in \mathbb{Z}. \end{cases} \tag{2.3}$$

Then the single-layer potential representation (1.1) becomes

$$u(z) := \int_0^1 \phi(t)\log|z-x(t)|\,dt \quad \forall\ z \in \mathbb{R},$$

and our boundary integral equation (1.2) becomes

$$A\phi(s) = g(s) \quad \forall\ s \in [0,1].$$

The operator V is the principal part of A, the remainder B having smooth kernel. The importance of this splitting is that the Fourier transform of $V\phi$ can be calculated analytically. This fact will be useful for proving the inf-sup condition for A in the finite-dimensional spaces and for the numerical implementation.

Let n be a positive odd number and

$$\Lambda_n := \{k \in \mathbb{Z} \ \Big|\ |k| \le (n-1)/2\}.$$

For $j = 1, \ldots, n$, let $\delta(t - j/n)$ denote the 1-periodic extension of the Dirac mass at j/n. As trial space we select

$$S_n = \operatorname{span}\{\delta(t-j/n) \ \Big|\ j = 1, \ldots, n\}.$$

This space can be characterized as

$$S_n = \{\rho \in H^{-1}([0,1]) \ \Big|\ \widehat{\rho}(m) = \widehat{\rho}(m+n) \quad \forall\ m \in \mathbb{Z}\}.$$

As test space, we choose

$$T_n := \ \operatorname{span}\{\exp(2\pi i k t) \ \Big|\ k \in \Lambda_n\},$$

the space of trigonometric polynomials with degree $\le n$.

The semidiscrete delta-trigonometric method seeks $\phi_n \in S_n$ such that

$$\int_0^1 A\phi_n(s)\psi(s)\,ds = \int_0^1 g(s)\psi(s)\,ds \quad \forall\ \psi \in T_n, \tag{2.4}$$

and takes as the approximate potential

$$u_n(z) = \int_0^1 \phi_n(t) \log|z - x(t)|\, dt \quad \forall\ z \in \mathbb{R}^2. \tag{2.5}$$

Since $\phi_n \in S_n$, the last integral is really the sum (1.3).

We now define the matrix equations with and without numerical quadratures for the delta-trigonometric method. To reduce (2.4) to a matrix equation we write the approximate density (trial function) as

$$\phi_n(t) = \sum_{j=1}^{n} \alpha_j \delta(t - j/n) \tag{2.6}$$

where the α_j are unknown coefficients, and we take as basis functions for test space T_n

$$\psi_k(s) = \exp(2\pi i k s),\ k \in \Lambda_n.$$

Define $n \times n$ matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{V}$ and an n-vector $\mathbf{g}$ by

$$\mathbf{A}_{kj} := \int_0^1 \log\big|x(s) - x(j/n)\big|\psi_k(s)\, ds, \tag{2.7}$$

$$\mathbf{B}_{kj} := \int_0^1 K(s, j/n)\psi_k(s)\, ds,$$

$$\mathbf{V}_{kj} := \int_0^1 \log\big|2\sin(\pi(s - j/n))\big|\psi_k(s)\, ds,$$

$$\mathbf{g}_k := \int_0^1 g(s)\psi_k(s)\, ds,$$

for $k \in \Lambda_n$, $j = 1, \ldots, n$. Then the matrix form of equation (2.4) is

$$\mathbf{A}\alpha = \mathbf{g}$$

(where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$) and the approximate potential given in (2.5) may be written

$$u_n(z) = \sum_{j=1}^{n} \alpha_j \log|z - x(j/n)| \quad \forall\ z \in \mathbb{R}.$$

Now $\mathbf{V}_{kj}$ can be calculated explicitly. The Fourier transform of

$$F(\theta) := \pi^{-1} \log|2\sin(\pi\theta)| + 1$$

is $\widehat{F}(k) = 1/\underline{k}$ (see [2, section 4]). Therefore,

$$\begin{aligned}
\mathbf{V}_{kj} &= \int_0^1 \log|2\sin(\pi(s - j/n))| \;\; \psi_k(s)\, ds \\
&= \int_0^1 \log|2\sin(\pi\theta)| \;\; \psi_k(\theta + j/n)\, d\theta \\
&= \int_0^1 \log|2\sin(\pi\theta)| \;\; \psi_k(\theta) \;\; \psi_k(j/n)\, d\theta \\
&= \int_0^1 -\pi F(\theta)\psi_k(\theta)\, d\theta \;\; \psi_k(j/n) + \pi \int_0^1 \psi_k(\theta)\, d\theta \;\; \psi_k(j/n) \\
&= \frac{-\pi}{\underline{k}}\psi_k(j/n) + \pi \int_0^1 \psi_k(\theta)\, d\theta \;\; \psi_k(j/n),
\end{aligned}$$

or

$$\mathbf{V}_{kj} = \begin{cases} -\psi_k(j/n)/(2|k|), & \text{if } k \neq 0, \\ 0, & \text{if } k = 0. \end{cases}$$

To obtain a fully discrete method we use the trapezoidal rule to evaluate $\mathbf{B}$ and $\mathbf{g}$. Thus set

$$\widetilde{\alpha} := (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)^T,$$

$$\widetilde{\mathbf{B}}_{kj} := \frac{1}{n} \sum_{l=1}^{n} K(l/n, j/n)\psi_k(l/n),$$

$$\widetilde{\mathbf{g}}_k := \frac{1}{n} \sum_{l=1}^{n} g(l/n)\psi_k(l/n),$$

$k \in \Lambda_n$, $j = 1, \ldots, n$.

The delta-trigonometric method with numerical quadrature defines the approximate density

$$\widetilde{\phi}_n(t) = \sum_{j=1}^{n} \widetilde{\alpha}_j \delta(t - j/n)$$

where $\widetilde{\alpha} \in \mathbb{R}^n$ is determined from the matrix equation

$$\widetilde{\mathbf{A}}\widetilde{\alpha} = \widetilde{\mathbf{B}}\widetilde{\alpha} + \mathbf{V}\widetilde{\alpha} = \widetilde{\mathbf{g}}.$$

The corresponding approximate potential is

$$\widetilde{u}_n(z) := \sum_{j=1}^{n} \widetilde{\alpha}_j \log|z - x(j/n)| \quad \forall \;\; z \in \mathbb{R}^2. \tag{2.8}$$

3. Convergence of the Semidiscrete Delta-Trigonometric Method

In this section, we show convergence for the approximate potentials produced by the delta-trigonometric method by extending the convergence analyses for the spline-trigonometric method given by Arnold [2]. (In [2], a constant is added to the single layer potential representation to handle the uniqueness problem, rather than scaling the domain. This involves only minor changes in the analyses.) We also present bounds for the condition numbers of the corresponding matrices. Since the analysis is a straightforward adaptation of [2], we present most proofs briefly, and refer the reader to [2, sections 4–6] and [7, sections 3.1 and 3.2] for details.

Since $\widehat{V\phi}(0)$ is zero whenever ϕ is a constant function, we need an additional term. Let

$$M\phi := \int_0^1 \phi(t)\,dt.$$

Theorems 3.1 and 3.3 state the inf-sup condition for the operators $V_1 := V - \pi M$ (see (2.2)) and A (see (2.1)). In theorems 3.5 and 3.6 we give exponential convergence results for the approximate densities and approximate potentials.

THEOREM 3.1 *Let $s \le s_0 < -1/2$. Then there exists a constant C depending only on s_0 such that*

$$\inf_{0\neq\rho\in S_n}\ \sup_{0\neq\sigma\in T_n} \frac{(V_1\rho,\sigma)}{\|\rho\|_{s,\epsilon}\|\sigma\|_{-s-1,\epsilon^{-1}}} \ge C$$

for all $\epsilon \in (0,1]$ and $n \in \mathbb{Z}^+$.

PROOF: The proof is similiar to [2]. We first show that there exists a constant C_1 depending only on s_0 such that

$$\|\rho\|_{s,\epsilon}^2 \le C_1 \sum_{\rho\in\Lambda_n} |\widehat{\rho}(p)|^2 \epsilon^{2|p|} \underline{p}^{2s} \quad \forall\ \rho \in S_n. \tag{3.1}$$

Recalling that $S_n = \{\rho \in H^{-1}([0,1])|\ \widehat{\rho}(m) = \widehat{\rho}(m+n),\ \forall\ m \in \mathbb{Z}\}$, we have for all $\rho \in S_n$,

$$\begin{aligned}\|\rho\|_{s,\epsilon}^2 &= \sum_{k\in\mathbb{Z}} |\widehat{\rho}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} = \sum_{p\in\Lambda_n}\sum_{m\in\mathbb{Z}} |\widehat{\rho}(p+mn)|^2 \epsilon^{2|p+mn|} (\underline{p+mn})^{2s} \\ &= \sum_{p\in\Lambda_n} |\widehat{\rho}(p)|^2 \epsilon^{2|p|} \underline{p}^{2s} \sum_{m\in\mathbb{Z}} \epsilon^{2|p+mn|-2|p|} (\underline{p+mn}/\underline{p})^{2s}.\end{aligned}$$

Note that $|p+mn| - |p| \ge 0$ and $\epsilon \in (0,1]$, so $\epsilon^{2|p+mn|-2|p|} \le 1$. Thus,

$$\|\rho\|_{s,\epsilon}^2 \le \sum_{p\in\Lambda_n} |\widehat{\rho}(p)|^2 \epsilon^{2|p|} \underline{p}^{2s} \sum_{m\in\mathbb{Z}} (\underline{p+mn}/\underline{p})^{2s}, \tag{3.2}$$

so to establish (3.1), it suffices to show that the final sum in (3.2) is bounded by a constant depending only on s_0. We consider two cases, in each using the fact that $s \le s_0 < -1/2$ and $p \in \Lambda_n$. If $p = 0$ then

$$\sum_{m\in\mathbb{Z}} (\underline{p+mn}/\underline{p})^{2s} = \sum_{m\in\mathbb{Z}} \underline{mn}^{2s} \le \sum_{m\in\mathbb{Z}} \underline{mn}^{2s_0} \le \sum_{m\in\mathbb{Z}} \underline{m}^{2s_0} \le C_2.$$

If $p \in \Lambda_n$ is nonzero, say positive, then $|n/p| > 2$, and we deduce that

$$\sum_{m\in\mathbb{Z}} (\underline{p+mn/p})^{2s} = \sum_{m\in\mathbb{Z}} |1+mn/p|^{2s} \le \sum_{m\in\mathbb{Z}} |1+mn/p|^{2s_0}$$
$$= \sum_{m=0}^{\infty} |1+mn/p|^{2s_0} + \sum_{m=-\infty}^{-1} |-1-mn/p|^{2s_0}$$
$$\le \sum_{m=0}^{\infty} |1+2m|^{2s_0} + \sum_{m=-\infty}^{-1} |-1-2m|^{2s_0} \le C_3.$$

This proves (3.1).

To complete the proof of the theorem we choose

$$\sigma(x) = -\sum_{k\in\Lambda_n} \widehat{\rho}(k)\epsilon^{2|k|}\underline{k}^{2s+1}\exp(2\pi ikx).$$

Then

$$\|\sigma\|^2_{-s-1,\epsilon^{-1}} = \sum_{k\in\Lambda_n} |\widehat{\rho}(k)|^2\epsilon^{2|k|}\underline{k}^{2s}, \tag{3.3}$$

and

$$(V_1\rho,\sigma) = \sum_{k\in\Lambda_n} \overline{\widehat{\rho}(k)}\epsilon^{2|k|}\underline{k}^{2s+1}\int_0^1 \rho(t)\int_0^1 [-\log|2\sin(\pi(s-t))|+\pi]\exp(-2\pi iks)\,ds\,dt$$
$$= \pi\sum_{k\in\Lambda_n} \overline{\widehat{\rho}(k)}\epsilon^{2|k|}\underline{k}^{2s}\int_0^1 \rho(t)\exp(-2\pi ikt)\,dt = \pi\sum_{k\in\Lambda_n} |\widehat{\rho}(k)|^2\epsilon^{2|k|}\underline{k}^{2s}.$$

By (3.3) and (3.1),

$$(V_1\rho,\sigma) = \pi\sqrt{\sum_{k\in\Lambda_n} |\widehat{\rho}(k)|^2\epsilon^{2|k|}\underline{k}^{2s}}\ \|\sigma\|_{-s-1,\epsilon^{-1}}$$
$$\ge \pi C_1^{-1/2}\|\rho\|_{s,\epsilon}\|\sigma\|_{-s-1,\epsilon^{-1}}. \ \square$$

The next lemma concerns the exponential decays of the Fourier coefficients of the analytic kernel K. This result will be useful in showing exponential convergence for the approximate densities and potentials.

LEMMA 3.2 *Let $S_\delta := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \delta\}$. Then the kernel K defined in (2.3) is a real 1-periodic analytic function in each variable and extends analytically to $S_\delta \times S_\delta$ for some $\delta > 0$. Moreover, there exists constants C and $\epsilon_K \in (0,1)$ such that*

$$|\widehat{K}(p,q)| \le C\epsilon_K^{|p|+|q|}, \qquad p,q \in \mathbb{Z}.$$

PROOF: This is an easy consequence of a lemma given in [10, section 2.1] on the exponential decay of the Fourier coefficients of analytic functions. $\square$

It follows from the lemma that B maps $X_{s,\epsilon}$ compactly into $X_{s+1,\epsilon}$ for all $s \in \mathbb{R}$, $\epsilon \in (\epsilon_K, 1]$. The same is then also true of $B_1 = B + \pi M$. Now, by theorem 3.1, there exists $\beta > 0$ such that for all n and $\rho \in S_n$, there exists $\sigma \in T_n$ satisfying

$$(A\rho, \sigma) \geq \beta \|\rho\|_{s,\epsilon} \|\sigma\|_{-s-1,\epsilon^{-1}} - (B_1\rho, \sigma).$$

The inf-sup condition for the operator A follows, using a compactness argument. (See, for example, [2] or [6].)

THEOREM 3.3 *Let $s \leq s_0 < -1/2$, $\epsilon \in (\epsilon_K, 1]$. Then for sufficiently large n, there exists a constant C depending only on s_0, and Γ such that*

$$\inf_{0 \neq \rho \in S_n} \sup_{0 \neq \sigma \in T_n} \frac{(A\rho, \sigma)}{\|\rho\|_{s,\epsilon} \|\sigma\|_{-s-1,\epsilon^{-1}}} \geq C.$$

REMARK: The constant in the previous theorem blows up as the conformal radius of Γ approaches 1. For a circular domain of radius r, this constant behaves like $1/\log(r)$.

In view of this stability result, the standard theory of Galerkin methods gives existence and quasioptimality of the approximate solution.

THEOREM 3.4 *There exists a constant N, depending only on Γ, such that for all $n \geq N$ and $g \in \bigcup\{X_{s,\epsilon} \big| s \in \mathbb{R}, \epsilon > 0\}$ the delta-trigonometric method (2.4) obtains unique solutions, $\phi_n \in S_n$. Moreover, if $s \in (-\infty, -1/2)$, $\epsilon \in (\epsilon_K, 1]$ (ϵ_K being determined in lemma 3.2), $g \in X_{s+1,\epsilon}$, and $n \geq N$, then there exists a constant C, depending only on ϵ, s, and Γ such that*

$$\|\phi - \phi_n\|_{s,\epsilon} \leq C \inf_{\rho \in S_n} \|\phi - \rho\|_{s,\epsilon} \ .$$

Because the approximate solution is quasioptimal, we establish its convergence by bounding the error in any approximation from S_n of the exact solution. A convenient choice is $P_n\phi \in S_n$ determined by the equations

$$\widehat{P_n\phi}(k) = \widehat{\phi}(k) \quad \forall \ k \in \Lambda_n.$$

Then

$$\|\phi - P_n\phi\|_{s,\epsilon}^2 = \sum_{k \notin \Lambda_n} |\widehat{\phi}(k) - \widehat{P_n\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} \leq 2 \sum_{k \notin \Lambda_n} [|\widehat{\phi}(k)|^2 + |\widehat{P_n\phi}(k)|^2] \epsilon^{2|k|} \underline{k}^{2s}.$$

Now, if $t \geq s$ and $\epsilon \leq 1$, then

$$\sum_{k \notin \Lambda_n} |\widehat{\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} \leq \epsilon^n \sum_{k \notin \Lambda_n} \underline{k}^{2s-2t} |\widehat{\phi}(k)|^2 \underline{k}^{2t} \leq \epsilon^n (\pi n)^{2s-2t} \|\phi\|_t^2.$$

If also $t \leq 0$, then, since $P_n\phi \in S_n$,

$$\begin{aligned}
\sum_{k \notin \Lambda_n} |\widehat{P_n\phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} &= \sum_{p \in \Lambda_n} \sum_{m \in \mathbb{Z}^*} |\widehat{P_n\phi}(p + mn)|^2 \epsilon^{2|p+mn|} (\underline{p + mn})^{2s} \\
&= \sum_{p \in \Lambda_n} \sum_{m \in \mathbb{Z}^*} |\widehat{P_n\phi}(p)|^2 \epsilon^{2|p+mn|} \Big(2\pi |p + mn|\Big)^{2s} \\
&= (\pi n)^{2s-2t} \sum_{p \in \Lambda_n} |\widehat{\phi}(p)|^2 \underline{p}^{2t} (\pi n/\underline{p})^{2t} \sum_{m \in \mathbb{Z}^*} \epsilon^{2|p+mn|} (2|p + mn|/n)^{2s} \\
&\leq (\pi n)^{2s-2t} \epsilon^n \sum_{p \in \Lambda_n} |\widehat{\phi}(p)|^2 \underline{p}^{2t} \sum_{m \in \mathbb{Z}^*} (2|p + mn|/n)^{2s}.
\end{aligned}$$

Using the fact that $p \in \Lambda_n$ it is easy to show that the final sum on the right hand side of the inequality is bounded as long as $s < -1/2$, whence

$$\sum_{k \notin \Lambda_n} |\widehat{P_n \phi}(k)|^2 \epsilon^{2|k|} \underline{k}^{2s} \leq C(\pi n)^{2s-2t} \epsilon^n \|\phi\|_t^2.$$

Combining these estimates, we get

$$\|\phi - P_n \phi\|_{s,\epsilon} \leq C \epsilon^{n/2} (\pi n)^{s-t} \|\phi\|_t \qquad \forall \ \phi \in H^t,$$

valid under the assumptions $s < -1/2$, $t \in [s, 0]$, $\epsilon \leq 1$. This approximation result, together with the quasioptimality asserted in Theorem 3.4, gives the basic convergence result for the approximate density computed by the semi-discrete delta-trigonometric method.

THEOREM 3.5 *Let $s < -1/2$, $t \in [s, 0]$, $n \geq N$, and suppose that $\phi \in H^t$. Then for $\epsilon \in (\epsilon_K, 1]$ (ϵ_K being determined in lemma 3.2), there exists a constant C depending only on ϵ, s, and Γ such that*

$$\|\phi - \phi_n\|_{s,\epsilon} \leq C \epsilon^{n/2} n^{s-t} \|\phi\|_t.$$

Once the density ϕ has been approximated, the potential u can be reconstructed by integrating ϕ against the appropriate kernel. Away from Γ, the kernels are smooth, so combining the exponential convergence rates of the previous theorem with a simple duality argument gives exponential convergence rates for the approximate potentials on compact sets disjoint from the boundary. For details see [2, theorem 5.3].

THEOREM 3.6 *Let $n \geq N$, $\phi \in H^t$, and Ω_K be a compact set in $\mathbb{R}^2 \backslash \Gamma$. Then, for any multiindex β, there exist constants C and $\epsilon \in (0, 1)$ depending only on t, N, Ω_K, and Γ, such that*

$$\|\partial^\beta (u - u_n)\|_{L^\infty(\Omega_K)} \leq C \epsilon^n \|\phi\|_t.$$

While convergence away from Γ is exponential, the approximate potential converges on all of $\mathbb{R}^2$ at an algebraic rate. Convergence of order 3/2 holds in L^2 on bounded sets. To cover the case of convergence near infinity as well, we introduce the weighted norm

$$|||v|||^2 = \int_{\mathbb{R}^2} \frac{|v(z)|^2}{1 + |z|^4} \, dz. \tag{3.4}$$

THEOREM 3.7 *Let $-3/2 \leq t \leq 0$, $n \geq N$, and $\phi \in H^t$. Then there exists a constant C depending only on Γ such that*

$$|||u - u_n||| \leq C n^{-t-3/2} \|\phi\|_t.$$

PROOF: Let Ω and Ω_c denote the bounded and unbounded components of $\mathbb{R}^2 \setminus \Gamma$, respectively. Without loss of generality, we may assume that $0 \in \Omega$. We shall show that

$$\int_\Omega |(u - u_n)(z)|^2 \, dz + \int_{\Omega_c} |(u - u_n)(z)|^2 \frac{dz}{|z|^4} \leq C \|\phi - \phi_n\|_{-3/2},$$

which, in view of Theorem 3.5, clearly implies the asserted estimate. Let $v = u - u_n$ and $q = v|_\Gamma$. Then $q \circ x = A(\phi - \phi_n)$, so $\|q\|_{H^{s+1}(\Gamma)} \le C\|\phi - \phi_n\|_s$, for all real s and some constant C (depending on s). In particular, $\|q\|_{H^{-1/2}(\Gamma)} \le C\|\phi - \phi_n\|_{-3/2}$. Now, v solves the Dirichlet problem

$$\Delta v = 0 \quad \text{on} \quad \mathbb{R}^2 \backslash \Gamma, \qquad v = q \quad \text{on} \quad \Gamma.$$

Therefore, for all real s, there is a constant C such that $\|v\|_{H^s(\Omega)} \le C\|q\|_{H^{s-1/2}(\Gamma)}$ (see, e.g., [14, Ch. 2, § 7.3]). In particular, $\|v\|_{L^2(\Omega)} \le C\|q\|_{H^{-1/2}(\Gamma)}$. Combining these estimates gives the desired estimate for $u - u_n$ on Ω.

To obtain the estimate on Ω_c, we use the Kelvin transform, $\kappa(z) = z/|z|^2$. Since $0 \in \Omega$, κ maps Γ analytically onto some simple closed curve $\bar{\Gamma}$. Let $\bar{\Omega}$ denote the bounded component of the complement of $\bar{\Gamma}$, and set $\bar{v} = v \circ \kappa$. Then $\bar{v}$ is harmonic on $\bar{\Gamma} \setminus \{0\}$. Moreover the singularity at the origin is removable (since v is bounded). The argument therefore implies that $\|\bar{v}\|_{L^2(\bar{\Omega})} \le C\|\bar{v}\|_{H^{-1/2}(\bar{\Gamma})}$. But clearly, $\|\bar{v}\|_{H^{-1/2}(\bar{\Gamma})} \le C\|q\|_{H^{-1/2}(\Gamma)}$, and a simple calculation shows that

$$\|\bar{v}\|_{L^2(\bar{\Omega})} = \int_{\Omega_c} |(u - u_n)(z)|^2 \frac{dz}{|z|^4}. \quad \square$$

To close this section, we show that the condition numbers of the discretization matrices are bounded linearly proportional to the numbers of subintervals. Recall that A (defined in (2.1)) represents the single-layer potential operator and $\mathbf{A}$ (defined in (2.7)) denotes the matrix arising from the delta-trigonometric method. In lemma 3.8, we state a relationship between $\|\phi_n\|_{-1}$ and $\|\alpha\|$ defined in (2.6). Then in theorem 3.9, we present bounds for $\|\mathbf{A}\|$, $\|\mathbf{A}^{-1}\|$. and the condition numbers of $\mathbf{A}$.

LEMMA 3.8 *There exists a constant C such that*

$$\|\phi_n\|_{-1} \le C\sqrt{n}\|\alpha\|$$

and

$$\|\alpha\| \le C\sqrt{n}\|\phi_n\|_{-1}.$$

PROOF: For the first half, note that

$$\begin{aligned} \|\phi_n\|_{-1}^2 = \sum_{k\in\mathbb{Z}} |\widehat{\phi}_n(k)|^2 \underline{k}^{-2} &= \sum_{k\in\mathbb{Z}} \Big|\sum_{j=1}^n \alpha_j \exp(2\pi ikj/n)\Big|^2 \underline{k}^{-2} \\ &\le \sum_{k\in\mathbb{Z}} \underline{k}^{-2} \Big(\sum_{j=1}^n |\alpha_j|\Big)^2 = C^2 \Big(\sum_{j=1}^n |\alpha_j|\Big)^2 \le C^2 n\|\alpha\|^2. \end{aligned}$$

For the second half, note that

$$\begin{aligned} \|\phi_n\|_{-1}^2 &\ge \sum_{p\in\Lambda_n} \underline{p}^{-2} \Big|\sum_{j=1}^n \alpha_j \exp(2\pi ipj/n)\Big|^2 \\ &\ge \sum_{p\in\Lambda_n} (\pi n)^{-2} \Big[\sum_{j=1}^n |\alpha_j|^2 + \sum_{j=1}^n \sum_{l=j+1}^n 2\alpha_j\alpha_l \exp(2\pi ip(j-l)/n)\Big]. \end{aligned}$$

Rearranging the summations, gives

$$\|\phi_n\|_{-1}^2 \geq (\pi n)^{-2}\Big[\sum_{p\in\Lambda_n}\sum_{j=1}^{n}|\alpha_j|^2 + \sum_{j=1}^{n}\sum_{l=j+1}^{n} 2\alpha_j\alpha_l \sum_{p\in\Lambda_n}\exp(2\pi ip(j-l)/n)\Big].$$

But $\sum_{p\in\Lambda_n}\exp(2\pi ip(j-l)/n) = 0$ for $l \neq j (\text{mod } n)$, so

$$\|\phi_n\|_{-1}^2 \geq (\pi n)^{-2}\sum_{p\in\Lambda_n}\sum_{j=1}^{n}|\alpha_j|^2 = Cn^{-1}\|\alpha\|^2,$$

as desired. □

THEOREM 3.9 *Let $\kappa(\mathbf{A})$ represents the condition number of the matrix $\mathbf{A}$. Then there exists a constant C depending only on Γ such that*

$$\|\mathbf{A}\| \leq C\sqrt{n}, \qquad \|\mathbf{A}^{-1}\| \leq C\sqrt{n},$$

and

$$\kappa(\mathbf{A}) \leq Cn.$$

PROOF: The first two inequalities follow from Theorem 3.8 by standard arguments, and the third is a consequence. See [2] or [7] for details. □

4. Convergence of the Fully Discrete Delta-Trigonometric Method

In this section, we adapt the results of the last section to the fully discrete delta-trigonometric method. A key step is the application of the Euler-MacLaurin theorem to estimate the integration error for the trapezoidal rule when the integral is a product of an analytic function and a trigonometric polynomial.

THEOREM 4.2 *Let f be an analytic 1-periodic function and define*

$$f_k := \int_0^1 f(s)\exp(2\pi iks)\,ds$$

and

$$\widetilde{f}_k := \frac{1}{n}\sum_{l=1}^{n} f(l/n)\exp(2\pi ikl/n) \quad \forall\ k \in \mathbb{Z}.$$

Then there exist constants C and $\epsilon \in (0,1)$ depending only on f such that

$$|f_k - \widetilde{f}_k| \leq C\epsilon^n \quad \forall\ k \in \Lambda_n.$$

PROOF: We can extend f to an analytic function in the complex strip $\overline{S}_\delta$ for some $\delta > 0$. Moreover, this extension is 1-periodic. In [10, pg. 490], Henrici shows that

$$|f_k - \widetilde{f}_k| \leq 2\|f\|_{L^\infty(S_\delta)}\cosh(2\pi k\delta)\frac{\exp(-2\pi n\delta)}{1-\exp(-2\pi n\delta)}.$$

Since

$$\cosh(2\pi k\delta)\exp(-2\pi n\delta) \le \exp(-\pi n\delta) \quad \forall\ k \in \Lambda_n,$$

the theorem follows with

$$C = \frac{2\|f\|_{L^\infty(S_\delta)}}{1-\exp(-2\pi\delta)}, \quad \epsilon = \exp(-\pi\delta). \quad \square$$

In theorem 4.3, we use theorem 4.2 to bound the perturbations due to numerical integration. Then we use theorem 4.3 to bound the approximate potential errors in theorem 4.4.

THEOREM 4.3 *There exists constants C and $\epsilon \in (0,1)$ depending only on g and Γ such that*

$$\|\mathbf{g} - \widetilde{\mathbf{g}}\| \le C\epsilon^n, \qquad \|\mathbf{B} - \widetilde{\mathbf{B}}\| \le C\epsilon^n.$$

PROOF: For the first estimate, note that by theorem 4.2,

$$\|\mathbf{g} - \widetilde{\mathbf{g}}\| \le \sqrt{n} \max_{k\in\Lambda_n} |g_k - \widetilde{g}_k| \le C\epsilon^n.$$

For the second estimate, recall that K is 1-periodic and analytic function with respect to both its variable (lemma 3.2). By theorem 4.2,

$$\begin{aligned} |\mathbf{B}_{kj} - \widetilde{\mathbf{B}}_{kj}| &= \Big| \int_0^1 K(s, j/n)\psi_k(s)\, ds - \frac{1}{n}\sum_{p=1}^{n} K(p/n, j/n)\psi_k(p/n) \Big| \\ &\le C\epsilon^n, \quad \forall\ j = 1,\ldots,n \quad \text{and} \quad k \in \Lambda_n. \end{aligned}$$

Therefore,

$$\|\mathbf{B} - \widetilde{\mathbf{B}}\| \le n \max_{j=1,\ldots,n} \max_{k\in\Lambda_n} |(\mathbf{B} - \widetilde{\mathbf{B}})_{kj}| \le C\epsilon_1^n. \qquad \square$$

We are now ready to establish the convergence of approximate potentials for the fully discrete method.

THEOREM 4.4 *Let Ω_K be a compact set disjoint from the boundary. Then, for any multiindex β, there exist constants C and $\epsilon \in (0,1)$ depending only on g, Γ, and Ω_K, such that*

$$\|\partial^\beta(u_n - \widetilde{u}_n)\|_{L^\infty(\Omega_K)} \le C\epsilon^n.$$

PROOF: Note that

$$\begin{aligned} \alpha - \widetilde{\alpha} &= \mathbf{A}^{-1}[\mathbf{g} - \widetilde{\mathbf{g}} - (\mathbf{A} - \widetilde{\mathbf{A}})\widetilde{\alpha}] \\ &= \mathbf{A}^{-1}[\mathbf{g} - \widetilde{\mathbf{g}} - (\mathbf{B} - \widetilde{\mathbf{B}})\widetilde{\alpha}]. \end{aligned}$$

Hence,

$$\|\alpha - \widetilde{\alpha}\| \le \|\mathbf{A}^{-1}\|(\|\mathbf{g} - \widetilde{\mathbf{g}}\| + \|\mathbf{B} - \widetilde{\mathbf{B}}\|\|\widetilde{\alpha}\|).$$

Using the fact that

$$\|\widetilde{\alpha}\| \le \|\alpha - \widetilde{\alpha}\| + \|\alpha\| \le \|\alpha - \widetilde{\alpha}\| + \|\mathbf{A}^{-1}\|\|\mathbf{g}\|$$

we derive

$$\|\alpha - \widetilde{\alpha}\| \leq \frac{\|\mathbf{A}^{-1}\|(\|\mathbf{g} - \widetilde{\mathbf{g}}\| + \|\mathbf{B} - \widetilde{\mathbf{B}}\|\|\mathbf{A}^{-1}\|\|\mathbf{g}\|)}{1 - \|\mathbf{A}^{-1}\|\|\mathbf{B} - \widetilde{\mathbf{B}}\|} \quad .$$

Applying theorems 3.9 and 4.3 we conclude that

$$\|\alpha - \widetilde{\alpha}\| \leq C\epsilon^n, \tag{4.1}$$

where $C > 0$ and $\epsilon \in (0,1)$ depend only on g and Γ. It follows that

$$\begin{aligned}\|\partial^\beta(u_n - \widetilde{u}_n)\|_{L^\infty(\Omega_K)} &= \Big\|\sum_{j=1}^n (\alpha_j - \widetilde{\alpha}_j)\partial^\beta \log|\,\cdot\, - x(j/n)|\Big\|_{L^\infty(\Omega_K)} \\ &\leq \Big(\sum_{j=1}^n |\alpha_j - \widetilde{\alpha}_j|\Big) \max_{z\in\Omega_K} \max_{s\in\mathbb{R}} \Big|\partial_z^\beta \log|z - x(s)|\Big| \\ &\leq C\sqrt{n}\|\alpha - \widetilde{\alpha}\| \leq C\epsilon^n.\end{aligned}$$

Combining this estimate with theorem 3.6 gives the theorem. ∎

We also prove that the use of numerical quadratures does not affect the convergence rates in the weighted Sobolev norm defined in (3.4).

THEOREM 4.5 *Let $-3/2 \leq t \leq 0$, $n \geq N$, and $\phi \in H^t$. Then there exists a constant C depending only on g and Γ such that*

$$|||u - \widetilde{u}_n||| \leq Cn^{-t-3/2}.$$

PROOF: Arguing as in the proof of theorem 3.7, we have $|||u_n - \widetilde{u}_n||| \leq C\|\phi_n - \widetilde{\phi}_n\|_{-3/2}$. But

$$\begin{aligned}\|\phi_n - \widetilde{\phi}_n\|_{-3/2} = \Big\|\sum_{j=1}^n (\alpha_j - \widetilde{\alpha}_j)\delta(\cdot - j/n)\Big\|_{-3/2} &\leq C\Big(\sum_{j=1}^n |\alpha_j - \widetilde{\alpha}_j|\Big)\|\delta\|_{-3/2} \\ &\leq C\sqrt{n}\|\alpha - \widetilde{\alpha}\| \leq C\epsilon^n,\end{aligned}$$

where we use (4.1) in the last step. Combining this result with theorem 3.7 completes the proof. ∎

5. Numerical Implementation and Computational Results

In this section, we discuss the implementation of our method using the fast Fourier transform and give operation counts. Then we present numerical results to confirm our theory and to test the method in cases where the data is less smooth than we have assumed for the analysis.

In our program we use real test functions rather than complex ones. As basis functions we use

$$\widetilde{\psi}_k(s) = \begin{cases} \sin(k\pi s), & \text{if } k = 2, 4, \ldots, n-1, \\ \cos((k-1)\pi s), & \text{if } k = 1, 3, \ldots, n. \end{cases}$$

Also, we allow M-point Gaussian quadrature instead of just the trapezoidal rule. The M-point Gaussian quadrature rule on n subintervals is applied on the right hand term to give

$$\widetilde{\mathbf{g}}_k = \frac{1}{n} \sum_{l=1}^{n} \sum_{m=1}^{M} q_m^M g\left(\frac{\xi_m^M + l - 3/2}{n}\right) \widetilde{\psi}_k \left(\frac{\xi_m^M + l - 3/2}{n}\right)$$

where q_m^M's are quadrature weights on [0,1] and ξ_m^M's are the quadrature points on [0,1]. For any even k, simple trigonometric identities imply

$$\widetilde{\psi}_k\left(\frac{\xi_m^M + l - 3/2}{n}\right) = \widetilde{\psi}_k\left(\frac{l-1}{n}\right)\widetilde{\psi}_{k+1}\left(\frac{\xi_m^M - 1/2}{n}\right) + \widetilde{\psi}_{k+1}\left(\frac{l-1}{n}\right)\widetilde{\psi}_k\left(\frac{\xi_m^M - 1/2}{n}\right) \tag{5.1}$$

and

$$\widetilde{\psi}_{k+1}\left(\frac{\xi_m^M + l - 3/2}{n}\right) = \widetilde{\psi}_{k+1}\left(\frac{l-1}{n}\right)\widetilde{\psi}_{k+1}\left(\frac{\xi_m^M - 1/2}{n}\right) - \widetilde{\psi}_k\left(\frac{l-1}{n}\right)\widetilde{\psi}_k\left(\frac{\xi_m^M - 1/2}{n}\right). \tag{5.2}$$

The sums,

$$\frac{1}{n}\sum_{l=1}^{n} q_m^M g\left(\frac{\xi_m^M + l - 3/2}{n}\right)\widetilde{\psi}_k\left(\frac{l-1}{n}\right) \quad \text{for} \quad m \in [1, M],\ k \in [1, n],$$

can be computed in $O(nM \log n)$ operations using the fast Fourier transform. Then $\widetilde{\mathbf{g}}$ can be computed using (5.1) and (5.2) in $O(nM)$ operations. Thus, the number of operations to calculate $\mathbf{g}$ is $O(nM \log n)$.

The remainder matrix $\widetilde{\mathbf{B}}$ is calculated similiarly. Applying M-point quadrature gives to get

$$\widetilde{\mathbf{B}}_{kj} = \frac{1}{n}\sum_{m=1}^{M}\left[\sum_{l=1}^{n} q_m^M \log\left|\frac{x((\xi_m^M + l - 3/2)/n)) - x(j/n)}{2\sin(\pi(\xi_m^M + l - 3/2 - j)/n)}\right|\widetilde{\psi}_k\left(\frac{\xi_m^M + l - 3/2}{n}\right)\right].$$

The sum in the brackets is calculated (for $k = 1, \ldots, n$) by the fast Fourier transform. The number of operations needed to calculate $\widetilde{\mathbf{B}}$ is $O(n^2 M \log n)$.

The principal part,

$$\mathbf{V}_{kj} := \int_0^1 \log|2\sin(\pi(s - j/n))|\ \widetilde{\psi}_k(s)\, ds,$$

is integrated exactly. This requires $O(n^2)$ operations.

In summary, $O(Mn^2 \log n)$ are required to calculate the matrix. The matrix solution requires $O(n^3/3)$ calculations. Computer analysis show that the solution step requires less than a third of the total time for n as large as 81. In other words, it is important to use the fast Fourier transform since the matrix formation requires a significant amount of time.

The program SPLTRG implements the delta- and spline-trigonometric methods with numerical quadratures. In this section, we present several sample problems and numerical results for the delta-trigonometric method. The first problem is an ideal problem in that the boundary and boundary data are analytic. Then we look at some problems where the boundary and/or boundary data is not so smooth. For more details about SPLTRG, see [7] and SPLTRG documentation.

For the following tables, we let

****	:=	no answer
ue_n	:=	the error for the approximate potential using n subintervals
r_{n-m}	:=	the convergence rate from n subintervals to m subintervals
1-pt	:=	1-point quadrature
3-pts	:=	3-point quadrature
8-pts	:=	8-point quadrature

We define the relative error to be the absolute error divided by the exact solution. In cases where the exact solution is near zero, SPLTRG gives the absolute error. All calculations were done in double precision on an Apollo 420PEB. Consequently, we can not expect the relative errors to be much smaller than 1.0E−14.

EXAMPLE 5.1 Ellipse with analytic data

The first example involves an elliptic boundary with analytic boundary data. In this example, we examine the effects of using different quadrature rules.

Boundary: $x^2/4 + y^2 = 1/25$

Data: $g = 5x/2$

Exact solution:

$$u = \begin{cases} 5x/2, & \text{if } (x,y) \in \text{ellipse}, \\ 5x - w, & \text{if } (x,y) \notin \text{ellipse and } x \geq 0, \\ 5x + w, & \text{if } (x,y) \notin \text{ellipse and } x \leq 0, \end{cases}$$

where

$$w = \sqrt{\frac{25(x^2 - y^2) - 3 + \sqrt{(25(x^2 - y^2) - 3)^2 + 2500x^2y^2}}{2}} \; .$$

For table 1A and 1B, we pick a typical interior point and present relative errors and convergence rates for the approximate potential using different quadrature rules. The numerical results for other points away from the boundary are similiar. The approximate potentials converge very fast, i.e., relative errors are about 10^{-14} for $n = 81$. There are very little error differences when using different quadrature rules. Note that the convergence rates appear to be exponential in table 1B.

	$\lvert\mathrm{ue}_3\rvert$	$\lvert\mathrm{ue}_9\rvert$	$\lvert\mathrm{ue}_{27}\rvert$	$\lvert\mathrm{ue}_{81}\rvert$	$\lvert\mathrm{ue}_{243}\rvert$
1-pt	7.41E−01	5.89E−03	1.52E−06	3.78E−15	2.44E−15
3-pts	7.30E−01	5.88E−03	1.52E−06	5.33E−15	1.55E−15
8-pts	7.29E−01	5.88E−03	1.52E−06	4.66E−15	****

Table 1A. *Relative errors at (0.10, 0.05).*

	r_{3-9}	r_{9-27}	r_{27-81}	r_{81-243}
1-pt	4.40	7.52	18.04	0.10
3-pts	4.39	7.52	17.72	1.12
8-pts	4.39	7.52	17.80	****

Table 1B. *Convergence rates at (0.10, 0.05).*

We also examine the errors in the approximate potential on the boundary. Note that the approximate potential in (2.8) has a logarithmic singularity at the quadrature points. Therefore we evaluate the maximum relative errors at points midway between consecutive quadrature points and present these results in table 1C. Table 1C shows that there are no improvements in the errors when higher quadrature rules are used, and therefore, it is best to use a low quadrature rule.

	$\|ue_3\|$	$\|ue_9\|$	$\|ue_{27}\|$	$\|ue_{81}\|$	$\|ue_{243}\|$
1-pt	1.05E+01	8.94E−01	1.08E−01	1.28E−02	4.28E−03
3-pts	1.10E+01	8.94E−01	1.08E−01	1.28E−02	4.28E−03
8-pts	1.10E+01	8.94E−01	1.08E−01	1.28E−02	****

Table 1C. *Maximum relative errors at the midpoints of boundary subintervals.*

In Table 1D, we present the matrix condition numbers for different quadrature rules. Note that in fact the condition numbers grow proportionally slower than the numbers of subintervals.

	$\|ue_3\|$	$\|ue_9\|$	$\|ue_{27}\|$	$\|ue_{81}\|$	$\|ue_{243}\|$
1-pt	0.59E+01	0.11E+02	0.21E+02	0.55E+02	0.92E+02
3-pts	0.52E+01	0.11E+02	0.21E+02	0.55E+02	0.92E+02
8-pts	0.52E+01	0.11E+02	0.21E+02	0.55E+02	****

Table 1D. *Matrix condition numbers.*

Table 1E shows the CPU time for each run on the Apollo 420PEB. From this table, we see that it is expensive to compute using a high quadrature rule. It is more efficient to use a low quadrature rule and more subintervals (larger n).

	$time_3$	$time_9$	$time_{27}$	$time_{81}$	$time_{243}$
1-pt	3.013	8.961	31.892	151.227	1369.002
3-pts	4.031	10.200	41.278	244.154	2158.307
8-pts	4.659	12.479	63.295	455.942	****

Table 1E. *CPU times.*

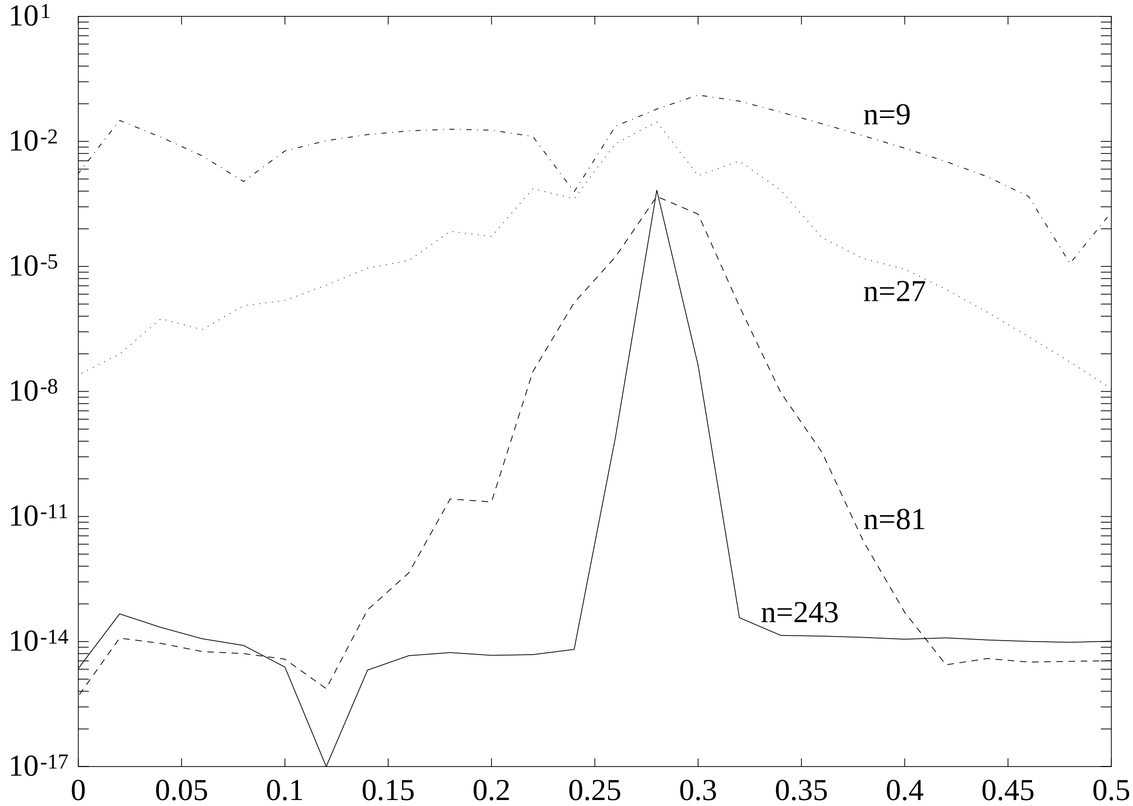

Fig. 1. *Relative error versus x on the line $x = 2y$ for example 5.1.*

We also examine the relative errors on a sample line. Figure 1 shows the relative errors on the line $x = 2y$ for different values of n.

Not suprisingly the relative errors are worst when the line crosses the boundary (about (x,y)=(0.283,0.141)).

For this example, we conclude that very fast convergence is indeed obtained for the approximate potentials on compact sets disjoint from the boundary using the delta-trigonometric method with numerical quadrature.

REMARK: Computations also showed that the approximate potentials produced by the spline-trigonometric method of [2] with numerical quadrature did not converge exponentially. The reason for this phenomenon is that the spline-trigonometric method involves numerical integration of non-analytic functions (ordinary splines) in (1.2) while the delta-trigonometric method avoids numerical integration of (1.2).

EXAMPLE 5.2 Ellipse with data of varying smoothness

This example involves the same elliptic boundary but with boundary data of different degrees of smoothness.

Boundary: $x^2/4 + y^2 = 1/25$

Data:

$$g = \begin{cases} 1.0, & \text{if } x \le 0, \\ 1.0 + x^s, & \text{if } x \ge 0, \end{cases} \quad \text{for} \quad s = 0, 1, 2, 3, 4, 5, \quad \text{and} \quad 6.$$

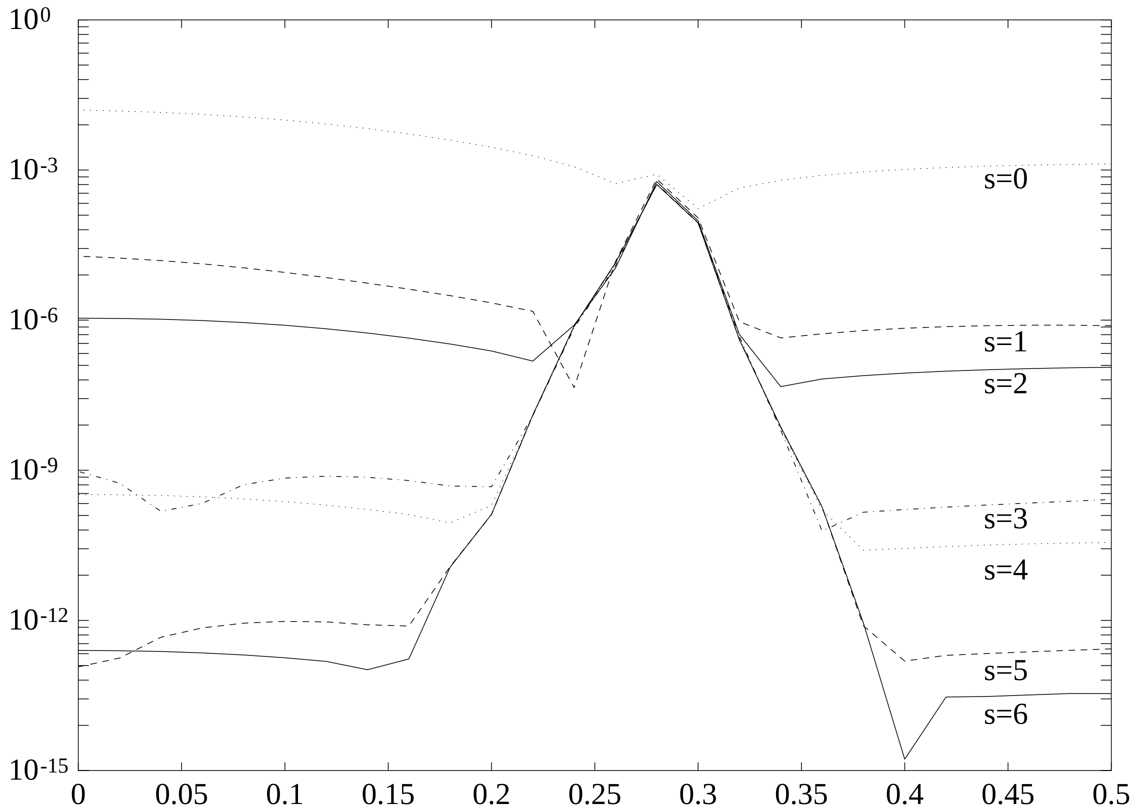

Fig. 2. *Relative error versus x on the line $x = 2y$ for example 5.2.*

The exact potential is not known, and therefore, the approximate relative errors are computed by using the approximate potentials for $n = 243$. For this problem, we only present results using trapezoidal quadrature. Table 2A compares the approximate relative errors at a typical interior point for different data smoothness. We see that the smoothness of the data affects the convergence rates significantly.

s	r_{3-9}	r_{9-27}	r_{27-81}	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$
0	1.20	1.26	0.60	2.86E-01	7.66E-02	1.92E-02	9.96E-03
1	3.14	2.12	3.01	7.89E-02	2.52E-03	2.46E-04	8.98E-06
2	6.68	0.79	2.90	6.98E-02	4.54E-05	1.90E-05	7.90E-07
3	3.63	6.70	6.58	8.18E-02	1.52E-03	9.62E-07	6.98E-10
4	3.70	7.04	7.25	8.94E-02	1.54E-03	6.75E-07	2.36E-10
5	3.80	7.02	12.21	9.31E-02	1.43E-03	6.39E-07	9.56E-13
6	3.85	6.99	13.74	9.47E-02	1.38E-03	6.41E-07	1.79E-13

Table 2A. *Approximate convergence rates and relative errors at (0.10, 0.05) using trapezoidal quadrature.*

Figure 2 show the approximate relative errors on the line $x = 2y$ using $n = 81$ for s equal 0, 1, 2, 3, 4, 5, and 6. It is interesting to note that the errors are about the same as the line crosses the boundary.

From this example, we conclude that the boundary data lack of smoothness affects the errors greatly. Note that we did obtain fair results at points away from the boundary for $s \geq 1$. The condition numbers depend only on the geometry of the domain and are exactly the same as in table 1D (example 5.1).

EXAMPLE 5.3 Rectangle with linear data

The third example involves a boundary with corners, but the boundary data is linear.

Domain: $(-0.1, 0.1) \times (-0.1, 0.1)$

Data: $g = 5x/2$

The exact solution is known in the interior region only and coincides with the formula given for g. We examine the effects of using two different quadrature rules. Tables 3A and 3B show the exact relative errors and exact convergence rates, respectively, at a sample interior point. Note the error depends only slightly on the quadrature rule used.

	$\lvert ue_3 \rvert$	$\lvert ue_9 \rvert$	$\lvert ue_{27} \rvert$	$\lvert ue_{81} \rvert$	$\lvert ue_{243} \rvert$
1-pt	3.29E−01	4.41E−02	1.53E−03	8.46E−05	4.43E−06
3-pts	3.14E−01	7.74E−02	1.91E−03	1.17E−04	6.20E−06

Table 3A. *Exact relative errors at (0.05, 0.05).*

	r_{3-9}	r_{9-27}	r_{27-81}	r_{81-243}
1-pt	1.83	3.06	2.63	2.69
3-pts	1.27	3.37	2.54	2.67

Table 3B. *Exact convergence rates at (0.05, 0.05).*

Figure 3 shows the exact relative errors (for different n) on a sample line from the origin to a corner of the rectangle using trapezoidal quadrature.

We see that the errors become worse as the line approaches the boundary.

Considering all three examples together, we recommend using trapezoidal quadrature. If the boundary and boundary data are analytic, then the delta-trigonometric method with trapezoidal quadrature obtains exponential convergence for the approximate potentials at points away from the boundary as we showed theoretically. In examples, where the boundary and/or boundary data are not smooth, the convergence is significantly slower.

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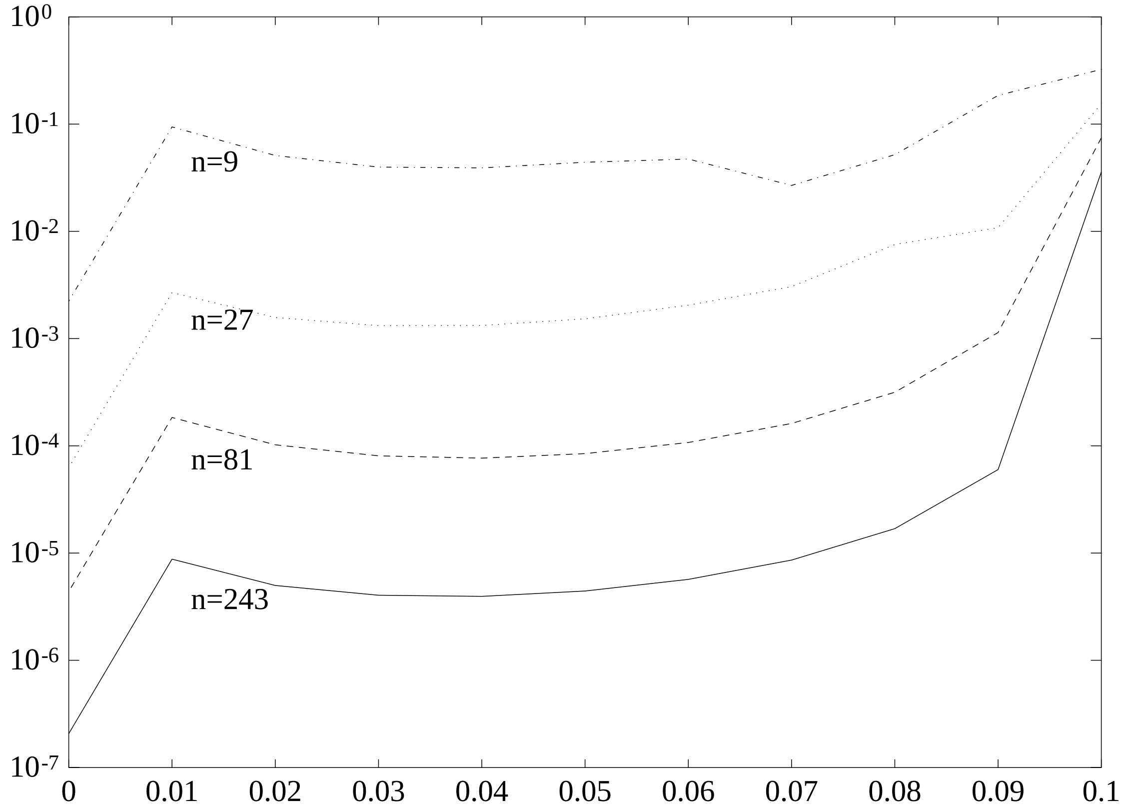

Fig. 3. *Relative error versus x on the line $x = y$ for example 5.3.*

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