NUMERICAL PROBLEMS IN GENERAL RELATIVITY

DOUGLAS N. ARNOLD
Department of Mathematics, Penn State University, University Park, PA 16802, USA

The construction of gravitational wave observatories is one of the greatest scientific efforts of our time. As a result, there is presently a strong need to numerically simulate the emission of gravitation radiation from massive astronomical events such as black hole collisions. This entails the numerical solution of the Einstein field equations. We briefly describe the field equations in their natural setting, namely as statements about the geometry of space time. Next we describe the complicated system that arises when the field equations are recast as partial differential equations, and discuss procedures for deriving from them a more tractable system consisting of constraint equations to be satisfied by initial data and together with evolution equations. We present some applications of modern finite element technology to the solution of the constraint equations in order to find initial data relevant to black hole collisions. We conclude by enumerating some of the many computational challenges that remain.

1 General relativity and gravity waves: the scientific problem

In the theory of general relativity spacetime is a smooth four-dimensional manifold endowed with a pseudo-Riemannian metric.[a] The metric determines the intrinsic curvature of the manifold, and the Einstein field equations relate the curvature of the metric at a point of spacetime to the mass-energy there. All bodies in free fall, that is, bodies not subject to any external forces other than gravity, trace out a geodesic in spacetime. Thus gravity—rather than being a force-field defined throughout space—is just a manifestation of the curvature of spacetime.

According to the Einstein equations, a massive object, for example a star, causes a large-scale deformation in spacetime. A common image is a stiff rubber membrane deformed under the weight of heavy objects placed upon it. A much more subtle consequence of the Einstein equations is that accelerating masses, e.g., binary stars in orbit around each other, generate small ripples on the curved surface of spacetime, and these propagate out through space at the speed of light. Because of the tiny amplitude of radiation from even very massive astronomical sources, no one has succeeded in detecting such *gravitational radiation*, despite a few decades of effort. A tremendous effort is underway to build more sensitive gravitational wave detectors based on laser

[a] Definitions will be given in the next section.

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interferometers. In such devices, mirrors are carefully suspended at the ends of two long evacuated tubes which emanate at right angles from a common endpoint. A laser beam is split at this vertex and sent down the tubes to the mirrors. After reflecting back and forth along the tubes a certain number of times, the beams are rejoined, and the resulting interference used to measure changes in lengths of the arms relative to one another. A passing gravitational wave will cause changes in length which fluctuate in time. These are tiny indeed: the collision of large stars or black holes in a nearby galaxy is expected to generate fluctuations in length of roughly one part in 10^{21}. The largest of the laser interferometers under construction, the Laser Gravitational Wave Observatory or LIGO, has arms four kilometers long, and has been designed to detect length changes on the order of one hundred millionth of a hydrogen atom diameter. The primary sources of detectable radiation are expected to be black holes collisions, neutron star collisions, black holes swallowing neutron stars, and supernovae explosions. It is hoped that detectable events will occur on the order of ten times per year, although such estimates are fraught with uncertainty.

While LIGO and similar projects are high risk ventures, the potential advances in gravitational physics and astronomy are tremendous. Gravitational waves carry information from the violent events that give them birth, and as such, are a sort of observatories. However, these "telescopes" see gravitational radiation, rather than light, x-rays, or radio waves. Indeed, they offer our first window on the universe that looks outside the electromagnetic spectrum. Since the universe is composed largely of dark matter, which doesn't emit electromagnetic radiation, the potential impact of the successful development of gravitational-wave observatories can be reasonably compared to that of the development of the telescope in the sixteenth century. Moreover, gravitational radiation can in principle be followed much further back in time toward the big bang, when dense clouds of matter existed which block electromagnetic radiation. Particular goals for LIGO include the ultimate conclusion of the gravity wave controversy, a stringent verification of the theory of general relativity, and the first direct evidence of black holes. But, just as for light, x-ray, and radio telescopy, the greatest payoff will probably come from the detection of objects whose existence hadn't been imagined before.

In order that gravitational wave observatories prove useful, numerical computation is needed to simulate the far-field gravitational radiation arising from massive cosmic events. In its purest form this is an inverse problem: given the waveform detected on earth, we wish to deduce the most likely astronomical event giving rise to it. We are nowhere near able to deal with such a problem at this point. As a key step, we must learn how to solve the forward problem: determine the spacetime metric arising from, for example, the collision of two black holes. This requires the determination of the relevant solutions of the Einstein equations, a massive computational challenge. Roughly speaking, simulating blackhole collisions seems to be about as hard as detecting them!

2 Geometry and the Einstein equations

Many applications of scientific computing simulate some aspect of the physical world in two main steps: first, the physical phenomena in question are modelled by differential equations, and, second, the differential equations are solved numerically. For general relativity it is essential to interpolate an additional step: first, the physics of gravity is modelled by *geometry*, then the geometry is converted (in a much less canonical fashion) to partial differential equations, and then the differential equations are solved numerically. In this section we describe the Einstein equations in their most natural form, as a statement about the geometry of spacetime. A full account can be found in many texts[6].

We begin by recalling some basic definitions. Spacetime is a smooth four-dimensional manifold, which we shall denote by M. To each point $m \in M$ is associated its four-dimensional tangent vectorspace, $T_m M$. For non-negative integers k and l, a tangent (k,l)-tensor at m is an element of the $4^{(k+l)}$-dimensional vectorspace $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$, $V = T_m M$, where there are k copies of V and l copies of its dual V^* in the product. Smooth functions which map each $m \in M$ to a tangent (k,l)-tensor at m are called (k,l)-tensorfields on M. In particular, $(0,0)$-tensorfields are smooth real-valued functions on M, $(1,0)$-tensorfields are tangent vectorfields, and $(0,1)$-tensorfields are covector fields.

A pseudo-Riemannian *metric* $\boldsymbol{g}$ on a manifold is a $(0,2)$-tensorfield which is symmetric and everywhere non-degenerate. Spacetime is endowed with a pseudo-Riemannian metric which is not Riemannian, but rather Lorentzian. That is, instead of being positive definite, at each point the bilinear form on the tangent space has signature $(-,+,+,+)$. Vectors $\boldsymbol{X}_m$ for which $\boldsymbol{g}_m(\boldsymbol{X}_m, \boldsymbol{X}_m) < 0$ are, by definition, those which point in a *timelike direction*. On a timelike curve (one for which the tangent vector $\boldsymbol{\tau}_m$ is always timelike), the integral of $\sqrt{-\boldsymbol{g}_m(\boldsymbol{\tau}_m, \boldsymbol{\tau}_m)}$ gives the *proper time* elapsed along the curve: the time as measured by an observer moving along the curve. The integral of the magnitude of the tangent for a spacelike curve gives the proper length of the curve, that is, the length measured by an observer for which all the events on the curve are simultaneous.

The intrinsic curvature of a pseudo-Riemannian manifold is completely captured by a $(1,3)$-tensorfield called the *Riemann curvature tensor*. Given three vectors $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ in T_mM, we parallel transport $\boldsymbol{Z}$ around a small curvilinear parallelogram determined by the tangent vectors $\epsilon\boldsymbol{X}$ and $\epsilon\boldsymbol{Y}$, to obtain a new tangent vector $\boldsymbol{Z}'$ at m which differs from $\boldsymbol{Z}$ by $O(\epsilon^2)$ (the metric determines a unique notion of parallel transport of a vector along a curve). Dividing by ϵ^2 and taking the limit as ϵ vanishes we obtain a new vector $\boldsymbol{W}$ which depends trilinearly on $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$: $\boldsymbol{W} = \mathbf{Riem}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$.

The Einstein equations relate some, but not all, of the curvature information in **Riem** to the mass and energy present. The relevant curvature information is the trace of the Riemann tensor, called the *Ricci tensor*: $\mathbf{Ric}(\boldsymbol{X}, \boldsymbol{Z}) = \operatorname{tr}[\mathbf{Riem}(\boldsymbol{X}, \cdot, \boldsymbol{Z})]$. To state the Einstein equations we introduce one more trace, that of the Ricci tensor, which is called the scalar curvature and denoted $\mathbf{R} = \operatorname{tr}\mathbf{Ric}$. The *Einstein tensor* is the combination $\boldsymbol{G} = \mathbf{Ric} - (1/2)\,\mathbf{R}\,g$ (then $\operatorname{tr}\boldsymbol{G} = -\operatorname{tr}\mathbf{Ric}$, so the Einstein tensor is often referred to as the trace-reversed Ricci tensor), and the Einstein equations read $\boldsymbol{G} = (8\pi k/c^2)\boldsymbol{T}$, where k is the gravitational constant, c the speed of light, $\boldsymbol{G}$ the Einstein tensor just constructed, and $\boldsymbol{T}$ the stress-energy tensor that measures the mass, energy, and momentum present. The constant entering this equation is tiny in human-sized units: $8\pi k/c^2 \approx 2 \times 10^{-27}$cm/gram, reflecting the fact that it takes a huge amount of matter to significantly effect the metric of space time. For convenience one often uses units in which $k = 1$ and $c = 1$, or, as here, $8\pi k = 1$, $c = 1$.

To summarize this section, the Einstein equations have a simple geometric content. From the metric of spacetime we can construct the Riemann curvature tensor. Taking its trace we get the Ricci tensor, and trace-reversing that gives the Einstein tensor $\boldsymbol{G}$. In appropriate units $\boldsymbol{G} = \boldsymbol{T}$, the stress-energy tensor.

3 Coordinatization of the Einstein equations

To write the Einstein equations as a system of partial differential equations we need to introduce a coordinate system on the spacetime manifold M. Of course, depending on the topology of M we might need to use multiple coordinate patches, but for our purposes we may assume that a global system of coordinates, x^α, $\alpha = 0, 1, 2, 3$, can be defined. Thus the quadruple (x^α) defines a diffeomorphism of M onto an open subset of $\mathbb{R}^4$. The differential of the inverse diffeomorphism then maps $\mathbb{R}^4$ onto T_mM for each $m \in M$, and so determines four vectorfields, $\boldsymbol{X}_0, \ldots, \boldsymbol{X}_3$ which form a basis for the tangent space at each point $m \in M$. We shall assume that the first of these vectorfields is always timelike and the remaining three always spacelike, so $t = x^0$ may be viewed as a temporal coordinate, and the x^i, $i = 1, 2, 3$, as spatial coordinates. In view of this basis, we can then specify any vectorfield $\boldsymbol{Y}$ by giving four coordinate functions $Y^\alpha : M \to \mathbb{R}$ so that the equation $\boldsymbol{Y} = Y^\alpha \boldsymbol{X}_\alpha$ holds at each point m (we use here the Einstein summation convention, so the repeated index α is summed from 0 to 3). The basis $\boldsymbol{X}_\alpha(m)$ for $T_m M$ determines a dual basis $\boldsymbol{X}^\alpha(m)$ for the space of tangent covectors $T_m^* M$, and, we can also express any covectorfield $\boldsymbol{\omega}$ as a linear combination of these basic covectorfields: $\boldsymbol{\omega} = \omega_\alpha \boldsymbol{X}^\alpha$. Appropriate tensor products give bases for spaces of tangent tensors of any order, and so higher order tensorfields are coordinatized as well. Thus the metric tensor g has coordinate functions $g_{\alpha\beta}$, $\alpha, \beta \in \{0, 1, 2, 3\}$ given by $g_{\alpha\beta} = \boldsymbol{g}(\boldsymbol{X}_\alpha, \boldsymbol{X}_\beta)$, and the Riemann tensor is determined by the coordinate functions $R^\alpha{}_{\beta\gamma\delta}$ such that $\mathbf{Riem}(\boldsymbol{X}_\beta, \boldsymbol{X}_\gamma, \boldsymbol{X}_\delta) = R^\alpha{}_{\beta\gamma\delta} \boldsymbol{X}_\alpha$.

Our next goal is to realize the Riemann curvature operator as a differential operator, that is, to give a formula relating its components, $R^\alpha{}_{\beta\gamma\delta}$, to the components $g_{\alpha\beta}$ of the metric tensor. We start with the expression for the Christoffel symbols

$$\Gamma^\beta_{\alpha\gamma} = \frac{g^{\beta\delta}}{2} \left(\frac{\partial g_{\alpha\delta}}{\partial x^\gamma} + \frac{\partial g_{\delta\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\delta} \right), \tag{1}$$

where the $g^{\alpha\delta}$ are the components the matrix inverse to $g_{\alpha\delta}$. These enter the formula for coordinatization of the covariant derivative operator:

$$\nabla \boldsymbol{Y} = \nabla_\alpha Y^\beta \, \boldsymbol{X}_\beta \otimes \boldsymbol{X}^\alpha, \quad \nabla_\alpha Y^\beta = \frac{\partial Y^\beta}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} Y^\gamma.$$

It can then be shown that

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\epsilon\gamma} \Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\epsilon\delta} \Gamma^\epsilon_{\beta\gamma}. \tag{2}$$

The coordinates of the Ricci tensor, scalar curvature, and Einstein tensor are then obtained through simple linear algebraic operations:

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}, \quad R = g^{\alpha\beta} R_{\alpha\beta}, \quad G_{\alpha\beta} = R_{\alpha\beta} - (1/2) R g_{\alpha\beta}. \tag{3}$$

From (1)–(3), we see that each of the ten components $G_{\alpha\beta}$ of the Einstein tensor is a second order quasilinear partial differential operator in the four independent variables x^α and the ten dependent variables $g_{\alpha\beta}$. Written out in full each contains over 1,000 terms!

If the components $T_{\alpha\beta}$ of the stress-energy tensor are given, then the Einstein equations $G_{\alpha\beta} = T_{\alpha\beta}$ form a systems of ten PDEs in ten unknowns and we might hope for existence and uniqueness with appropriate boundary conditions. However, this is not the case. The *Bianchi identities* imply that Einstein tensor is always divergence-free, i.e., the differential equations $\nabla_\alpha(g^{\alpha\beta}G_{\beta\gamma})$ are satisfied no matter what the metric coefficients $g_{\alpha\beta}$. Thus the ten possible components $T_{\alpha\beta}$ cannot be chosen arbitrary, but must satisfy four compatibility conditions. Roughly only six may be given. Correspondingly, we may expect that only six of the metric components may be determined from the equations, and the other four are not determined (or must be set "arbitrarily"). This indeterminacy results from the use of a coordinate description of the Einstein equations. The spacetime manifold may always be reparametrized, leading to a different set of coordinate functions, and thus a totally different solution to the PDEs, but still representing the same geometric object. The choice of parametrization is often referred to as *gauge freedom.*

4 The ADM $3+1$ Decomposition

Arnowitt, Deser, and Misner[2] proposed a strategy for developing a formally well-posed system of equations to determine the metric for all $t \geq 0$. Consider spacetime as foliated by the the slices $t =$ constant. Then the spatial components g_{ij}, $1 \leq i, j \leq 3$, of the metric restricted to a slice of the foliation are the components of a Riemannian metric on that slice; these of course vary with t. It is these components of the metric that will be determined from the Einstein equations, while the remaining four components, g_{00} and $\beta_i := g_{0i}$ will be assigned otherwise (by convention Roman indices vary from 1 to 3 and Greek indices from 0 to 3). We introduce the alternate notation γ_{ij} for g_{ij}, the components of a 3×3 symmetric, positive definite matrix-valued function, and let γ^{ij} denote the components of the inverse matrix. Finally we set $\alpha = \sqrt{-g_{00} + |\beta|^2}$ where $|\beta|^2 = \gamma^{ij}\beta_i\beta_j$. In short we have partitioned the metric tensor as

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \left(\begin{array}{c|ccc} |\beta|^2 - \alpha^2 & \beta_1 & \beta_2 & \beta_3 \\ \hline \beta_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \beta_2 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \beta_3 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{array}\right).$$

The quantities α and β_i, called the *lapse* and *shift*, will be determined independently of the Einstein equations, and the equations will be used to determine the time-varying spatial metric γ_{ij}.

Next we partition the Einstein equations in the same way. The four components G_{00} and G_{0i} give the *constraint equations* and the remaining six components G_{ij} the *evolution equations*. It turns out that the constraint equations don't involve second derivatives with respect to time. Hence if we introduce the time-derivative of the spatial metric, or more precisely, define

$$K_{ij} = -\frac{1}{2\alpha}\left(\frac{\partial \gamma_{ij}}{\partial t} - \tilde{\nabla}_i \beta_j - \tilde{\nabla}_j \beta_i\right), \tag{4}$$

the *extrinsic curvature*, then the constraint equations may be written as differential equations in only the three spatial independent coordinates, with time entering as a parameter. In the definition of K_{ij}, $\tilde{\nabla}$ denotes the covariant differentiation operator with respect to the spatial metric γ_{ij}. In a similar way, below we will write $\tilde{R}_{ij}$ and $\tilde{R}$ for the spatial Ricci tensor and scalar curvature.

Specifically the constraint equations are

$$\tilde{R} + (\operatorname{tr} K)^2 - |K|^2 = 2\rho, \tag{5}$$

$$\tilde{\nabla}_j (K_{ij} - \gamma_{ij} \operatorname{tr} K) = J^i, \tag{6}$$

where $\tilde{R}$ is the scalar curvature of the spatial metric, $\operatorname{tr} K = \gamma^{ij} K_{ij}$, $|K|^2 = \gamma^{ik}\gamma^{jl} K_{ij} K_{kl}$, and *matter energy density* and *matter momentum density* ρ and J^i are derived from the stress-energy tensor. This is a system of four quasilinear spatial differential operators in the twelve unknowns γ_{ij}, K_{ij} which must be satisfied on each time slice.

The evolution equations may then be written

$$\frac{\partial \gamma_{ij}}{\partial t} = -2\alpha K_{ij} + \tilde{\nabla}_i \beta_j + \tilde{\nabla}_j \beta_i, \tag{7}$$

$$\frac{\partial K_{ij}}{\partial t} = \alpha \tilde{R}_{ij} - \gamma^{kl} \beta_k \tilde{\nabla}_l K_{ij} + \text{numerous additional terms.} \tag{8}$$

The first of these is just a rewriting of the equation (4) defining the extrinsic curvature. For the full form of the second equation see, e.g., Cook–Teukolsky[3].

We can now summarize the ADM solution procedure for determining a spacetime metric for all time $t > 0$.

1. Choose the lapse and shift in advance or as the computational progresses.
2. Determine values for γ_{ij} and K_{ij} on the slice $t = 0$ satisfying the four constraint equations given in (5) and (6).
3. Evolve the initial data for $t > 0$ using the evolution equations (7) and (8).

A solution so determined will satisfy the Einstein equations since, as a consequence of the Bianchi identities, if the constraint equations are satisfied for

$t = 0$ and the evolution equations are satisfied, then the constraint equations are satisfied for all $t > 0$.

5 The York-Lichnerowicz conformal decomposition

The problem of determining compatible initial data involves finding twelve functions γ_{ij} and K_{ij} at time $t = 0$, satisfying the four constraint equations. The work of York[7] and Lichnerowicz[4] furnishes a strategy to divide the unknown quantities into freely-specifiable functions together with four functions to be determined by the equations.

First, the spatial metric γ_{ij} is multiplicatively decomposed as $\psi^4\hat{\gamma}_{ij}$ where the *background metric* $\hat{\gamma}_{ij}$ is to be specified and the *conformal factor* ψ is to be determined. Second, the extrinsic curvature K_{ij} is additively decomposed into its trace-free part and a multiple of the spatial metric: $K_{ij} = \psi^{-2}\hat{A}_{ij} + (1/3)\,\mathrm{tr}(K)\gamma_{ij}$, where $\mathrm{tr}\,K$ is to be specified. Finally, $\hat{A}_{ij}$ is additively decomposed into a divergence-free trace-free tensor $\hat{A}^*_{ij}$, to be specified, and the symmetric trace-free gradient of a vector potential W^i:

$$\hat{A}_{ij} = \hat{A}^*_{ij} + [\hat{\gamma}_{ik}\hat{\nabla}_j W^k + \hat{\gamma}_{jk}\hat{\nabla}_i W^k - \frac{2}{3}\hat{\gamma}_{ij}\hat{\nabla}_k W^k]. \tag{9}$$

The circumflex over an operator means that it is formed with respect to the background metric. With $\hat{\gamma}_{ij}$, $\mathrm{tr}(K)$, and $\hat{A}^*_{ij}$ given, the constraint equations reduce to a system of four coupled second order PDEs for the conformal factor ψ and the vector potential W^i:

$$\hat{\Delta}\psi - \frac{1}{8}\hat{R}\psi - \frac{1}{12}(\mathrm{tr}\,K)^2\psi^5 + H\psi^{-7} = 2\rho\psi^5,$$
$$\hat{\Delta}W^i + \frac{1}{3}\hat{\gamma}^{il}\hat{\nabla}_l\hat{\nabla}_j W^j - \frac{2}{3}\hat{\gamma}^{ij}\hat{\nabla}_j\,\mathrm{tr}\,K\,\psi^6 + \hat{\gamma}^{ij}\hat{R}_{jk}W^k = J^i\psi^{10},$$

where $H = \hat{\gamma}^{ik}\hat{\gamma}^{jl}\hat{A}_{ij}\hat{A}_{kl}$, and so depends on W^i through (9).

6 Computing black hole initial data

An important problem is to find initial data which represents two black holes which, when evolved, orbit about each other and eventually collide and merge into a single black hole, spewing forth gravity waves along the way. There is a great deal of freedom in developing initial data compatible with the constraints, but it is not so clear how to find data which is physically relevant to black hole collisions.

For one of the simplest physically relevant cases, we take the conformal metric to be flat ($\gamma_{ij} = \delta_{ij}$), and take both the trace and the trace-free, divergence-free portion of the extrinsic curvature to vanish. We look for solutions of the homogeneous constraint equations on the domain $\Omega = \mathbb{R}^3 \setminus (B_1 \cup B_2)$ where the B_i are disjoint balls related to the horizons of the black holes. In this case, the equation for the vector potential is linear and decoupled from the nonlinear equation for the conformal factor, and we may solve it explicitly. It remains to compute the conformal factor, which satisfies the semilinear elliptic equation

$$\Delta\psi + H\psi^{-7} = 0,$$

where the function H is computable from the vector potential. This PDE is supplemented by Robin boundary conditions on the black hole horizons and the condition that ψ tends towards unity at infinity, which is itself usually approximated by an artificial boundary condition on a suitably large sphere.

Figures 1 and 2 are visualizations of results of such a computation. (For color versions, see http://www.math.psu.edu/dna/publications.html.) The black holes radii were taken to be $a = \sqrt{3}/2$ and $2a$ and their centers placed at $(0, 0, -b)$ and $(0, 0, b)$ respectively, where $b = 2\sqrt{3}$. The artificial outer boundary is a sphere centered at origin with radius $128a$. The holes were given linear momenta of $(0, 0, 15)$ and $(0, 0, -15)$ respectively, and no angular momenta. Starting with a coarse mesh of 585 vertices and 2,892 tetrahedra, we solved the scalar elliptic boundary value problem using a program based on piecewise linear finite elements on adaptively generated tetrahedral meshes with Newton's method as the nonlinear iteration and full multigrid as the linear solver. The code is described more fully in A. Mukherjee's thesis[5]. The finest adaptively generated mesh we computed had 63,133 vertices and 346,084 tetrahedra, while the figures show results computed on an intermediate mesh with 13,899 vertices and 75,300 tetrahedra. Specifically, they show a contour plot of ψ on the plane $x = y$ (the plot shade is keyed to the value of ψ), superimposed with the intersection of the mesh with the plane (shrunk to improve visibility) and the mesh edges on the boundary.

To obtain more physically relevant initial data we have recently taken another approach. The metric for a single black hole with a constant spin and linear momentum (a *boosted Kerr black hole*), is known analytically. We write this metric in appropriate coordinates (Kerr-Schild coordinates), and linearly superimpose two such solutions. Of course, the constraint equations are not linear, and so the superposition does not solve them. Therefore, we define only the freely specifiable quantities, $\hat{\gamma}_{ij}$, $\operatorname{tr} K$, and A^*_{ij} from this superposition and use all four constraint equations to do determine the conformal factor and the vector potential. In Figures 3 and 4 we show results of one such computation. Here the hole boundaries were centered at $(0, 0, \pm 2\sqrt{3})$ and the hole radii were both 1. The outer boundary was placed at $32\sqrt{3}$, and holes were given initial velocities chosen to be approximately equal to those for a circular orbit in Newtonian mechanics. Figure 3 shows the conformal factor on the plane $x = 0$, and Figure 4 shows the norm of the vector potential. These computations were done using the finite MC by Michael Holst. That code is also based on piecewise linear elements and adaptive tetrahedral meshes, but permits systems of equations in addition to just scalar equations. A more complete description of our methodology and further computations can be found in a forthcoming paper[1].

7 Challenges ahead

We have briefly described the scientific problem of gravitational wave detection and the complex systems of PDEs it leads to. Some successful numerical computations have been made, especially for the constraint equations, but a great many challenges remain. I close with a brief listing of some of these, already enough to occupy many computational scientists for a long time to come.

- While first steps have been made toward numerical solution of the evolution equations, they have been plagued by poorly understood instabilities. This remains the foremost challenge for the future.

- There are many ways to cast the Einstein equations as PDEs, and the most appropriate formulation for computations is still a matter for research.

- There are many ways to deal with the gauge freedom afforded by the Einstein equations when they are written as PDEs. How this is done will certainly have large implications for the numerical solution and this too is a matter for research. In particular, a good choice of the lapse and shift is important if the numerical problem is to be tractable, but it is not at all clear how to make one.

- I have barely mentioned boundary conditions, but there are many theoretical and computational issues associated with them, both at the horizons of black holes and at the artificial outer boundary, both for the constraint equations and the evolution equations.

- Since the evolution equations cannot be solved exactly, satisfaction of the constraints at the initial time will not guarantee it at all future times. It is not clear to what extent it will be necessary to reimpose the constraints during the course of evolution, nor how to do it.
- If the only matter present is inside the horizons of black holes, it is not necessary to deal with matter terms. But for computations involving stars, the Einstein solvers will have to be coupled with hydrodynamic codes to treat the movement of matter.
- We need to extract far-field behavior from the solution.
- Ultimately we need to solve the inverse problem of deducing the astronomical events from partial knowledge of the far-field.

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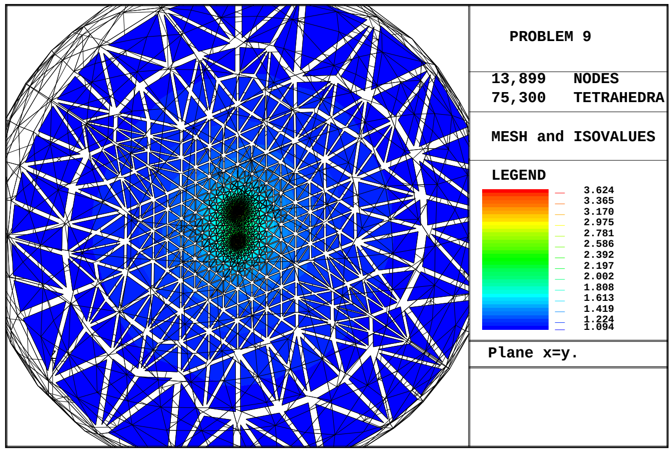

Figure 1. Conformal factor for conformally flat binary black hole initial data.

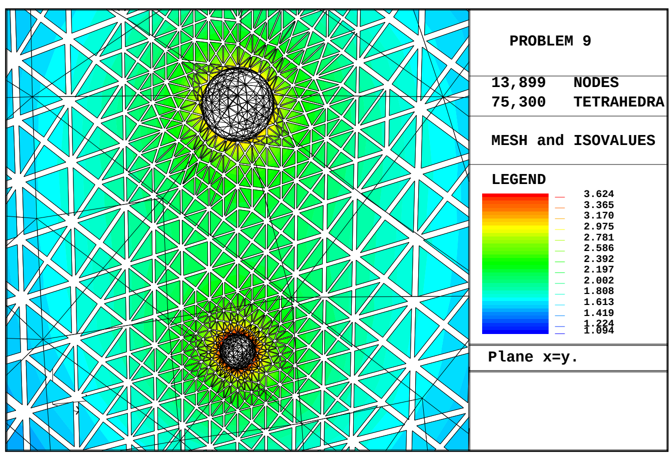

Figure 2. Zoom of the previous figure.

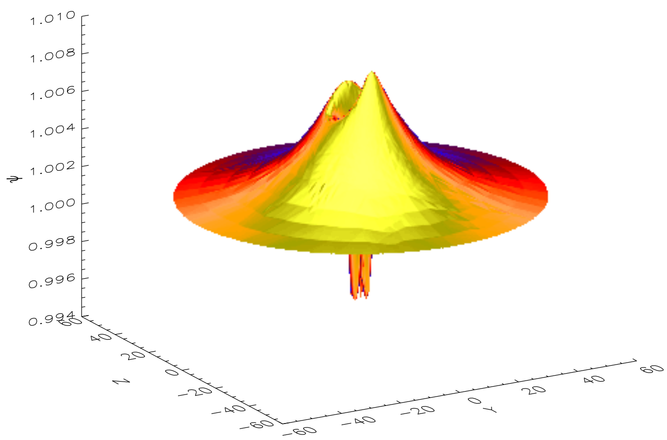

Figure 3. The conformal factor on the plane $x = 0$.

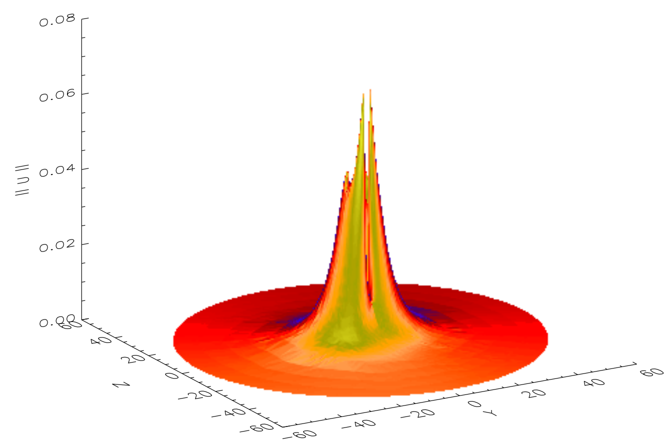

Figure 4. The norm of the vector potential on the plane $x = 0$.