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Finite element differential forms

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A differential form is a field which assigns to each point of a domain an alternating multilinear form on its tangent space. The exterior derivative operation, which maps differential forms to differential forms of the next higher order, unifies the basic first order differential operators of calculus, and is a building block for a great variety of differential equations. When discretizing such differential equations by finite element methods, stable discretization depends on the development of spaces of finite element differential forms. As revealed recently through the finite element exterior calculus, for each order of differential form, there are two natural families of finite element subspaces associated to a simplicial triangulation. In the case of forms of order zero, which are simply functions, these two families reduce to one, which is simply the well-known family of Lagrange finite element subspaces of the first order Sobolev space. For forms of degree 1 and of degree $n-1$ (where n is the space dimension), we obtain two natural families of finite element subspaces, unifying many of the known mixed finite element spaces developed over the last decades.

This note reports on some results which form part of the finite element exterior calculus (FEEC) developed in [1], to which we refer for more complete results, details, and further references. FEEC is a new way of looking at finite element spaces used to discretize some of the most fundamental differential operators. It has brought great clarity and unity to the development and analysis of mixed finite elements for a variety of problems, and has enabled major advances in finite elements for elasticity, preconditioning, a posteriori error estimates, and implementation. The fundamental idea of FEEC is to mimic the framework of exterior calculus by developing finite element spaces of differential forms which exactly transfer key geometrical properties of de Rham theory and Hodge theory from the continuous to the discrete level.

We recall the basic framework. Let Ω denote a bounded region in $\mathbb{R}^n$ (for the start it could be a smooth n-dimensional manifold). For each point x of Ω, the tangent space $T_x\Omega$ is an n-dimensional vector space. If f is a smooth real-valued function of Ω, then df_x is a linear map from $T_x\Omega$ to $\mathbb{R}$. The differential df is a covector field or a 1-form. More generally, an exterior k-form on Ω is a field ω for which ω_x is a skew-symmetric k-linear form on $T_x\Omega \times \cdots \times T_x\Omega \to \mathbb{R}$ for each $x \in \Omega$. We denote the vector space of all smooth k-forms on Ω by $\Lambda^k(\Omega)$. The exterior derivative of such a k-form ω is a $(k+1)$-form $d^k\omega$. It satisfies $d^{k+1} \circ d^k = 0$, so the de Rham complex

$$0 \to \Lambda^0(\Omega) \xrightarrow{d^0} \Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n(\Omega) \to 0$$

is indeed a complex, and we may define the de Rham cohomology spaces $\mathfrak{H}^k = \ker(d^k)/\operatorname{range}(d^{k-1})$, which capture the topology of the domain. For Ω a bounded region in $\mathbb{R}^3$, $\dim \mathfrak{H}^k$ is equal to the number of components, tunnels, and voids in the domain, for $k = 0, 1, 2$, respectively. In this case, the exterior derivative corresponds to the differential operators grad, curl, and div, respectively, which draws a connection between exterior calculus and many of the most basic partial differential equations of mathematical physics, such as the Laplacian or Maxwell's equation.

A key motivation of FEEC is that, even for well-posed boundary value problems involving these operators, simple finite element schemes are often unstable. A simple example is the variational problem of minimizing $\int_\Omega (|\sigma|^2/2 + \operatorname{div}\sigma u + fu)\,dx$ over $\sigma \in H(\operatorname{div})$ and $u \in L^2$, which is a mixed formulation of the Laplacian. For a low order case, suppose we approximate the space L^2 by piecewise constants. A naive finite element discretization of $H(\operatorname{div})$ using continuous piecewise linear functions will then lead to an unstable approximation, while the more sophisticated Raviart–Thomas elements leads to stable approximation, as illustrated in Fig. 1.

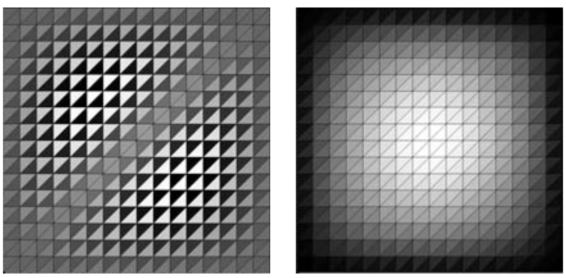

Fig. 1 Piecewise constant approximations of a smooth solution to $\Delta u = 1$ from a mixed formulation using continuous piecewise linear and Raviart–Thomas elements, respectively.

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The spaces H^1, $H(\mathrm{div})$, and $H(\mathrm{curl})$ on a domain in $\mathbb{R}^3$ are examples of the space $H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega) \,|\, d\omega \in L^2\Lambda^{k+1}(\Omega) \,\}$. An important observation of FEEC is that when discretizing the spaces $H\Lambda^k(\Omega)$ with finite dimensional subspaces Λ_h^k, in order to obtain stable discretizations, we should use spaces for which $d^k\Lambda_h^k \subset \Lambda_h^{k+1}$ and for which there exists a *bounded cochain projection*, i.e., a sequence of bounded projection operators $\pi_h^k : H\Lambda^k(\Omega) \to \Lambda_h^k$ for which $d^k\pi_h^k = \pi^{k+1}d^k$. This property has many consequences. For instance, it ensures, with some weak approximation properties, that the induced map on cohomology from the continuous to the finite dimensional case is an isomorphism. This means the cohomology is captured on the finite dimensional level, which is crucial to the stability.

Our goal is to construct finite dimensional spaces of differential forms with this bounded cochain projection property using the finite element assembly process. That is, we assume our domain is triangulated by simplices T. For each simplex we need to specify a finite dimensional space $V(T)$ of polynomial differential forms on T, and a set of degrees of freedom, i.e., a decomposition of its dual space into subspaces associated to the subsimplices. In the prototypical case of Lagrange elements of degree $r > 0$, the space $V(T) = \mathcal{P}_r(T)$, the space of polynomials of degree at most r on T, and, to each subsimplex f of T, we associate the space of functionals

$$W(T,f) = \{\, u \mapsto \textstyle\int_f \mathrm{tr}_{T,f}\, uv\, dx \;:\; v \in \mathcal{P}_{r-1-\dim f}(f) \,\}.$$

A major result of FEEC is that there are two inter-related families of polynomial differential forms $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$, which lead to the natural finite element discretizations of $H\Lambda^k(\Omega)$. They can be assembled into discrete subcomplexes of the de Rham complex in numerous ways, admitting bounded cochain projections. In the case $k = 0$, i.e., discretization of the Sobolev space H^1, the two families coincide with each other and with the Lagrange finite elements. For $k = n$, i.e., discretization of L^2, the spaces coincide with an index shift: $\mathcal{P}_r\Lambda^n = \mathcal{P}_{r+1}^-\Lambda^n$. These spaces assemble to give the space of all piecewise polynomials of degree at most r without inter-element continuity constraints. But for $0 < k < n$ the two families are distinct. In 2 and 3 dimensions they encompass the leading families of mixed finite elements: the Raviart–Thomas and Brezzi–Douglas–Marini elements in $2D$ and the Nedelec edge and face elements of the first and second kind in $3D$. In any number of dimensions, the spaces $\mathcal{P}_1^-\Lambda^k$ are the Whitney forms.

The key to the construction of $\mathcal{P}_r^-\Lambda^k$ is the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$ given by $(\kappa\omega)_x(v^1, \ldots, v^{k-1}) = \omega_x(X, v^1, \ldots, v^{k-1})$, where $X = x - x_0$ for some fixed $x_0 \in \Omega$. This operator satisfies the homotopy relation $(d\kappa + \kappa d)\omega = (r+k)\omega$ for $\omega \in \mathcal{H}_r\Lambda^k$, the space of homogeneous polynomial k-forms of degree r. This immediately implies that $\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$, and so we may define $\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$.

To complete the specification of the finite element spaces, we specify the degrees of freedom. For $\mathcal{P}_r^-\Lambda^k(T)$ and $\mathcal{P}_r\Lambda^k(T)$, respectively, we associate to a subsimplex f of T, the functionals

$$\omega \in \mathcal{P}_r^-\Lambda^k(T) \mapsto \int_f \mathrm{tr}_f\, \omega \wedge \eta,\ \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f); \qquad \omega \in \mathcal{P}_r\Lambda^k(T) \mapsto \int_f \mathrm{tr}_f\, \omega \wedge \eta,\ \eta \in \mathcal{P}_{r+k-d}^-\Lambda^{d-k}(f).$$

These degrees of freedom lead to projection operators from sufficiently smooth forms onto these finite dimensional spaces, and these projections commute with exterior differentiation. These operators are not bounded, but they can be combined with a classical smoothing operator to lead to bounded cochain projections.

These spaces can be assembled into subcomplexes of the de Rham complex in numerous ways. Using only the pure polynomial spaces, we get sequences with decreasing polynomial degree:

$$0 \to \mathcal{P}_s\Lambda^0 \xrightarrow{\ d\ } \mathcal{P}_{s-1}\Lambda^1 \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \mathcal{P}_{s-n}\Lambda^n \to 0.$$

Using only the $\mathcal{P}_r^-\Lambda^k$ spaces, we get another sequence, this time with degree constant

$$0 \to \mathcal{P}_r^-\Lambda^0 \xrightarrow{\ d\ } \mathcal{P}_r^-\Lambda^1 \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \mathcal{P}_r^-\Lambda^n \to 0.$$

If $s = r + n - 1$, then the final spaces $\mathcal{P}_{s-n}\Lambda^n$ and $\mathcal{P}_r^-\Lambda^n$ coincide. In fact, these are only the two extreme cases of 2^{n-1} subcomplexes of the de Rham complex. For each of these, the smoothed projections mentioned above provide bounded cochain projections.

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References

[1] D. Arnold, R. Falk and R. Winther, Acta Numerica **15**, 1–55.