4 From Flight Dynamics to Control Algorithms

In a natural environment, insects are constantly being knocked about by wind or visual and mechanical perturbations. And yet they appear to be unperturbed and are able to correct their course with ease. The halteres, mentioned earlier, provide a fast gyroscopic sensor that enables a fruit fly to keep track of its angular rotational rate. Recent work has found that when a fruit fly's body orientation is perturbed with a torque impulse, it automatically adjusts its wing motion to create a corrective torque. If the perturbation is small, the correction is almost perfect.

Exactly how their brains orchestrate this is a question for neural science as well as for mathematical modeling of the whole organism. By examining how insects turn and respond to external perturbations, we can begin to learn about their thoughts.

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VI.6 The Flight of a Golf Ball

Douglas N. Arnold

A skilled golfer hitting a drive can accelerate his club head from zero to 120 miles per hour in the quarter of a second before making contact with the ball. As a result, the ball leaves the tee with a typical speed of 175 miles per hour and at an angle of 11° to the ground. From that moment the golfer no longer exercises control. The trajectory of the ball is determined by the laws of physics.

Figure 1 The actual trajectory of a golf ball is far from parabolic.

In elementary calculus we learn to model the trajectory of an object under the influence of gravity. The horizontal component of its velocity is constant, while it experiences a vertical acceleration down toward Earth at 32.2 feet per second per second. This results in a parabolic trajectory that can be described exactly. Over a flat course, a ball traveling with the initial speed and launch angle mentioned above would return to Earth at a point 256 yards from the tee. In fact, observation of golf ball trajectories reveals that their shape is far from parabolic, as illustrated in figure 1, and that golfers often drive the ball significantly higher and farther than the simple formulas from calculus predict, even on a windless day. The discrepancy can be attributed to the fact that these formulas assume that gravity is the only force acting on the ball during its flight. They neglect the forces that the atmosphere exerts on the ball passing through it. Surprisingly, this air resistance can help to *increase* the range of the ball.

1 Drag and Lift

Instead of decomposing the air resistance force vector into its horizontal and vertical components, it is more convenient to make a different choice of coordinate directions: namely, the direction opposite to the motion of the ball, and the direction orthogonal to that and directed skyward (see figure 2). The corresponding components of the force of air resistance are then called the *drag* and the *lift*, respectively. Drag is the same force you feel pushing on your arm if you stick it

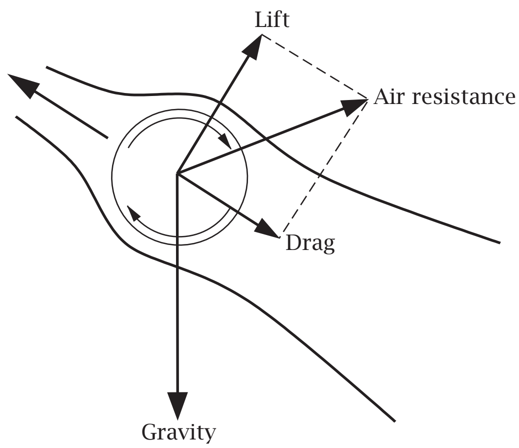

Figure 2 Air resistance is decomposed into drag and lift.

out of the window of a moving car. Golfers want to minimize it, so their ball will travel farther. Lift is largely a consequence of the back spin of the ball, which speeds the air passing over the top of the ball and slows the air passing under it. By Bernoulli's principle, the result is lower pressure above and therefore an upward force on the ball. Lift is advantageous to golfers, since it keeps the ball aloft far longer than would otherwise be the case, allowing it to achieve more distance.

Drag and lift are very much affected by how the air interacts with the surface of the ball. In the middle of the nineteenth century, when rubber golf balls were introduced, golfers noticed that old scuffed golf balls traveled farther than new smooth balls, although no one could explain this unintuitive behavior. This eventually gave rise to the modern dimpled golf ball. Along the way a great deal was learned about aerodynamics and its mathematical modeling. Hundreds of different dimple patterns have been devised, marketed, and patented. However, even today the optimal dimple pattern lies beyond our reach, and its discovery remains a tough challenge for applied mathematics and computational science.

2 Reynolds Number

Drag and lift—which are also essential to the design of aircraft and ships, the swimming of fish and the flight of birds, the circulation of blood cells, and many other systems—are not easy to model mathematically. In this article, we shall concentrate on drag. It is caused by two main sources: the friction between the ball's surface and the air, and the difference in pressure ahead of and behind the ball. The size and relative importance of these contributions depends greatly on the flow regime. In the second half of the nineteenth century, George Stokes and Osborne Reynolds realized that a single number could be assigned to a flow that captured a great deal about its qualitative behavior. Low *Reynolds number* flows are slow, orderly, and laminar. Flows with high Reynolds number are fast, turbulent, and mixing.

The Reynolds number has a simple formula in terms of four fundamental characteristics of the flow: (1) the diameter of the key features (e.g., of the golf ball), (2) the flow speed, (3) the fluid density, and (4) the fluid viscosity. The formula is simple: the Reynolds number is simply the product of the first three of these divided by the fourth. This results in a dimensionless quantity: it does not matter what units you use to compute the four fundamental characteristics as long they are used consistently. The *viscosity*, which enters the Reynolds number, measures how thick the fluid is: water, for example, is a moderately thin fluid and has viscosity 5×10^{-4} lb/ft s, while honey, which is much thicker, has a viscosity of 5 in the same units, and pitch, which is practically solid, has a viscosity of about 200 000 000.

Using the diameter of a golf ball (0.14 feet), its speed (257 feet per second), and the density (0.74 pounds per cubic foot) and viscosity (0.000012 lb/ft s) of air, we compute the Reynolds number for a professionally hit golf ball in flight as about 220 000, much more than a butterfly flying (4000) or a minnow swimming (1), but much less than a Boeing 747 (2 000 000 000).

3 The Mysterious Drag Crisis

At the very beginning of the twentieth century, as the Wright brothers made the first successful airplane flight, aerodynamics was a subject of intense interest. The French engineer Alexandre Gustave Eiffel, renowned for his famous tower, dedicated his later life to the study of aerodynamics. He built a laboratory in the Eiffel tower and a wind tunnel on its grounds and measured the drag on various objects at various Reynolds numbers. In 1912 Eiffel made a shocking discovery: the *drag crisis*. Although one would expect that drag increases with increasing speed, Eiffel found that for flow around a smooth sphere, there is a paradoxical *drop* in drag as the flow speed increases past Reynolds number 200 000. This is illustrated in figure 3. Of great importance in some aerodynamical regimes, the drag crisis begged for an explanation.

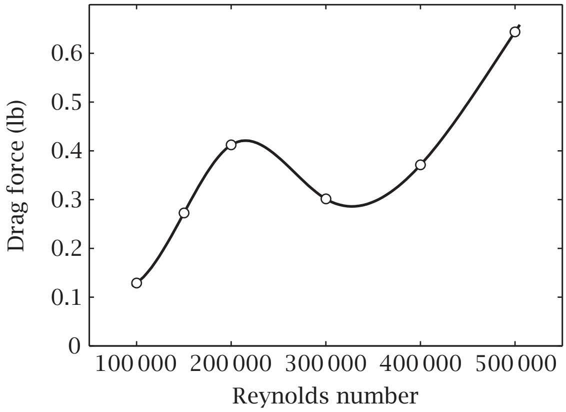

Figure 3 A smooth sphere moving through a fluid exhibits the drag crisis: between Reynolds numbers of approximately 200 000 and 300 000, the drag decreases as the speed increases.

4 The Drag Crisis Resolved

The person who was eventually to explain the drag crisis was Ludwig Prandtl. Eight years before Eiffel discovered the crisis, Prandtl had presented one of the most important papers in the field of fluid dynamics at the International Congress of Mathematicians. In his paper he showed how to mathematically model flow in the *boundary layer*. As a ball flies through the air, a very accurate mathematical model of the flow is given by the system of partial differential equations known as the NAVIER–STOKES EQUATIONS [III.23]. If we could solve these equations, we could compute the drag and thereby elucidate the drag crisis. But the solution of the Navier–Stokes equations is too difficult. Prandtl showed how parts of the equations could be safely ignored in certain parts of the flow: namely, in the extremely thin layer where the air comes into contact with the ball. His equations demonstrated how the air speed increased rapidly from zero (relative to the ball) at the surface of the ball to the ball speed outside a thin layer around the ball surface. Prandtl also described very accurately the phenomenon of *boundary-layer separation*, by which higher pressure behind the ball (the pressure being lower on the top and bottom of the ball, by Bernoulli's principle) forces the boundary layer off the ball and leads to a low-pressure trailing wake behind the ball, much like the wake left behind by a ship. This low-pressure trailing wake is a major source of drag.

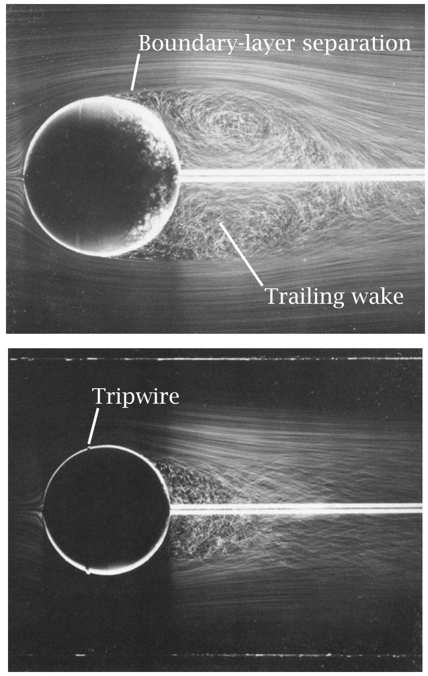

Figure 4 Flow past a smooth sphere, clearly exhibiting boundary-layer separation and the resulting trailing wake. A tripwire has been added to the lower sphere. The resulting turbulence in the boundary layer delays separation and so leads to a smaller trailing wake. (Photos from *An Album of Fluid Motion*, Milton Van Dyke.)

In 1914 Prandtl used these tools to give the following explanation of the drag crisis.

(1) At high speed, the boundary layer become turbulent. For a smooth sphere, this happens at a Reynolds number of about 250 000.
(2) The turbulence mixes fast-moving air outside the boundary layer into the slow air of the boundary layer, thereby speeding it up.
(3) The air in the boundary layer can therefore resist the high-pressure air from behind the ball for longer, and boundary-layer separation occurs farther downwind.
(4) The low-pressure trailing wake is therefore narrower, reducing drag.

Prandtl validated this subtle line of reasoning experimentally by measuring the drag on a sphere in an air stream and then adding a small tripwire to the sphere to induce turbulence. As you can see in a reproduction of this experiment shown in figure 4, the result is indeed a much smaller trailing wake.

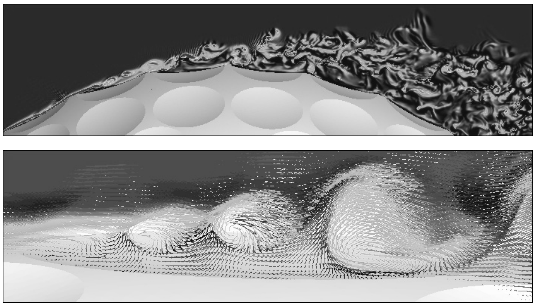

Figure 5 Simulation of flow over dimples.

5 The Role of Dimples

The drag crisis means that when a smooth sphere reaches a Reynolds number of 250 000 or so, it experiences a large decrease in drag and can travel farther. This would be a great boon to golfers were it not for one fact: a golf ball–size sphere would need to travel over 200 miles per hour to achieve that Reynolds number, a speed that is not attained in golf. So why is the drag crisis relevant to golfers? The answer lies in the dimples. Just as a tripwire can be added to a smooth sphere to induce turbulence and precipitate the drag crisis, so can other perturbations of the surface. By suitably roughening the surface of a golf ball, e.g., by adding dimples, the Reynolds number at which the drag crisis occurs can be lowered to about 50 000, well within the range of any golfer. The resulting drag reduction doubles the distance flown by the ball over what can be achieved with a smooth ball.

6 Stalking the Optimal Golf Ball

As we have seen, dimples dramatically affect the flight of a golf ball, so a natural question is how to design an optimally dimpled ball. How many dimples should there be and in what pattern should they be arranged? What shape of dimple is best: round, hexagonal, triangular, ..., some combination? What size should they be? How deep and with what profile? There are countless possibilities, and the thousands of dimple patterns that have been tested, patented, and marketed encompass only a small portion of the relevant design space. Modern computational science offers the promise that this space can be explored in depth with computational simulation, and indeed great progress has been made. For example, in 2010 a detailed simulation of flow over a golf ball with about 300 spherical dimples at a Reynolds number of 110 000 was carried out by Smith et al. (2012). The computation was based on a finite-difference discretization of the Navier–Stokes equations using about a billion unknowns and it required hundreds of hours on a massive computing cluster to solve. It furnished fascinating insights into the role of the dimples in boundary-layer detachment and reattachment, hinted at in figure 5. But even such an impressive computation neglects some important and difficult aspects, such as the spin of the golf ball, and once those issues have been addressed the coupling of the simulation to effective optimization procedures will be no small task. The understanding of the flight of a golf ball has challenged applied mathematicians for over a century, and the end is not yet in sight.

Further Reading

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VI.7 Automatic Differentiation

Andreas Griewank

1 From Analysis to Algebra

In school, many people have suffered the pain of having to find derivatives of algebraic formulas. As in some other domains of human endeavor, everything begins with just a few simple rules:

$$(u + cv)' = u' + cv', \qquad (uv)' = u'v + uv'. \qquad (1)$$

With a constant factor c, the first identity means that differentiation is a *linear process*; the second identity is known as the *product rule*. Here we have assumed that u and v are smooth functions of some variable x, and differentiation with respect to x is denoted by a prime. Alternatively, one writes $u' = u'(x) = \mathrm{d}u/\mathrm{d}x$ and also calls the derivative a differential quotient. To differentiate composite functions, suppose the independent variable x is first mapped into an intermediate variable $z = f(x)$ by the function f, and then z is mapped by some function g into the dependent variable y. One then obtains, for the composite function $y = h(x) \equiv g(f(x))$,

$$h'(x) = g'(f(x))f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x}. \qquad (2)$$

This expression for $h'(x)$ as the product of the derivatives g' and f' evaluated at $z = f(x)$ and x, respectively, is known as the *chain rule*. One also needs to