COERCIVITY OF THE SINGLE LAYER HEAT POTENTIAL

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Abstract

The single layer heat potential operator, $\mathcal{K}$, arises in the solution of initial-boundary value problems for the heat equation using boundary integral methods. In this note we show that $\mathcal{K}$ maps a certain anisotropic Sobolev space isomorphically onto its dual, and, moreover, satisfies the coercivity inequality $< \mathcal{K}_{q,q} >\geq c\|q\|^2$. We thereby establish the well-posedness of the operator equation $\mathcal{K}_q = f$ and provide a basis for the analysis of the discretizations.

§1. Introduction

If $u(x,t)$ solves the homogeneous heat equation for x in a smoothly bounded domain Ω in $\mathbf{R}^3$ and $t > 0$ and vanishes when $t = 0$, then u may be expressed in terms of its Cauchy data on $\Gamma \times \mathbf{R}_+$ as

$$u(x,t) = \int_0^t \int_\Gamma \left[\frac{\partial u}{\partial n}(y,s)K(x-y,t-s) - \frac{\partial K}{\partial n}(x-y,t-s)u(y,s)\right] dyds$$

where $\Gamma = \partial\Omega$ and $K(x,t)$ denotes the fundamental solution to the heat equation,

$$K(x,t) = \begin{cases} \dfrac{\exp(-|x|^2/4t)}{(4\pi t)^{3/2}}, & x \in \mathbf{R}^3, t > 0, \\ 0, & x \in \mathbf{R}^3, t \leq 0. \end{cases}$$

Letting x tend to Γ and empolying a simp jump relation [**5**, p. 137] leads to a mixed Volterra-Fredholm integral equation relating the values of u and $\partial u/\partial n$ on the boundary of the space-time cylinder. Together with given boundary conditions, this integral equation can be used to solve for the Cauchy data of u and then u can be determined globally from the representation formula. Thus the solution of boundary value problems for the heat equation can be reduced to the solution of an integral equation posed on the boundary of the cylinder. This approach is employed with increasing frequency for the numerical solution of transient heat conduction problems [2, 4, 7, 9, 12].

The form of the integral equation to be solved depends on the boundary value problem considered. If Neumann or Robin conditions are specified for u, then the unknown Dirichlet data of u is determined by an integral equation of the second kind. This second kind equation has been studied in depth by Pogorzelski [**10, 11**] and his results have recently been applied to the analysis of numerical methods based on this boundary integral formulation [**3**]. On the other hand, if u solves a Dirichlet problem, then the representation formula leads to an integral equation of the first kind of the form

$$\mathcal{K}_q(x,t) := \int_0^t \int_\Gamma q(y,s)K(x-y,t-s)dyds = f(x,t), \quad x \in \Gamma, t > 0, \tag{1}$$

where q is the unknown flux and f is known. Our aim in this note to prove a simple coercivity estimate for this operator which implies the well-posedness of the integral equation and provides a basis for the analysis of discretizations. It is appropriate to work with the anisotropic Sobolev space $H^{r,s}(S,I) := L^2(I,H^r(S)) \cap H^s(I,L^2(S))$ and its dual $H^{-r,-s}(S,I)(r,s \geq 0, S$ a smooth submanifold of $\mathbf{R}^3, I$ an interval.) Cf. [**6**, Ch. 4]. Our major result may be stated as follows:

Theorem 1. *The single layer heat potential operator $\mathcal{K}$ defines an isomorphism from the Hilbert space $H^{-1/2,-1/4}(\gamma,\mathbf{R}_+)$ onto its dual. Moreover the coercivity estimate*

$$< \mathcal{K}_{q,q} > \quad \geq c\|q\|^2_{-1/2,-1/4} \qquad \text{for all } q \in H^{-1/2,-1/4}(\Gamma,\mathbf{R}_+)$$

holds for some positive constant c.

Remarks. 1. Nedelec and Planchard [**8**] proved a similar result for the electrostatic single layer potential equation for the Laplace equation. It is remarkable that such a coercivity estimate, typical of elliptic operators, also holds for the single layer heat potential. 2. R. Brown [**1**] has recently shown that $\mathcal{K}$ maps $L^2(\Gamma \times \mathbf{R}_+)$ isomorphically onto $H^{1,1/2}(\Gamma,\mathbf{R}_+)$. In the present context, this may be viewed as one of many possible regularity results.

§2. Factorization of the heat potential

For u in the Schwartz class $\mathcal{S}(\mathbf{R}^3 \times \mathbf{R})$ set

$$\begin{aligned}
\|u\|^2_{\mathcal{W}^{1,0}} &= \int_\mathbf{R}\int_{\mathbf{R}^3} |\xi|^2|\hat{u}(\xi,\tau)|^2 d\xi d\tau,\\
\|u\|^2_{\mathcal{W}^{1,1/2}} &= \int_\mathbf{R}\int_{\mathbf{R}^3} (|\xi|^2+|\tau|)|\hat{u}(\xi,\tau)|^2 d\xi d\tau,\\
\|u\|^2_{\mathcal{V}} &= \int_\mathbf{R}\int_{\mathbf{R}^3} (|\xi|^2+|\tau|^2|\xi|^{-2})|\hat{u}(\xi,\tau)|^2 d\xi d\tau,
\end{aligned}$$

where $\hat{u}$ denotes the Fourier transform of u in both space and time. We denote by $\mathcal{W}^{1,0}$, $\mathcal{W}^{1,1/2}$, and $\mathcal{V}$ the completions of $S(\mathbf{R}^3 \times \mathbf{R})$ in the respective norms. These are Hilbert spaces of locally integrable functions, each continuously and densely included in the preceeding. Their dual spaces, $\mathcal{W}^{-1,0}, \mathcal{W}^{-1,-1/2}$, and $\mathcal{V}^*$ are spaces of Schwarz distributions. It is easy to establish the following mapping properties of the heat operator.

Theorem 2. *The heat operator $\Lambda = \partial/\partial t - \Delta$ defines to an isomorphism of $\mathcal{W}^{1,0}$ onto $\mathcal{V}^*$ and of $\mathcal{W}^{1,1/2}$ onto $\mathcal{W}^{-1,-1/2}$. Moreover*

$$\langle \Lambda u, u\rangle = \|u\|^2_{\mathcal{W}^{1,0}} \quad \textit{for all} \quad u \in \mathcal{W}^{1,1/2}.$$

We require certain trace properties of the spaces $\mathcal{W}^{1,1/2}$ and $\mathcal{V}$. It is known that the restriction operator $S(\mathbf{R}^3 \times \mathbf{R}) \to S(\Gamma \times \mathbf{R})$ extends to a bounded surjection of $H^{1,1/2}(\mathbf{R}^3, \mathbf{R})$ onto $H^{1/2,1/4}(\Gamma, \mathbf{R})$ [**6**, Ch. 4, Thm. 4.1]. Now the space $\mathcal{W}^{1,1/2}$ differs from $H^{1,1/2}(R^3, R)$ only in its behavior for $|x|$ near infinity, i.e., if $u \in \mathcal{W}^{1,1/2}$ and $\phi \in S(\mathbf{R}^3)$ then $\phi u \in H^{1,1/2}(\mathbf{R}^3, \mathbf{R})$. Therefore the restriction also maps $\mathcal{W}^{1,1/2}$ boundedly onto $H^{1/2,1/4}(\Gamma, \mathbf{R})$. Similar techniques can be applied to show that even $\mathcal{V}$ is mapped boundedly onto $H^{1/2,1/4}(\Gamma, \mathbf{R})$. Composing with the operation of restriction to the positive time axis, which is a surjection from $H^{1/2,1/4}(\Gamma, \mathbf{R})$ to $H^{1/2,1/4}(\Gamma, \mathbf{R}_+)$, we may conclude the following theorem.

Theorem 3. *The trace operator $\gamma : \mathcal{W}^{1,1/2} \to H^{1/2,1/4}(\Gamma, \mathbf{R}_+)$ is bounded and maps $\mathcal{V}$ onto $H^{1/2,1/4}(\Gamma, \mathbf{R}_+)$.*

Identifying $\mathcal{W}^{-1,-1/2}$ with a subspace of $\mathcal{V}^*$, it follows that the adjoint operator $\gamma^* : H^{-1/2,-1/4}(\Gamma, \mathbf{R}_+) \to \mathcal{W}^{-1,-1/2}$ is an injection of $H^{-1/2,-1/4}(\Gamma, \mathbf{R}_+)$ onto a closed subspace of $\mathcal{V}^*$, so

$$\|\gamma^* q\|_{\mathcal{V}^*} \geq c\|q\|_{H^{-1/2,-1/4}(\Gamma, \mathbf{R}_+)} \qquad \text{for all} \quad q \in H^{-1/2,-1/4}(\Gamma, \mathbf{R}_+) \tag{2}$$

for some positive constant c.

Now let $q \in H^{-1/2,-1/4}(\Gamma, \mathbf{R}_+)$ be a smooth function. Then $\gamma^* q$ is the distribution given by

$$\langle \gamma^* q, \phi\rangle = \int_{\mathbf{R}_+} \int_\Gamma q(y,s)\phi(y,s)\,dyds, \qquad \phi \in S(\mathbf{R}^3 \times R).$$

Therefore,

$$\Lambda^{-1}\gamma^* q(x,t) = (K * \gamma^* q)(x,t) = \int_0^t \int_\Gamma q(y,s) K(x-y, t-s)\,dyds.$$

Comparing with (1) we conclude the following basic factorization result for the single layer heat potential operator.

Theorem 4. *Let $\Lambda : \mathcal{W}^{1,1/2} \to \mathcal{W}^{-1,-1/2}$ denote the heat operator and $\gamma : \mathcal{W}^{1,1/2} \to H^{1/2,1/4}(\Gamma, \mathbf{R}_+)$ the trace operator. The single layer heat potential defined* *on* $S(\Gamma\times\mathbf{R}_+)$ *by* (1) *extends to a bounded linear operator of* $H^{-1/2,-1/4}(\Gamma,\mathbf{R}_+)$ *into* $H^{1/2,1/4}(\Gamma,\mathbf{R}_+)$ *and coincides with the composition* $\gamma\circ\Lambda^{-1}\circ\gamma^*$.

The estimate in Theorem 1 follows easily from the preceeding considerations : for $q\in H^{-1/2,-1/4}(\Gamma,\mathbf{R}_+)$

$$\begin{aligned}\langle q,\mathcal{K}q\rangle &= \langle\Lambda\Lambda^{-1}\gamma^*q,\Lambda^{-1}\gamma^*q\rangle && \text{by Theorem 4}\\ &\geq \|\Lambda^{-1}\gamma^*q\|^2_{\mathcal{W}^{1,0}} && \text{by Theorem 2}\\ &\geq c\|\gamma^*q\|^2_{\mathcal{V}^*} && \text{by Theorem 2}\\ &\geq c\|q\|^2_{H^{1-2/,-1/4}(\Gamma,\mathbf{R}_+)} && \text{by (2)}\ .\end{aligned}$$

By the Lax-Milgram theorem, $\mathcal{K}$ is an isomorphism.

§3. Discretization

Theorem 1 implies quasioptimality of Galerkin approximations to (1). That is, if $q_h\in Q_h$ is defined by the Galerkin equations

$$\langle p,Kp_h\rangle=\langle p,f\rangle \qquad \text{for all } p\in Q_h, \tag{3}$$

for some closed subspace $Q_h\subset H^{-1/2,-1/4}(\Gamma,\mathbf{R}_+)$, then

$$\|q-q_h\|_{H^{-1/2,-1/4}(\Gamma,\mathbf{R}_+)}\leq C\inf_{r\in Q_h}\|q-r\|_{H^{-1/2,-1/4}(\Gamma,\mathbf{R}_+)}, \tag{4}$$

where C is independent of the particular subspace. Typically Q_h is chosen as a tensor product $V_h\otimes S_h$ where $V_h\subset L^2(\Gamma)$ and $S_h\subset L^2(\mathbf{R}_+)$ are piecewise polynomial spaces. The Galerkin equations can then be solved by a time marching algorithm. For example if S_h consists of piecewise constant functions subordinate to a given partition of $\mathbf{R}_+$, the solution of (3) reduces to the solution of a system of dim V_h linear equations for each time step.

Starting from the quasioptimality estimate (4) the derivation of rates of convergence in various norms is standard. For example, suppose V_h consists of piecewise linear functions subordinate to a quasiuniform triangulation of Γ (say Γ is polyhedral) with triangles of size at most h, and that S_h consists of piecewise constant functions with timestep h^2. Then one can show the estimates

$$\|q-q_h\|_{-1/2,-1/4}\leq Ch^{5/2}\|q\|_{2,1}, \qquad \|q-q_h\|_0\leq Ch^2\|q\|_{2,1}.$$

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