# QUADRATIC VELOCITY/LINEAR PRESSURE STOKES ELEMENTS*

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**Abstract.** We study the finite element approximation of the stationary Stokes equations in the velocity-pressure formulation using continuous piecewise quadratic functions for velocity and discontinuous piecewise linear functions for pressure. For some meshes this method is unstable, even after spurious pressure modes are removed. For other meshes there are spurious local pressure modes, but once they are removed the method is stable, and in particular, the velocity converges with optimal order. On yet other meshes there are no spurious pressure modes and the method is stable and optimally convergent for both pressure and velocity.

**1. Introduction.** We study the approximation of the Stokes equations with no-slip boundary conditions:

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \qquad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} = 0 \quad \text{on } \partial\Omega. \tag{1}$$

A finite element discretization of (1) is based on the weak formulation which seeks $(\boldsymbol{u}, p) \in \boldsymbol{V} \times P := \mathring{\boldsymbol{H}}^1(\Omega) \times L^2(\Omega)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \qquad b(\boldsymbol{u}, q) = 0 \quad \forall q \in P,$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}, \quad b(\boldsymbol{v}, q) := \int_\Omega q \operatorname{div} \boldsymbol{v}, \quad (\boldsymbol{f}, \boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}.$$

The solution exists and is unique except for the addition of an undetermined constant to the pressure.

Let $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $P_h \subset P$ be finite dimensional spaces parametrized by $h$ (typically a mesh-size). The approximate solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$ satisfies

$$a(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \qquad b(\boldsymbol{u}_h, q) = 0 \quad \forall q \in P_h. \tag{2}$$

The system (2) always admits solutions. The velocity $\boldsymbol{u}_h$ is always uniquely determined, while the pressure $p_h$ is determined only up to the addition of a pressure mode, i.e., an element of the kernel $N_h$ of the discrete gradient operator, defined by

$$N_h = \{\, p \in P_h \mid b(\boldsymbol{v}, p) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \,\}.$$

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While the constant functions are always pressure modes (assuming, as we may without loss of generality, that $P_h$ contains the constant functions), it may happen that there are additional spurious pressure modes, i.e., that $\dim N_h > 1$. Note that, even though solutions of (2) are not unique, it is possible to compute them, for example by using a penalty method or certain iterative procedures. We shall not discuss this aspect in the present paper.

If Brezzi's condition [6]

$$\gamma_h := \inf_{0 \neq p \in \hat{P}_h} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{1,\Omega} \|p\|_{0,\Omega}} > 0$$

holds, then there are no spurious pressure modes, so that $p_h$ is determined up to an additive constant. (The circumflex in $\hat{P}_h$ indicates the subspace of $P_h$ consisting of functions of mean value zero.) Moreover

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leq C( \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{1,\Omega} + \inf_{q \in P_h} \|p - q\|_{0,\Omega})$$

with $C$ depending on $\gamma_h$, but otherwise independent of the particular subspaces $\boldsymbol{V}_h$ and $P_h$. If $\gamma_h$ remains bounded below by a positive constant as the mesh is refined, then the corresponding finite element method is said to be stable. In this case the convergence of the method (in the energy space $\mathring{\boldsymbol{H}}^1 \times L^2$) depends only on the approximation properties of the subspaces.

Let $M_h$ denote the $L^2$-orthogonal complement of $N_h$ in $P_h$. Then always

$$\bar{\gamma}_h := \inf_{0 \neq p \in M_h} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{1,\Omega} \|p\|_{0,\Omega}} > 0.$$

Consequently the pair $(\bar{\boldsymbol{u}}_h, \bar{p}_h) \in \boldsymbol{V}_h \times M_h$ is uniquely determined by (2) with $P_h$ replaced by $M_h$ and

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}_h\|_{1,\Omega} + \|p - \bar{p}_h\|_{0,\Omega} \leq C( \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{1,\Omega} + \inf_{q \in M_h} \|p - q\|_{0,\Omega}) \tag{3}$$

with $C$ depending only on $\bar{\gamma}_h$. For this estimate to be useful we need to bound $\bar{\gamma}_h$ below and to establish the approximation properties of $M_h$. Note that, while the space $M_h$ may not be easy to compute with in practice, it is easy to see that $\bar{\boldsymbol{u}}_h = \boldsymbol{u}_h$, so (3) gives us an estimate for the velocity error in the original approximation. The pressure approximation $\bar{p}_h$ is the $L^2$-projection of $p_h$ onto $M_h$ (although $p_h$ is not necessarily uniquely determined its projection onto $M_h$ is). Thus we obtain a practical estimate for the pressure from (3) as long as there is an inexpensive post-processing procedure (or filtering) to compute $\bar{p}_h$ from $p_h$. This is indeed the case in the examples considered below.

Although a variety of stable finite element methods have been devised for the Stokes equations (see, e.g., [10], [7]), many of the most natural choices of finite element spaces do not give stable methods. Here we study the $\mathcal{P}^2$–$\mathcal{P}^1$ finite element, i.e., continuous piecewise quadratic approximation to the velocity and discontinuous piecewise linear approximation of the pressure on a partition of triangles. Since for this choice of finite elements, $\operatorname{div} \boldsymbol{V}_h \subset P_h$, it follows that the velocity approximation $\boldsymbol{u}_h$ is divergence-free, and indeed, is the $\mathring{\boldsymbol{H}}^1$-projection of $\boldsymbol{u}$ into the space of divergence-free continuous piecewise quadratic functions. The approximation order for this finite element space is 2; that is, the error in the best approximation of $(\boldsymbol{u}, p)$ in $\mathring{\boldsymbol{H}}^1 \times L^2$ by functions in $\boldsymbol{V}_h \times P_h$ tends to zero like $O(h^2)$. However, the convergence and stability properties of the $\mathcal{P}^2$–$\mathcal{P}^1$ finite element method depend essentially on the mesh family used.

Introducing the stream function, we see that the rate of convergence of the velocity in any Sobolev space $H^s$ for the $\mathcal{P}^2$–$\mathcal{P}^1$ method is the same as the rate of best approximation by $C^1$ piecewise cubic functions in $H^{s+1}$. Based on this observation and the analysis of the Fraeijs de Veubeke–Sander element by Ciavaldini and Nedelec [9], Mercier [11] observed that for a triangulation composed of convex quadrilaterals each of which is partitioned into four triangles by its two diagonals, $\boldsymbol{u}_h$ converges to $\boldsymbol{u}$ with optimal order $h^2$ in $H^1$. Below we shall show strengthen this result by showing that for such meshes the pair $\boldsymbol{V}_h \times M_h$ is actually stable, and that the subspace $M_h$ gives the same order of approximation as $P_h$.

A negative result can be deduced from the work of de Boor and Höllig [5]. See also [4]. They considered the triangulation of the unit square obtained by subdividing into equal subsquares and then bisecting each of these by its positively sloped diagonal. For this mesh family they showed that the space of $C^1$ piecewise cubic polynomials has approximation properties which are one order suboptimal, which is equivalent to suboptimal approximation of $\boldsymbol{u}$ by $\boldsymbol{u}_h$. (The norms considered by de Boor and Höllig are not exactly those pertinent to this discussion. However their argument can be adapted. Cf., [1].)

In this paper, we show that for some mesh families the $\mathcal{P}^2$–$\mathcal{P}^1$ finite element method is stable, and consequently the velocity and the pressure converge with optimal order. For some other mesh families, $\boldsymbol{V}_h \times P_h$ is not stable due to the presence of spurious pressure modes, but we show that $\boldsymbol{V}_h \times M_h$ is stable and obtain optimal order convergence of the velocity. Moreover we show how to obtain an optimally convergent pressure approximation in this case as well.

Results of the sort discussed here can be derived for other finite element choices as well, in particular, for the linear/constant triangle ($\mathcal{P}^1$–$\mathcal{P}^0$) and the isoparametric bilinear/constant quadrilateral ($\mathcal{Q}^1$–$\mathcal{P}^0$). These results will be discussed in forthcoming work.

**2. The macroelement technique.** The key technique of analysis will be the use of macroelements to localize the stability conditions. Many variation of the macroelement technique have been introduced. See for example, [2], [3], [13], [14], and [7, § VI.5.3]. Here we state two precise results, one for the case of a partition into macroelement, and one for the case of a covering by overlapping macroelements.

Given a triangulation $\mathcal{T}_h$ of a polygonal domain $\Omega$, a *macroelement* with respect to

$\mathcal{T}_h$ is a polygonal region $U$ formed by the union of some of the triangles in $\mathcal{T}_h$. The triangles of $\mathcal{T}_h$ which are contained in $U$ form a triangulation of $U$, which we denote by $\mathcal{T}_h^U$. A *macroelement covering* is simply a covering of $\Omega$ by macroelements. Such a covering is called *macroelement partition* if the intersection of a pair of distinct, nondisjoint macroelements is either a single vertex of the triangulation $\mathcal{T}_h$ or a connected set consisting of some edges of the triangulation.

Given a macroelement $U$ we define the localizations of the finite element spaces $\boldsymbol{V}_h$, $P_h$, $N_h$, and $M_h$ to $U$ as follows:

$$\begin{aligned}
\boldsymbol{V}_h^U &= \{\, \boldsymbol{v} \in \boldsymbol{V}_h \mid \operatorname{spt} \boldsymbol{v} \subset U \,\},\\
P_h^U &= \{\, \chi^U p \mid p \in P_h \,\},\\
N_h^U &= \{\, p \in P_h^U \mid b(\boldsymbol{v}, p) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h^U \,\},\\
M_h^U &= \text{ the } L^2\text{-orthogonal complement of } N_h^U \text{ in } P_h^U.
\end{aligned}$$

Here $\chi^U$ denotes the characteristic function of $U$.

The first theorem relates global stability to stability on each macroelement of a macroelement partition.

Macroelement partition theorem. *Let $\mathcal{U}_h$ be a macroelement partition of $\Omega$ with respect to some triangulation $\mathcal{T}_h$. For each $U \in \mathcal{U}_h$ set*

$$\bar{\gamma}_h^U = \inf_{0 \neq p \in M_h^U} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h^U} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{1,\Omega} \|p\|_{0,\Omega}}, \tag{4}$$

*and*

$$\beta_h = \inf_{0 \neq p \in Q_h} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{1,\Omega} \|p\|_{0,\Omega}},$$

*where*

$$Q_h := M_h \cap \sum_{U \in \mathcal{U}_h} N_h^U. \tag{5}$$

*Note that $\bar{\gamma}_h^U > 0$. If $\beta_h > 0$, then $\bar{\gamma}_h > 0$. Moreover if $\alpha$ is a positive lower bound for $\beta_h$ and all the $\bar{\gamma}_h^U$, then $\bar{\gamma}_h$ can be bounded below by a positive constant depending only on $\alpha$.*

For $\mathcal{U}_h$ a macroelement covering of $\Omega$ and $e \in \mathcal{E}_h$, the set of edges of triangles in $\mathcal{T}_h$, we define $L_e$ to be the number of macroelements in $\mathcal{U}_h$ which contain $e$ in their interior. The covering is said to possess the *overlap property* if $L_e \geq 1$ for all edges $e$ of $\mathcal{T}_h$. The quantity $\max_e L_e$ is called the *covering constant* of $\mathcal{U}_h$. To state the result in the case of overlapping coverings we need to state an approximation property of the velocity space. We assume that for all $\boldsymbol{w} \in \mathring{\boldsymbol{H}}^1(\Omega)$ there exists $\boldsymbol{w}_h \in \boldsymbol{V}_h$ such that

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{0,T}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{0,e}^2 + \|\boldsymbol{w}_h\|_1^2 \leq c \|\boldsymbol{w}\|_1^2, \tag{6}$$

where $h_T$ is the diameter of $T$ and $h_e$ is the length of $e$. The verification of this approximation property for piecewise polynomial spaces is standard.

Macroelement covering theorem. *Let $\mathcal{U}_h$ be a macroelement covering satisfying the overlap property. For each $U \in \mathcal{U}_h$ define $\bar{\gamma}_h^U > 0$ by (4). Assume that $\boldsymbol{V}_h$ satisfies (6) and that*

$$\chi^U p \in M_h^U + \mathbb{R}\chi^U, \quad \forall p \in M_h. \tag{7}$$

*Then $\bar{\gamma}_h > 0$. Moreover if $\alpha$ is a positive lower bound for all the $\bar{\gamma}_h^U$, then $\bar{\gamma}_h$ can be bounded below by a positive constant depending only on $\alpha$ and the covering constant for $\mathcal{U}_h$.*

**3. The $\mathcal{P}^2$–$\mathcal{P}^1$ element.** We now specialize to the case of $\mathcal{P}^2$–$\mathcal{P}^1$ elements. First we fix some notations for piecewise polynomials. For $r$ a nonnegative integer we denote by $\mathcal{P}^r(\Omega)$ the space of polynomial functions on $\Omega$ of degree at most $r$. If $\mathcal{T}_h$ is a partition of $\Omega$ we set

$$\begin{aligned} M^r(\mathcal{T}_h) &= \left\{\, q \in L^2 \mid q|_T \in \mathcal{P}^r(T) \quad \forall T \in \mathcal{T}_h \,\right\}, \\ M_0^r(\mathcal{T}_h) &= C^0(\Omega) \cap M^r(\mathcal{T}_h), \\ \mathring{M}_0^r(\mathcal{T}_h) &= \mathring{\boldsymbol{H}}^1(\Omega) \cap M_0^r(\mathcal{T}_h). \end{aligned}$$

The $\mathcal{P}^2$–$\mathcal{P}^1$ finite element spaces are

$$\boldsymbol{V}_h = \mathring{M}_0^2(\mathcal{T}_h), \quad P_h = M^1(\mathcal{T}_h),$$

where $\mathcal{T}_h$ is a triangulation of the polygonal domain $\Omega$.

For the $\mathcal{P}^2$–$\mathcal{P}^1$ element every singular vertex of the triangulation determines a locally supported pressure mode. A vertex is called *singular* if the edges meeting there lie on two straight lines. An interior singular vertex is contained in exactly four triangles of the triangulation, while one, two, or three triangles may meet at a boundary singular vertex. For each singular vertex there is a spurious pressure mode whose support is the union $U$ of the triangles containing it. We denote this function by $p^U$. The vertex values of $p^U$ are given in terms of the lengths of the edges meeting at the singular vertex in figure 1.

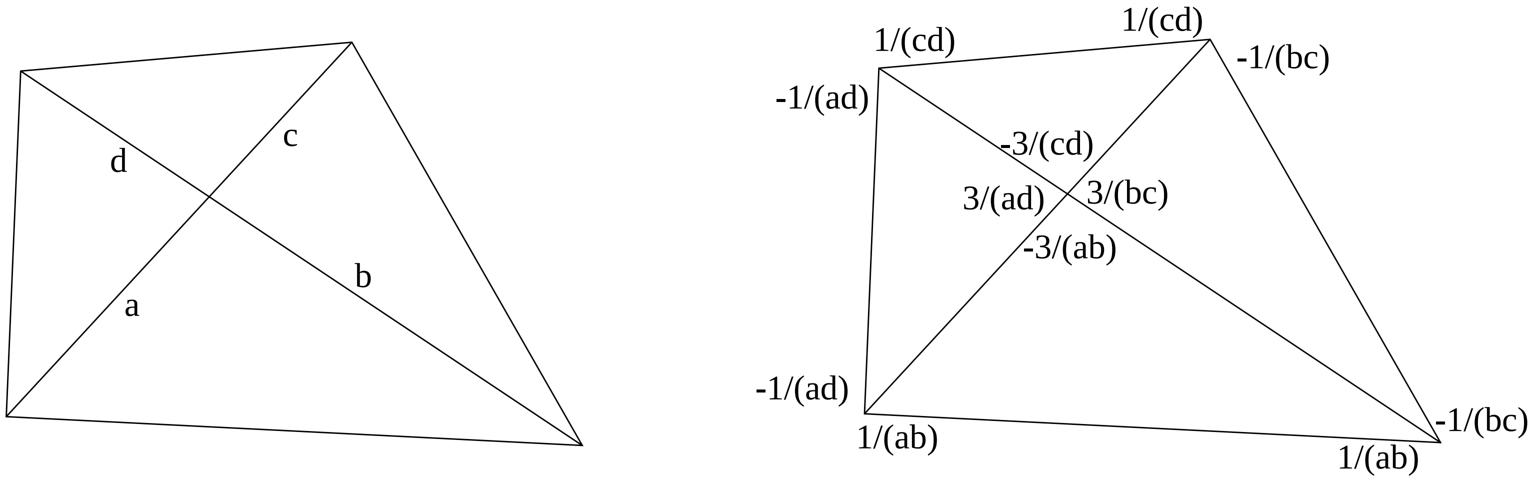


Figure 1. A singular vertex and the vertex values of a spurious pressure mode there.

**Stability of $\mathcal{P}2$–$\mathcal{P}1$ on irregular crisscross meshes.** We begin by analyzing stability in the case considered by Mercier [11]. That is we suppose that $\mathcal{T}_h$ admits a macroelement partition for which each $U \in \mathcal{U}_h$ is a convex quadrilateral, and $\mathcal{T}_h^U$ is simply the decomposition of $U$ into four triangles by its diagonals. Figure 2 shows such a triangulation and figure 1 a typical macroelement. Clearly the spurious pressure modes $p^U$ are linearly independent (having disjoint supports). In fact, together with the constant function they form a basis for $N_h$. The complement $M_h$ is thus the space of piecewise linear pressure fields of mean value zero which on each $U$ are orthogonal to $p^U$. In order to establish stability of $\boldsymbol{V}_h \times M_h$ in this case we apply the macroelement partition theorem. For $U \in \mathcal{U}_h$, $\boldsymbol{V}_h^U$ is the 10-dimensional space of continuous vectorfields which are quadratic on each of the four triangles in $U$ and vanish on $\partial U$, $P_h^U$ is the 12-dimensional space of discontinous piecewise linear fields, $N_h^U$ is the two-dimensional space spanned by $p^U$ and $\chi^U$, and consequently $M_h^U$ is a subspace of $P_h^U$ of dimension 10. Using continuity and scaling we can verify that the local inf-sup constant $\bar{\gamma}_h^U$ is bounded below by a constant that depends only on the minimum angle of the triangles in $\mathcal{T}_h$. Now $Q_h$, defined in (5), is simply $\hat{M}^0(\mathcal{U}_h)$, the space of functions which are constant on each macroelement and have average value zero on $\Omega$. Since this is a subspace of $\hat{M}^0(\mathcal{T}_h)$ and the pair $V_h \times M^0(\mathcal{T}_h)$ is well-known to be stable for the Stokes problem, we see that $\beta_h$ can be bounded below by a positive constant independent of $h$. In this way we establish the stability.

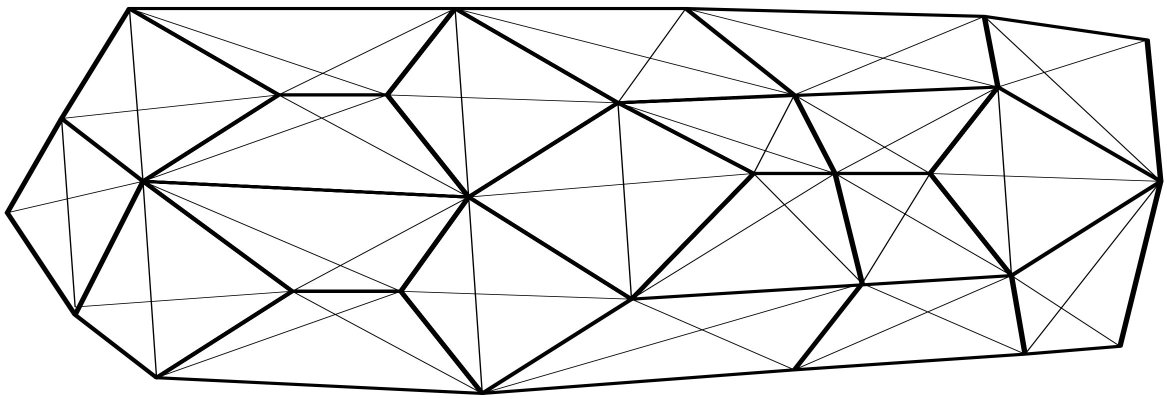

Figure 2. An irregular crisscross triangulation.
Dark lines indicate macroelement boundaries.

In order to recover Mercier's result on the convergence of the velocity, we also need to verify that

$$\inf_{q\in M_h} \|p-q\|_0 \le Ch^2\|p\|_2.$$

Now if $q \in \mathcal{P}^1(U)$, then it is easy to check that

$$\int_U q\, p^U = 0.$$

Thus $M_h$ contains $\hat{M}^1(\mathcal{U}_h)$, and the approximation follows easily.

For such crisscross meshes it is also an easy matter to compute $\bar{p}_h$ once $p_h$ is known, namely, assuming that $p_h$ has already been normalized to have mean value zero, $\bar{p}_h$ is given on each macroelement $U$ by

$$\bar{p}_h = p_h - \lambda^U p^U, \qquad \lambda^U = \frac{\int_U p_h p^U}{\int_U (p^U)^2}. \tag{8}$$

The pressure approximation $\bar{p}_h$ determined by this post-processing procedure converges with optimal order.

**Stability of mixed diagonal and crisscross squares.** We now consider a particular class of triangulations of the unit square $\Omega$. Let $\mathcal{Q}_h$ denote the partition of $\Omega$ into $n^2$ equal subsquares of side length $h = 1/n$ with $n > 2$ an integer. The meshes we consider in this subsection are obtained from $\mathcal{Q}_h$ by subdividing each of the squares by some of their diagonals. Specifically we assume that each square is either subdivided into two triangles by its positively sloped diagonal, which we shall call a diagonal subdivision, or into four triangles by both its diagonals, a crisscross subdivision. For such meshes the space $N_h$ of pressure modes can be determined explicitly.

If every square is subject to a diagonal subdivision, then $N_h$ has dimension 6 with a basis consisting of one function supported in each of the upper left and lower right corners (since these corner vertices are singular), the constant function, and three globally supported spurious modes (cf. [8]and [12]). From the work of de Boor and Höllig cited in the introduction we can deduce that even after eliminating these modes, the method is unstable. That is, $\bar{\gamma}_h$ tends to 0 with $h$. On the other extreme, if every square is subdivided by two diagonals then we have seen that the method is stable after the pressure modes associated with the singular vertices have been removed; i.e., $\bar{\gamma}_h$ remains bounded away from zero.

We now consider what happens if there is a mixture of diagonal and crisscross subdivisions. Our main result is that the resulting method is stable after the removal of local pressure modes associated with the singular vertices, as long as the proportion of crisscross subdivisions is not vanishingly small in any part of the domain.

Let us call a macroelement with respect to the triangulation $\mathcal{T}_h$ a $k$-square if it is a square of side $kh$. Thus $\mathcal{Q}_h$ is the set of all 1-squares. Note that there must always be at least two singular vertices in the triangulation, since either the upper left and lower right corners are singular vertices, or the 1-squares containing them are crisscross subdivided and therefore contain singular vertices.

Theorem.

(1) *Suppose that the triangulation $\mathcal{T}_h$ is obtained from the uniform square partition $\mathcal{Q}_h$ by diagonal and crisscross subdivisions. If the number $\sigma$ of singular vertices of $\mathcal{T}_h$ is at least three, then* $\dim N_h = \sigma + 1$, *and $N_h$ is spanned by the constant function and the locally supported pressure modes associated with the singular vertices.*

(2) *Consider a family of such triangulations $\mathcal{T}_h$ parametrized by $h$ tending to zero. Suppose that there exists a number $k$ such that every $k$-square contains at least one 1-square which is crisscross subdivided. Then there exists a positive constant $\gamma$ depending only on $k$ such that $\bar{\gamma}_h \geq \gamma$.*

Let us sketch the proof of this theorem. For simplicity we assume that the 1-squares at the upper left and lower right corners of the domain are crisscross subdivided, so all the singular vertices occur in the interior. (This assumption can be avoided with some slight additional considerations.) Let $R_h$ be the subspace of $N_h$ spanned by locally supported spurious modes arising from the singular vertices, and let $R_h^\perp$ be the $L^2$-orthogonal complement of $R_h$ in $N_h$. If $p \in R_h^\perp$ then $p$ is constant on each crisscross subdivided square. Hence $p$ belongs to $M^1(\mathcal{T}_h')$ where $\mathcal{T}_h'$ is the triangulation obtained by diagonally subdividing every square $S \in \mathcal{Q}_h$ (so $\mathcal{T}_h$ is a proper refinement of $\mathcal{T}_h'$). We know the kernel space $N_h'$ for this mesh has dimension 6 and we have an explicit basis for it, so $p$ is constrained to be of a very particular form. From this a fairly straightforward argumentation allows us to conclude that $p$ is constant, thus establishing the first part of the theorem.

The proof of the second part is based on the macroelement covering theorem with a slightly complicated choice of macroelements. We associate a macroelement $U$ to each

$(k+2)$-square $S$ as follows. The vertices on the boundary of $S$ are never singular, except possibly the vertices at the upper left and lower right corners of $S$, which are singular or not depending on whether a diagonal or crisscross subdivision is applied to the 1-square in $S$ which contains them. If neither corner vertex is singular then we take $U = S$. If both corner vertices are singular we take $U = S \setminus (T_1 \cup T_2)$ where $T_1$ and $T_2$ are the triangles of $\mathcal{T}_h$ in $S$ which contain these corners. If only one corner vertex is singular we only excise the corresponding triangle from $S$. In this way we obtain a macroelement covering $\mathcal{U}_h$ which satisfies the overlap property. Figure 3 displays several macroelement configurations which may arise for $k = 2$.

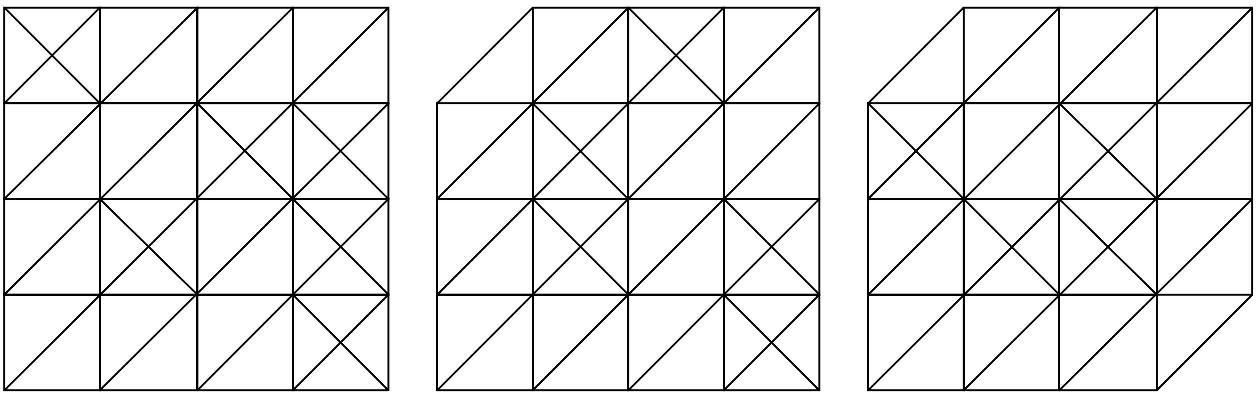

Figure 3. Typical macroelements used to analyze a mixed diagonal/crisscross mesh in the case $k = 2$.

Macroelements constructed in this way may differ due to which corners are excised and the particular pattern of diagonal and crisscross subdivisions. However, since $k$ is a fixed integer, there is a fixed finite number of macroelement configurations such that every macroelement in $\mathcal{U}_h$, for any $h$, is obtained by dilation and translation from one of these. Since $\bar{\gamma}_h^U$ is invariant under dilation and translation, we can bound it below independent of $h$.

It remains to verify hypothesis (7) from the macroelement covering theorem. Now every macroelement $U \in \mathcal{U}_h$ contains a $k$-square in its interior, and hence, by hypothesis, contains an interior 1-square which is crisscross subdivided. Reasoning as in the proof of the first part of the theorem we find that $N_h^U$ is spanned by $\chi^U$ and the locally supported pressure modes from the singular vertices. From this the verification follows easily.

In this case as well we get an optimal estimate for the pressure after a simple post-process. The post-processing procedure is again given by (8), and only needs to be applied on squares $U$ which are crisscross subdivided.

**Stable mesh configurations.** Because of the prominent role played by the local pressure modes at the singular vertices in foregoing analysis, one might think that singular vertices are necessary for stability of the $\mathcal{P}^2$–$\mathcal{P}^1$ element. However, this is not true. We now give an example of a simple mesh family for which there are no spurious pressure modes and the $\mathcal{P}^2$–$\mathcal{P}^1$ element is stable. The triangulation $\mathcal{T}_h$ is determined by from the uniform square partition $\mathcal{Q}_h$ of the unit square by subdividing each 1-square into four triangles by connecting its vertices to the point midway between the center of the subsquare and its bottom edge. See figure 4. Then the stability constant $\gamma_h$ is positive and can be bounded below independent of $h$.

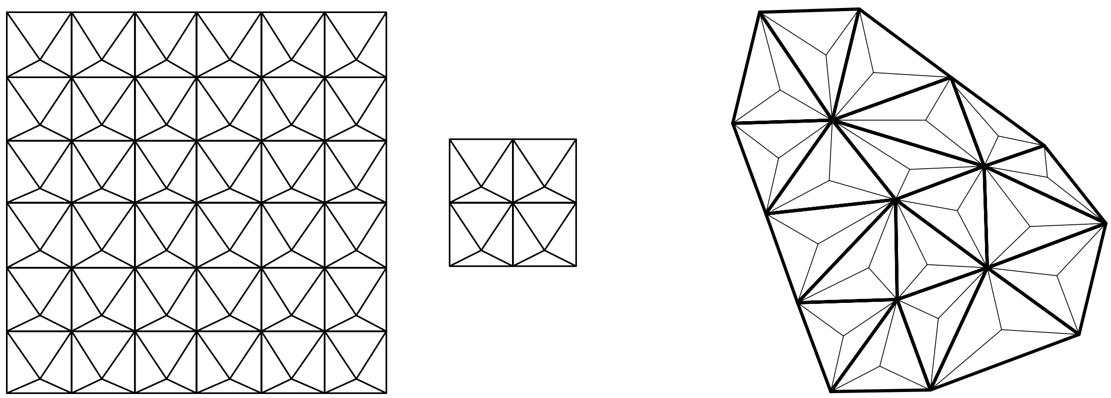

Figure 4. Two triangulations for which the $\mathcal{P}^2$–$\mathcal{P}^1$ element is stable and a typical macroelement for the first triangulation. The macroelements boundaries are marked with dark lines for the second triangulation.

To prove this we shall use the macroelement covering theorem. As macroelement covering we do not use $\mathcal{Q}_h$, but rather take $\mathcal{U}_h$ to be the set of all 2-squares. Note that $\mathcal{U}_h$ satisfies the overlap property. For $U \in \mathcal{U}_h$, we have $\dim \boldsymbol{V}_h^U = 50$ and $\dim P_h^U = 48$. Direct algebraic calculation (assisted by a computer algebra package) shows that $N_h^U$ consists only of the trivial mode $\chi^U$. Now if $q \in N_h$, then $\chi^U q \in N_h^U$, and hence $q$ is constant on $U$. Since this must be true for every $U \in \mathcal{U}_h$, and they overlap, we conclude that $q$ is constant on $\Omega$. This establishes the claim that there are no spurious pressure modes.

Since the local inf-sup constant $\bar{\gamma}_h^U$ is clearly invariant under translation and dilation of $U$, it is immediate that it is independent of $U$ and $h$. To apply the macroelement covering theorem it remains to verify the hypothesis (7). This is trivial in this case, since $M_h^U + \mathbb{R}\chi^U = M_h^U + N_h^U = P_h^U$. Thus we obtain optimal convergence of both the velocity approximation and the raw (unfiltered) pressure approximation.

Another stable mesh configuration can be obtained from any triangulation by connecting the vertices of each triangle to the barycenter, thereby subdividing the triangle into three. A simple computation shows that there are no spurious pressure modes. Taking the triangles of the original triangulation as macroelements it is easy to deduce stability from the macroelement partition theorem, and then optimal convergence of velocity and pressure. Optimal convergence of the velocity for this element can be obtained using a stream function since the $C^1$ cubics on this mesh form the Clough-Tocher finite element space, which has optimal approximation properties.

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