ON NONCONFORMING LINEAR–CONSTANT ELEMENTS FOR SOME VARIANTS OF THE STOKES EQUATIONS

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Sunto. L'uso di elementi finiti lineari a tratti nonconformi per il campo di velocità e costanti a tratti per il campo di pressione, come suggerito da Crouzeix e Raviart, fornisce una approssimazione semplice, stabile e di ordine ottimale per le equazioni di Stokes. Tuttavia, come è noto, questo metodo non è stabile per un sistema molto simile che descrive la deformazione di un solido incomprimibile, linearmente elastico, che differisce dalle equazioni di Stokes solo per la sostituzione dell'operatore di Laplace vettoriale con l'operatore di Lamé dell'elasticità lineare. Dopo un richiamo introduttivo, consideriamo le varianti di questi due sistemi di equazioni ottenute ruotando la variabile vettoriale di un angolo opportuno o, equivalentemente, sostituendo la divergenza con il rotore. Si dimostra che in questo caso l'elemento nonconforme lineare–costante è stabile sia per il sistema che usa il Laplaciano sia per quello con l'operatore di Lamé. Infine, si analizza una applicazione all'analisi di stabilità di una semplice procedura agli elementi finiti per la piastra di Reissner–Mindlin.

Abstract. The use of nonconforming piecewise linear finite elements for the velocity field and piecewise constant finite elements for the pressure fields, as suggested by Crouzeix and Raviart, gives a simple stable, optimal order approximation to the Stokes equations. As is well-known, however, this method is not stable for the very similar system describing the deformation of an incompressible linearly elastic solid, which differs from the Stokes equations only in that the vector Laplace operator is replaced by the Lamé operator of linear elasticity. After reviewing this situation we consider the variants of these two systems of equations obtained by rotating the vector variable through a right angle, or equivalently by replacing the divergence by the rotation. We show that in this case the nonconforming linear–constant element is stable both for the system involving the Laplacian and for that involving the Lamé operator. Finally we discuss an application to the stability analysis of a simple finite element procedure the Reissner–Mindlin plate.

1. Nonconforming linear velocity–constant pressure Stokes element. The weak formulation of the Stokes system determines the velocity field field $\boldsymbol{u}$ and the pressure field p in the spaces $\mathring{\boldsymbol{H}}^1(\Omega)$ and $\hat{L}^2(\Omega)$ (the subspace of $L^2(\Omega)$ consisting of function of mean value zero), from the equations

$$(1)\qquad \begin{aligned} a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)&=f(\boldsymbol{v}) \text{ for all } \boldsymbol{v}\in\mathring{\boldsymbol{H}}^1(\Omega),\\ b(\boldsymbol{u},q)&=0 \qquad \text{for all } q\in\hat{L}^2(\Omega).\end{aligned}$$

Here

$$a(\boldsymbol{u},\boldsymbol{v})=a_1(\boldsymbol{u},\boldsymbol{v}):=\int_\Omega \mathcal{D}\,\boldsymbol{u}:\mathcal{D}\,\boldsymbol{v},\quad b(\boldsymbol{v},q)=b_1(\boldsymbol{v},q):=\int_\Omega q\,\operatorname{div}\boldsymbol{v},\quad f(\boldsymbol{v}):=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v},$$

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with Ω denoting the domain, which we assume to be a plane polygon, and $\boldsymbol{f}$ a given function in $\boldsymbol{L}^2(\Omega)$. Note that we use boldface type to denote 2-vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions, and use script type in a similar way for 2×2–matrix objects. In particular $\mathcal{D}\,\boldsymbol{u}$ denotes the matrix of partial derivatives of the vector-valued function $\boldsymbol{u}$, that is, the rows of $\mathcal{D}\,\boldsymbol{u}$ are the gradients of the components of $\boldsymbol{u}$. For simplicity we have assumed homogeneous Dirichlet boundary conditions.

Let $\mathcal{T}_h$ denote a triangulation of Ω and define

$$\boldsymbol{H}^1(\mathcal{T}_h) = \left\{\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,|\, \boldsymbol{v}|_T \in \boldsymbol{H}^1(T) \text{ for all } T \in \mathcal{T}_h \,\right\},$$

which is a Hilbert space with norm

$$\|v\|_{1h} := \left(\sum_{T \in \mathcal{T}_h} \|v\|^2_{H^1(T)} \right)^{1/2}.$$

For $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$ define $\mathcal{D}_h\,\boldsymbol{v} \in \mathcal{L}^2(\Omega)$ and $\operatorname{div}_h \boldsymbol{v} \in L^2(\Omega)$ by applying the differential operators piecewise and set

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = a_{1h}(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \mathcal{D}_h\,\boldsymbol{u} : \mathcal{D}_h\,\boldsymbol{v}, \quad b_h(\boldsymbol{v}, q) = b_{1h}(\boldsymbol{v}, q) := \int_\Omega q \operatorname{div}_h \boldsymbol{v},$$

for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$, $q \in L^2(\Omega)$. To define a (possibly nonconforming) finite element method for the Stokes system we choose $\boldsymbol{V}_h \subset \boldsymbol{H}^1(\mathcal{T}_h)$ and $W_h \subset \hat{L}^2(\Omega)$ and seek $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p_h \in W_h$ such that

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}) + b_h(\boldsymbol{v}, p_h) &= f(\boldsymbol{v}) \text{ for all } \boldsymbol{v} \in \boldsymbol{V}_h, \\ b_h(\boldsymbol{u}_h, q) &= 0 \qquad \text{for all } q \in W_h. \end{aligned} \tag{2}$$

The convergence of such a procedure depends on three factors: stability, consistency, and approximability. This is expressed precisely in the following basic theorem [2, Ch. II.2.6].

Theorem 1. *Let*

$$\boldsymbol{Z}_h = \{\, \boldsymbol{v} \in \boldsymbol{V}_h \,|\, b_h(\boldsymbol{v}, q) = 0 \textit{ for all } q \in W_h \,\} \tag{3}$$

and set

$$\begin{aligned} \gamma_h^1 &= \inf_{\substack{\boldsymbol{v} \in \boldsymbol{Z}_h \\ \boldsymbol{v} \neq 0}} \frac{a_h(\boldsymbol{v}, \boldsymbol{v})}{\|\boldsymbol{v}\|^2_{1h}}, \\ \gamma_h^2 &= \inf_{\substack{q \in W_h \\ q \neq 0}} \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{b_h(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{1h} \|q\|_0}, \\ M_h &= \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_h \\ \boldsymbol{v} \neq 0}} \frac{a_h(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - (\boldsymbol{f}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1h}}. \end{aligned} \tag{4}$$

If γ_h^1 and γ_h^2 are positive then the discrete problem (2) has a unique solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p_h \in W_h$. Moreover there exists a constant C depending only a positive lower bound for γ_h^1 and γ_h^2, but otherwise independent of h, for which

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1h} + \|p - p_h\|_0 \le C(\inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{1h} + \inf_{q \in W_h} \|p - q\|_0 + M_h).$$

The quantities γ_h^1 and γ_h^2 determine the stability of the discretization. The assumptions that they are bounded above zero uniformly in h are Brezzi's first and second conditions respectively. The quantity M_h measures the consistency error. For a conforming finite element method, i.e., one for which $\boldsymbol{V}_h \subset \mathring{\boldsymbol{H}}^1(\Omega)$, $M_h = 0$. In this case we also have

$$\gamma_h^1 \ge \gamma := \inf_{\substack{\boldsymbol{v} \in \mathring{\boldsymbol{H}}^1(\Omega) \\ \boldsymbol{v} \ne 0}} \frac{a(\boldsymbol{v}, \boldsymbol{v})}{\|\boldsymbol{v}\|_1^2}.$$

Note that γ is independent of the choice of conforming finite element spaces and is positive by virtue of Poincaré's inequality. Thus the first Brezzi condition is automatic for conforming finite element methods for the Stokes problem, and stability depends only on γ_h^2 (which is often referred to as the LBB constant in this context).

A simple and seemingly natural choice of conforming finite element spaces is the subspace of piecewise linear functions in $\mathring{\boldsymbol{H}}^1(\Omega)$ for $\boldsymbol{V}_h$ and the subspace of piecewise constant functions in $\hat{L}^2(\Omega)$ for W_h. The nodal diagram for this element choice is shown in Figure 1a.

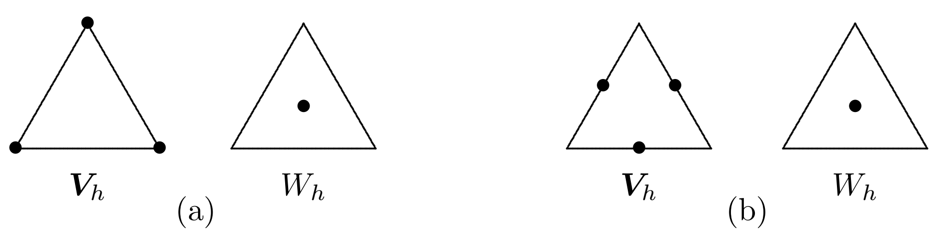

FIG. 1. (a) The conforming linear–constant element. (b) The nonconforming linear–constant element.

Unfortunately, this simplest possible Stokes element is notoriously unstable. On any triangulation with at least three vertices on the boundary the dimension of the pressure space exceeds that of the velocity space, from which it follows that $\gamma_h^2 = 0$ and the finite dimensional problem is singular. Moreover, while the discrete velocity field $\boldsymbol{u}_h$ is uniquely determined (as it is for any conforming method for the Stokes problem), for this choice of elements $\boldsymbol{u}_h$ belongs to the space of divergence-free fields piecewise linear fields, and on many meshes, for example on a uniform diagonal mesh of the square as shown in Figure 2, this space is known to reduce to zero. So even after accounting for the indeterminancy of the pressure we have no convergence.

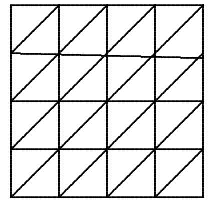

FIG. 2. A uniform diagonal mesh of the unit square.

While the simple conforming linear velocity–constant pressure Stokes element is not stable, Crouzeix and Raviart [4] showed that a stable element is obtained by simply enlarging $\boldsymbol{V}_h$ to be the *nonconforming* piecewise linear approximation of $\mathring{\boldsymbol{H}}^1(\Omega)$. That is, $\boldsymbol{V}_h$ is defined to be the space of functions which are linear on each triangle, continuous at midpoints of element edges, and zero at midpoints of element edges contained in $\partial\Omega$, while again W_h is taken to be the space of piecewise constant functions in $\hat{L}^2(\Omega)$. See Figure 1b. Although $\boldsymbol{V}_h \not\subseteq \mathring{\boldsymbol{H}}^1(\Omega)$ a discrete analogue of Poincaré's inequality holds, namely there exists a positive constant γ independent of h for which

$$\inf_{\substack{\boldsymbol{v}\in\boldsymbol{V}_h\\ \boldsymbol{v}\neq 0}} \frac{a_{1h}(\boldsymbol{v},\boldsymbol{v})}{\|\boldsymbol{v}\|_{1h}^2} \geq \gamma. \tag{5}$$

It follows immediately that γ_h^1 is bounded above zero uniformly in h. Moreover, it can be shown that γ_h^2 is bounded above zero uniformly, and that the consistency error satisfies

$$|M_h| \leq Ch(\|\boldsymbol{u}\|_2 + \|p\|_1). \tag{6}$$

Consequently the method is convergent with optimal order:

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1h} + \|p-p_h\|_0 \leq Ch(\|\boldsymbol{u}\|_2 + \|p\|_1).$$

2. Some variants of the Stokes system. An important variant of the Stokes system arises as the equations for an incompressible linearly elastic material. This is given by (1) with the same meaning for the form b, but with

$$a(\boldsymbol{u},\boldsymbol{v}) = a_2(\boldsymbol{u},\boldsymbol{v}) := (\mathcal{C}\,\mathcal{E}\,\boldsymbol{u}, \mathcal{E}\,\boldsymbol{v}) \tag{7}$$

where $\mathcal{E}\,\boldsymbol{v}$ denotes the symmetric part of the gradient $\mathcal{D}\,\boldsymbol{v}$ and $\mathcal{C}$ denotes the elasticity tensor, a positive-definite linear operator from the space of symmetric 2×2 real matrices to itself. The nonconforming linear Stokes element is not suitable for this variant Stokes system. While clearly the second stability constant γ_h^2 and the consistency error M_h are bounded as before, it is no longer true that γ_h^1 is bounded above zero. In fact, in [5] Falk and Morley show that the discrete analogue of Korn's first inequality

$$\inf_{\substack{\boldsymbol{v}\in\boldsymbol{V}_h\\ \boldsymbol{v}\neq 0}} \frac{a_{2h}(\boldsymbol{v},\boldsymbol{v})}{\|\boldsymbol{v}\|_{1h}^2} \geq \gamma > 0 \tag{8}$$

does not hold. Specifically on the triangulation of the square consisting of four triangles obtained by subdividing with both diagonals they exhibit a nonzero $\boldsymbol{v} \in \boldsymbol{V}_h$ with $a_{2h}(\boldsymbol{v}, \boldsymbol{v}) = 0$. We now describe a generalization of their construction. For this purpose we define a function $\boldsymbol{z} \in \boldsymbol{L}^\infty(\mathbb{R}^2)$ by

$$\boldsymbol{z}(x, y) = (-1)^{i+j} \begin{pmatrix} y - j + 1/2 \\ i - 1/2 - x \end{pmatrix} \text{ for } i - 1 < x < i,\ j - 1 < y < j,\ i, j \in \mathbb{Z}.$$

Note that the restriction of $\boldsymbol{z}$ to each of the lattice squares $(i-1, i) \times (j-1, j)$ belongs to the space of linearized rigid motions, i.e., satisfies $\mathcal{E}\,\boldsymbol{z} \equiv 0$ there. Moreover, $\boldsymbol{z}$ vanishes at the centers of all the squares and, though not continuous across the square edges, is continuous at their midpoints. This information is summarized in Figure 3.

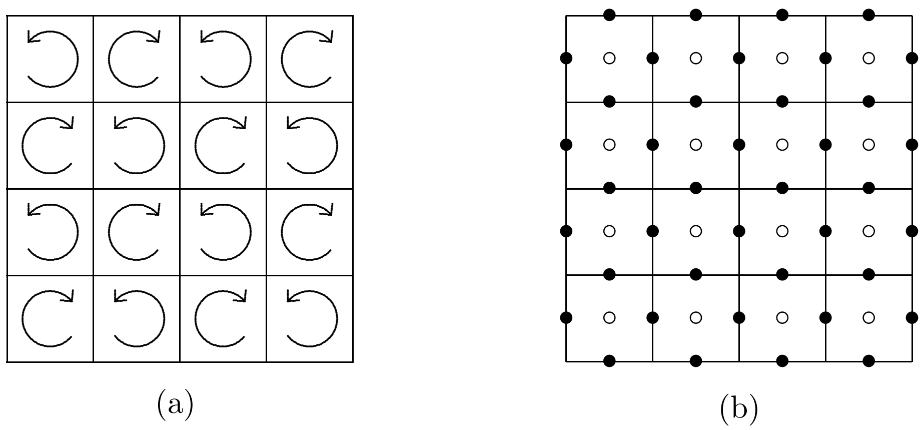

FIG. 3. (a) Schematic of the function $\boldsymbol{z}$. (b) Points where $\boldsymbol{z}$ vanishes ($\circ$) and points where $\boldsymbol{z}$ is continuous ($\bullet$).

Now take Ω to be the square with vertices $(\pm n, 0)$, $(0, \pm n)$ for some $n \in \mathbb{N}$ (more generally we could let Ω be any polygon with vertices at integer lattice points and sides of slope ± 1). The nonempty intersections of the lattice squares with Ω gives a partition of Ω into squares and triangles. We construct a triangulation $\mathcal{T}_h$ by subdividing each of the squares by either one or both of its diagonals. For $n = 3$, two of the many possible triangulations which arise in this way are shown in Figure 4.

Now let $\boldsymbol{v} = \boldsymbol{z}|_\Omega$. Then $\boldsymbol{v}$ belongs to the space $\boldsymbol{V}_h$ of nonconforming piecewise linear functions subordinate to the triangulation $\mathcal{T}_h$ vanishing at the boundary nodes, and $\mathcal{E}_h\,\boldsymbol{v} = 0$. Since $\operatorname{div}_h \boldsymbol{v} = \operatorname{tr}(\mathcal{E}_h\,\boldsymbol{v})$ vanishes, $\boldsymbol{v}$ belongs to $\boldsymbol{Z}_h$. Thus for these meshes of the square (which can of course be dilated to give arbitrarily fine meshes of the unit square) we have $\gamma_h^1 = 0$. For other simple mesh families γ_h^1 does not vanish. For the uniform diagonal mesh family pictured in Figure 2, Falk and Morley show that the left hand side of (8) is nonzero, but construct functions to show that it tends to zero with h. These functions are, moreover, piecewise divergence free, and so belong to $\boldsymbol{Z}_h$. Hence, for the uniform diagonal mesh family γ_h^1 tends to zero with h.

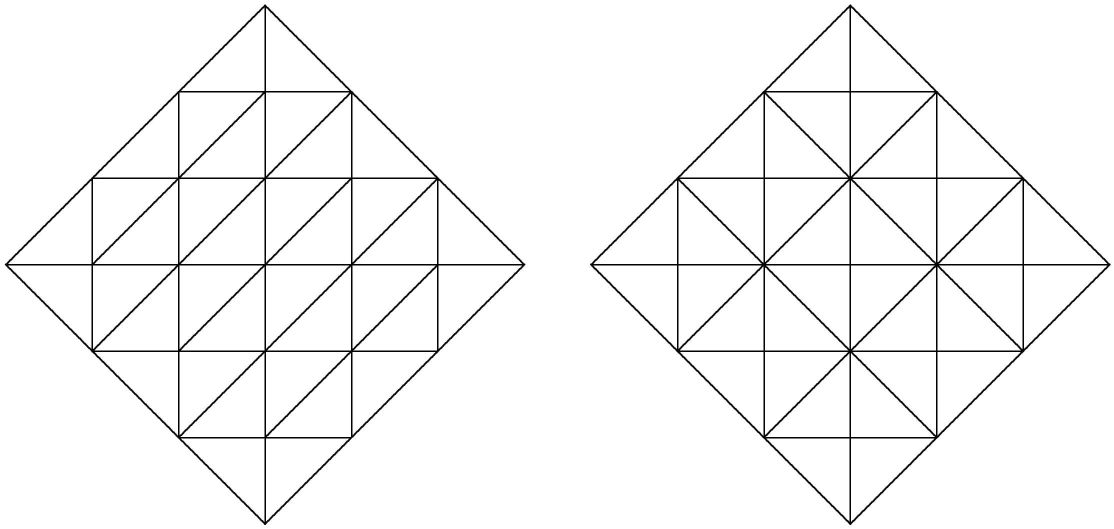

FIG. 4. Two meshes for which their exist non-vanishing functions in $\boldsymbol{V}_h$ in the kernel of $\mathcal{E}_h$.

Next we consider two additional variants of the Stokes system by taking

$$b(\boldsymbol{v}, q) = b_2(\boldsymbol{v}, q) = (\operatorname{rot} \boldsymbol{v}, q). \tag{9}$$

Here $\operatorname{rot} \boldsymbol{v} = \operatorname{div}(\mathcal{X} v)$ with

$$\mathcal{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The system with $a = a_2$, $b = b_2$ arises in the analysis of finite element approximations of the Reissner–Mindlin plate model and is the main motivation of this note.

For these two systems, as well as for the two considered previously, it is easy to see that the consistency error M_h for the nonconforming linear–constant element is bounded as in (6). It is also clear that the lower bound for γ_h^2 which holds for $b = b_1$ applies as well for $b = b_2$. Thus the convergence of the method depends on Brezzi's first condition, a positive uniform lower bound for γ_h^1. We have seen that when $b = b_1$, this condition holds for the standard Stokes system $a = a_1$, but not for the incompressible elasticity system $a = a_2$. When $b = b_2$ and $a = a_1$ we can again conclude the first Brezzi condition from the discrete Poincaré inequality (5) and so obtain optimal order convergence. One might suspect that when $a = a_2$ and $b = b_2$, the failure of the discrete Korn inequality (8) would lead to the failure of the first Brezzi condition. In fact this is not true as is shown by the following theorem, which is the principal observation of this note.

THEOREM 2. *Let $\boldsymbol{V}_h$ denote the nonconforming piecewise linear approximation of $\mathring{\boldsymbol{H}}^1(\Omega)$, and W_h the space of piecewise constant functions of mean value zero. Define bilinear forms a and b by (7) and (9), respectively, and let a_h and b_h denote the corresponding piecewise integrated forms. Then γ_h^1 defined by (3) and (4) is bounded below by a positive constant γ independent of the triangulation $\mathcal{T}_h$.*

Proof. To prove this result we note that the antisymmetric part of the gradient of any vectorfield $\boldsymbol{v}$ can be written as

$$\frac{1}{2}[\mathcal{D}\,\boldsymbol{v} - (\mathcal{D}\,\boldsymbol{v})^T] = \frac{1}{2}(\operatorname{rot}\boldsymbol{v})\mathcal{X}.$$

It follows that

$$\mathcal{D}_h\,\boldsymbol{v} = \mathcal{E}_h\,\boldsymbol{v} + \frac{1}{2}(\operatorname{rot}_h\boldsymbol{v})\mathcal{X} \text{ for all } \boldsymbol{v} \in \mathring{\boldsymbol{H}}^1(\mathcal{T}_h).$$

Now if $\boldsymbol{v} \in \boldsymbol{Z}_h$, then $\operatorname{rot}_h \boldsymbol{v} = 0$, so $\mathcal{D}_h\,\boldsymbol{v} = \mathcal{E}_h\,\boldsymbol{v}$. Therefore

$$a_{2h}(\boldsymbol{v}, \boldsymbol{v}) \geq \lambda \|\, \mathcal{E}_h\,\boldsymbol{v}\|_0^2 = \lambda \|\, \mathcal{D}_h\,\boldsymbol{v}\|_0^2 \geq \lambda\gamma \|\boldsymbol{v}\|_{1h}^2$$

where λ is the least eigenvalue of the positive definite operator C and γ is the constant of the discrete Poincaré inequality (5). Thus γ_h^1 is bounded below by $\lambda\gamma$. ∎

3. Application to the approximation of plates. In this section we refer briefly to work-in-progess which motivated the present study. In [3] Brezzi, Fortin, and Stenberg present a procedure for deriving approximation schemes for the Reissner–Mindlin plate model which are stable and optimal order uniformly in the plate thickness. Essential to their approach is a stable choice of finite element spaces $\boldsymbol{V}_h$, W_h for the Stokes equations, such that the pressure space W_h can also be paired with a finite element approximation Γ_h of $\mathring{\boldsymbol{H}}(\operatorname{rot}, \Omega)$ to give a stable choice of elements for second order scalar elliptic problems. Exploiting the many known stable mixed finite element methods for the Stokes equations and for second order elliptic problems they obtain several families of Reissner–Mindlin elements in this manner. The lowest order method obtained with this approach uses piecewise linear functions for the transverse displacement and piecewise quadratic functions for the rotation. As the authors of [3] point out, a method which uses linear shape functions for both displacement and rotation does not work because of the instability of the linear–constant Stokes element. Indeed, of the Reissner–Mindlin elements which have heretofore been shown to be stable the one which comes closest to using only linear shape functions arises from a different approach altogether. This is the element due to Falk and the author [1] which uses conforming linear elements augmented by a bubble function for the rotation and *nonconforming* linear elements for the transverse displacement.

Admitting the possibility of nonconforming elements in the approach of [3] suggests a Reissner–Mindlin element with nonconforming linear rotations and conforming linear displacements (note that the nonconformity is reversed with respect to that in [1]), based on the nonconforming linear–constant Stokes element and the lowest order Raviart–Thomas mixed element. In fact, precisely such an element has been recently suggested by Oñate, Zarate, and Flores [6]. For the analysis of this method it is essential to note that the variant of the Stokes system which arises in the analysis of the Reissner–Mindlin model is the one discussed in § 2 with $a = a_2$, $b = b_2$. As we have seen, the linear–constant Stokes element is stable for this system, despite the failure of the discrete Korn's inequality. Combining this fact with the approach of [3] it is straightforward to analyze Oñate's method, at least in the limiting (and typically most difficult) case of plate thickness zero. A full analysis in the general case will by Falk and the author elsewhere.

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