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# A NEW MIXED FORMULATION FOR THE NUMERICAL SOLUTION OF ELASTICITY PROBLEMS

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## Summary

A mixed formulation for boundary value problems in linear elastostatics is presented. This formulation differs slightly from the classical Hellinger-Reissner formulation. The unknown fields are the displacement and a tensor related but not equal to the stress. The tensors appearing in the formulation need not be symmetric, and consequently mixed finite elements developed for scalar second order elliptic problems may be applied directly.

## Introduction

This note reports on continuing work of the author and R. S. Falk of Rutgers University. A more complete account is in preparation.

The following notational conventions will be employed. Lower case letters with undertildes are used to denote 3-vectors, lower case letters with double undertildes denote $3\times 3$ tensors. Fourth order tensors are denoted by capital letters, their components by the corresponding lower case letters. The product of a fourth order tensor and a second order tensor is second order; thus $\tau = A\sigma$ means

$$\tau_{ij} = \sum_{k,\ell=1}^{3} a_{ijk\ell}\sigma_{k\ell}, \quad i,j = 1,2,3.$$

If $X$ is a space of scalars, $X$ denotes the space of vectors with components in $X$. If $Y$ is a space of vectors, $Y$ denotes the space of tensors with rows in $Y$. The subspace of symmetric tensors in $Y$ is denoted $Y_s$. We will use the space $H(\mathrm{div},\Omega)$ of square integrable vector-valued functions on a domain $\Omega$ with square integrable divergences and the corresponding spaces $H(\mathrm{div},\Omega)$ and $H_s(\mathrm{div},\Omega)$.

The system of (anisotropic,inhomogeneous) linear elasticity consists of the constitutive equations

$$\sigma = C\varepsilon(u) \tag{1}$$

and the equilibrium equations

$$\mathrm{div}\,\sigma = f. \tag{2}$$

These equations hold in the domain $\Omega \subseteq \mathbb{R}^3$ occupied by the elastic material and must be supplemented by appropriate boundary conditions. The vector-valued functions $u$ and $f$ give the displacement and imposed force, respectively and the strain tensor $\varepsilon(u)$ is defined as $[\mathrm{grad}\,u + (\mathrm{grad}\,u)^T]/2$. The coefficients of the elasticity tensor $C$ are given functions on $\Omega$ satisfying

$$c_{ijk\ell} = c_{k\ell ij} = c_{jik\ell}. \tag{3}$$

Consequently the stress tensor $\sigma$ is symmetric. The elasticity tensor satisfies the positivity condition:

$$\gamma_0|\tau|^2 \le \tau : C\tau \le c_0|\tau|^2, \quad \tau \in \mathbb{R}_s, \tag{4}$$

where $\gamma_0$ and $c_0$ are positive constants (independent of the point $x \in \Omega$ where the coefficients are evaluated) and

$$|\tau|^2 = \tau : \tau = \sum_{i,j} \tau_{ij}^2.$$

Therefore, for each $x$ the mapping $\tau \mapsto C\tau$, viewed as a linear operator on the six dimensional space $\mathbb{R}_s$, is invertible. Its inverse may also be written as $\tau \mapsto A\tau$ with $A$ a fourth order tensor whose coefficients satisfy

$$a_{ijk\ell} = a_{k\ell ij} = a_{jik\ell}. \tag{5}$$

These coefficients form the <u>compliance</u> <u>tensor</u>. From (4) it follows that

$$\gamma_1|\tau|^2 \le \tau : A\tau \le c_1|\tau|^2, \quad \tau \in \mathbb{R}_s \tag{6}$$

with $\gamma_1 = c_0^{-1}$, $c_1 = \gamma_0^{-1}$ positive constants.

For simplicity, we consider the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ but this restriction is inessential. To obtain a weak form of the resulting boundary value problem we invert the constitutive equations (1), multiply by a tensor $\tau \in H_s(\mathrm{div},\Omega)$, and use the identity

$$\int_\Omega \varepsilon(u) : \tau = \int_\Omega \mathrm{grad}\,u : \tau = -\int_\Omega u \cdot \mathrm{div}\,\tau.$$

The first equality holds since $\tau$ is symmetric, the second is Green's formula. Also testing the equilibrium equations against a function in $v \in L^2(\Omega)$ we arrive at the weak formulation, which is known as the Hellinger-Reissner principle:

Find $(\sigma,u) \in H_s(\mathrm{div}) \times L^2$ such that

$$\int A\sigma:\tau + \int u\cdot \mathrm{div}\,\tau = 0, \quad \tau \in H_s(\mathrm{div}), \tag{7}$$

$$\int v\cdot \mathrm{div}\,\sigma = \int f\cdot v, \quad v \in L^2. \tag{8}$$

(Note that we henceforth suppress explicit notation of the domain $\Omega$ from the function spaces and integrals.) The variational formulation (7,8) is *mixed* in that both stress and displacement fields are present.

To define a mixed finite element method based on this formulation we must specify finite element spaces $S_h \subset H_s(\mathrm{div})$, $V_h \subset L^2$. The approximate solution $(\sigma_h,u_h) \in S_h \times V_h$ is then determined by the equations analogous to (7,8) with $(\tau,v)$ restricted to $S_h \times V_h$. As is well-known the choice of the mixed finite elements (i.e., of the spaces $S_h$ and $V_h$) is a delicate one: the approximate solution need not approximate well even if the finite element spaces afford good approximation. Necessary and sufficient conditions for the quasioptimal estimate

$$\|\sigma-\sigma_h\|_{H(\mathrm{div})} + \|u-u_h\|_{L^2} \le c \inf_{\substack{\tau\in S_h \\ v\in V_h}} [\|\sigma-\tau\|_{H(\mathrm{div})} + \|u-v\|_{L^2}] \qquad (9)$$

are given by Brezzi [4]. In particular, in light of (6) the following conditions are sufficient (the second is also necessary):

$$\mathrm{div}\, S_h \subset V_h, \qquad (10)$$

$$\inf_{0\neq v\in V_h} \sup_{0\neq\tau\in S_h} \frac{\int v\cdot \mathrm{div}\,\tau}{\|v\|_{L^2}\|\tau\|_{H(\mathrm{div})}} \ge \gamma > 0. \qquad (11)$$

These conditions imply (9) with $c$ depending only on $c_1$ and $\gamma_1$ in (6) and $\gamma$ in (11).

The analogous problem of finding convenient, accurate, stable finite elements for mixed formulations of scalar second order problems has been effectively solved. Here one needs spaces $S_h \subset H(\mathrm{div})$ (vector-valued) and $V_h \subset L^2$ (scalar-valued) satisfying in analogy to (10,11)

$$\mathrm{div}\, S_h \subset V_h, \qquad (12)$$

$$\inf_{0\neq v\in V_h} \sup_{0\neq\tau\in S_h} \frac{\int v\, \mathrm{div}\,\tau}{\|v\|_{L^2}\|\tau\|_{H(\mathrm{div})}} \ge \gamma > 0. \qquad (13)$$

For example, we recall the definition of Raviart-Thomas-Nedelec spaces of order $k \ge 0$, [11,12,15]. Let a regular family of triangulations $\{\Delta_h\}_h$ of $\Omega$ be given, with meshsize $|h|$ tending to zero. Then the spaces

$$S_h = \{\tau \in H(\mathrm{div}) \mid \forall T\in\Delta_h\ \exists p \in P_k,\ q\in P_k \ni \tau(x) = p(x) + q(x)x \text{ on } \mathring{T}\},$$

$$V_h = \{v \in L^2 \mid \forall T\in\Delta_h\ \exists p\in P_k \ni v(x) = p(x) \text{ on } \mathring{T}\},$$

satisfy (12,13) with $\gamma$ independent of $h$. These spaces approximate to $O(|h|^{k+1})$ in $H(\mathrm{div}) \times L^2$, and admit simple nodal bases. They have been successfully used in many computations, and thoroughly analyzed [5-9,11-15]. For second order scalar elliptic problems energy estimates, $L^2$ estimates in the separate variables, negative norm estimates, $L^\infty$ estimates, superconvergence, and interior estimates have all been shown.

Several authors have tried to adapt these elements to the elasticity equations, but such an adaptation is not straightforward. The difficulty arises from the requirement that $S_h$ consist of symmetric tensors. (Equation (7) does not remain valid for $\tau$ asymmetric.) In particular one cannot simply choose $S_h = S_h \times S_h$ (the space of all tensors with rows in the Raviart-Thomas-Nedelec space $S_h$) and $V_h = V_h \times V_h$, although this choice clearly satisfies (10,11).

A number of ways around this difficulty have been proposed, at least in the plane homogeneous isotropic case. Johnson and Mercier [10] developed a composite linear triangular element sharing many, though not all, of the desirable properties of the Raviart-Thomas elements and showed that it gave quasioptimal approximation. In [3] Arnold, Douglas, and Gupta proposed a family of composite elements of quadratic and higher orders, with stability properties similar to those of the Raviart-Thomas Nedelec elements and in particular satisfying (10) and (11). Another approach was followed by Arnold, Brezzi, and Douglas in [2]. In that paper the symmetry of the stress tensor is enforced only weakly via a Lagrange multiplier. Finite element spaces for the stress, displacement, and Lagrange multiplier based on, (although somewhat more complex than) the lowest degree Raviart-Thomas spaces were proposed, and the appropriate analogues of (10,11) were shown.

Each of these approaches apparently offers an acceptable mixed finite element method for elasticity, at least in the plane, but none fully shares the simplicity and desirable convergence properties of the Raviart-Thomas-Nedelec elements for scalar equations. Here we propose a new mixed formulation for elasticity in two or three dimensions to which the Raviart-Thomas-Nedelec (or other) elements may be applied directly. Our formulation differs slightly from (7,8). The unknowns are the displacement $u$ and a tensor $\rho$ of the form

$$\rho = E\,\mathrm{grad}\, u$$

with the coefficient tensor $E$ derived from the elasticity tensor as explained below. The tensor $\rho$ is not symmetric and so does not coincide with the stress tensor, but the stress components may be deduced from the components of $\rho$ simply by linear combinations and so this formulation preserves the advantageous property of the usual mixed formulation that the stress may be derived from the computed unknowns without differentiating.

## A New Mixed Formulation

We define the auxiliary variable $\rho$ by

$$\rho = \sigma + \beta(\mathrm{div}\, u)\delta - \beta(\mathrm{grad}\, u)^T, \qquad (14)$$

where $\delta$ is the identity tensor. There is some freedom in the selection of the constant $\beta$; we take $\beta = \gamma_0/3$ where $\gamma_0$ is a positive constant so that (4) holds. Since

$$\mathrm{div}[(\mathrm{div}\, u)\delta] = \mathrm{grad}\,\mathrm{div}\, u = \mathrm{div}(\mathrm{grad}\, u)^T,$$

the equilibrium equation (2) implies that $\mathrm{div}\,\rho = f$.

Let us show how (14) may be inverted. Clearly $\rho = (C+\beta D)\,\mathrm{grad}\, u$ where $C$ is the elasticity tensor and $D$ is defined by

$$D\tau = \mathrm{tr}(\tau)\delta - \tau^T.$$

If $\tau$ is any tensor we may express it in terms of three vectors, $x$, $y$, $z$ thus:

$$\tau = \begin{bmatrix} x_1 & y_3 & y_2 \\ z_3 & x_2 & y_1 \\ z_2 & z_1 & x_3 \end{bmatrix}.$$

Then

$$\tau : D\tau = (\sum x_i)^2 - |x|^2 - 2y \cdot z$$

$$= 2 \sum_{i<j} x_i x_j - 2y \cdot z \geq -2|x|^2 - 2y \cdot z.$$

Also by (3) and (4)

$$\tau : C\tau = [(\tau+\tau^T)/2] : C[(\tau+\tau^T)/2] \geq \gamma_0 |(\tau+\tau^T)/2|^2$$

$$= \gamma_0 (|x|^2 + 2|(y+z)/2|^2)$$

$$= \gamma_0 (|x|^2 + |y|^2/2 + |z|^2/2 + y \cdot z).$$

Thus

$$\tau : (C+\beta D)\tau \geq (\gamma_0 - 2\beta)|x|^2 + \gamma_0(|y|^2+|z|^2)/2 + (\gamma_0 - 2\beta) y \cdot z$$

$$\geq (\gamma_0 - 2\beta)|x|^2 + \beta(|y|^2+|z|^2) = \frac{\gamma_0}{3}|\tau|^2.$$

It follows that $C + \beta D$ is positive definite on $\mathbb{R}$ (not just on $\mathbb{R}_s$). Let $B$ denote its inverse. Then

$$\gamma_2 |\tau|^2 \leq \tau : B\tau \leq c_2 |\tau|^2, \qquad \tau \in \mathbb{R}, \tag{15}$$

with $c_2 = 3/\gamma_0$, $\gamma_2 = (c_0 + 2\gamma_0/3)^{-1}$. Also $B\rho = \text{grad } u$.

We can now state our mixed formulation of the elasticity problem

Find $(\rho, u) \in H(\text{div}) \times L^2$ such that

$$\int B\rho : \tau + \int u \cdot \text{div } \tau = 0, \qquad \tau \in H(\text{div}), \tag{16}$$

$$\int v \cdot \text{div } \rho = f \cdot v, \qquad v \in L^2. \tag{17}$$

Clearly the pair $(\rho, v)$ solves this problem, and from (15) we easily infer that this solution is unique. Note that we can recover $\sigma$ from $\rho$ simply as $C[B\rho + (B\rho)^T]/2$.

Because asymmetric tensors are admissible as both trial and test functions in our formulation, discretization is straightforward. Let $S_h \subset H(\text{div})$, $V_h \subset L^2$ be any spaces satisfying (12) and (13). Then the corresponding spaces $S_h$ and $V_h$ satisfy (10) and (11), and so satisfy the Brezzi conditions for the discretization of (16,17), and quasioptimal approximation occurs. In particular we may use any of Reviart-Thomas-Nedelec spaces. The various error analyses for these elements carry over directly.

## The Isotropic Case

In the case of an isotropic material the constitutive law (1) has the form

$$\sigma = 2\mu\varepsilon(u) + \lambda(\text{div } u)\delta, \tag{16}$$

where $\mu > 0$ and $\lambda \geq 0$ are the Lamé constants. In this case we choose $\beta = \mu$ and (14) gives

$$\rho = \mu \text{ grad } u + (\mu+\lambda)(\text{div } u)\delta. \tag{17}$$

The analysis sketched in the last section shows that a mixed method based on our formulation will achieve quasioptimal approximation uniformly with respect to $\mu$ and $\lambda$ in any compact subinterval of $(0,\infty)$ and $[0,\infty)$ respectively. However as the material tends to incompressibility the coefficient $\lambda$ tends to infinity, and one of the reasons that mixed methods are used for elasticity problems is that for many elements (any which satisfy (10) and (11) for example), the convergence is uniform with respect to $\lambda \in [0,\infty)$. (In contrast the convergence of most displacement methods with low order elements degenerates in the case of a nearly incompressible material.) Let us recall how the uniformity with respect to $\lambda$ of the standard mixed methods is proven and show that the same reasoning applies to our method.

First recall that despite the coefficient $\lambda$ in (16), $\sigma$ as well as $u$, remains bounded and regular as $\lambda$ tends to infinity. The quantity $\lambda \text{ div } u$ tends to a limit, see, e.g., [3]. From (17) we see that $\rho$ remains bounded and regular also.

For the compliance tensor in the isotropic case we have the equations

$$\tau : A\tau = \frac{1}{2\mu}|\tau|^2 - \frac{\lambda}{2\mu(2\mu+3\lambda)}|\text{tr}(\tau)|^2. \tag{18}$$

Hence one may easily check that

$$\tau : A\tau \geq \frac{1}{2\mu+3\lambda}|\tau|^2, \qquad \tau \in \mathbb{R}_s.$$

The constant $(2\mu+3\lambda)^{-1}$ appearing here is the best possible, and so the constant $\gamma_1$ in (6) tends to zero as $\lambda$ tends to infinity. Nonetheless, as stated, the convergence of the mixed methods does not degenerate as $\lambda$ tends to infinity. This is because dependence of the constant $c$ in (9) on $\gamma_1$ can be weakened to dependence on $\gamma_1'$ given by the bound

$$\tau : A\tau \geq \gamma_1' \|\tau^D\|^2, \qquad \tau \in \mathbb{R}_s, \tag{19}$$

where $\tau^D = \tau - \text{tr}(\tau)\delta/3$ is the deviatoric of $\tau$. (For $\tau \in H(\text{div})$ with $\text{div } \tau = 0$, $\int \text{tr}(\tau) = 0$, the $L^2$ norm of $\tau$ is bounded by a constant multiple of the $L^2$ norm of $\tau^D$. Hence (19) together with (10) is sufficient to establish the first Brezzi condition. See [3] for details.) Finally, since

$$|\tau|^2 = |\tau^D|^2 + \frac{1}{3}|\text{tr}(\tau)|^2,$$

(18) implies (19) with $\gamma' = 1/2\mu$ independent of $\lambda$.

Now the tensor $B$ inverting (17) is given by

$$\text{grad } u = B\rho = \frac{1}{\mu}\rho - \frac{\mu+\lambda}{\mu(4\mu+3\lambda)} \text{tr}(\rho)\delta.$$

Thus

$$\underset{\approx}{\tau} : B\underset{\approx}{\tau} = \frac{1}{\mu}|\underset{\approx}{\tau}|^2 - \frac{\mu+\lambda}{\mu(4\mu+3\lambda)}|\operatorname{tr}(\underset{\approx}{\tau})|^2$$

$$\geq \frac{1}{\mu}|\underset{\approx}{t}^D|^2, \qquad \underset{\approx}{\tau} \in \underset{\approx}{\mathbb{R}}.$$

The constant $\mu^{-1}$ in this bound does not depend on $\lambda$. Consequently as long as the finite element spaces used with our formulation satisfy (10) the first condition of Brezzi holds uniformly in $\lambda$. We conclude that the finite element approximation of an isotropic elastic material via the formulation (16,17) and spaces satisfying (10,11) is quasioptimal uniformly with respect to the Lamé constant $\lambda$.

An extension of the uniformity result to some classes of anisotropic materials will appear.

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