CONTINUOUS DEPENDENCE ON THE ELASTIC COEFFICIENTS FOR A CLASS OF ANISOTROPIC MATERIALS

Douglas N. Arnold
Department of Mathematics
University of Maryland
College Park, MD 20742

Richard S. Falk
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903

Abstract

We prove apriori estimates and continuous dependence on the elastic moduli for the equations of homogeneous orthotropic elasticity. These results are uniform with respect to the three Poisson ratios, Young's moduli, and shear moduli of the material for certain ranges of these constants. These ranges include the possibility that the compliance tensor is singular such as occurs for incompressible materials.

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1. Introduction

The equations of anisotropic elasticity are

$$A\underset{\approx}{\sigma} = \underset{\approx}{\epsilon}(\underset{\sim}{u}) \quad \text{in } \Omega, \tag{1.1}$$

$$\underset{\sim}{\mathrm{div}}\ \underset{\approx}{\sigma} = \underset{\sim}{f} \quad \text{in } \Omega. \tag{1.2}$$

where $\underset{\approx}{\sigma} = (\sigma_{kl})$ is a 3×3 symmetric tensor of unknown stresses, $A\underset{\approx}{\sigma}$ is the tensor $\sum_{k,l} a_{ijkl}\sigma_{kl}$, $\underset{\sim}{u}$ is a 3 vector of unknown displacements, and $\underset{\sim}{f}$ is a given 3 vector of forces, all defined on a smoothly bounded domain $\Omega \in \mathbf{R}^3$. We also use the notations

$$\epsilon_{ij}(\underset{\sim}{u}) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$$

and

$$\underset{\sim}{\mathrm{div}}\ \underset{\approx}{\sigma} = (\sum_{j=1}^{3} \partial\sigma_{1j}/\partial x_j, \sum_{j=1}^{3} \partial\sigma_{2j}/\partial x_j, \sum_{j=1}^{3} \partial\sigma_{3j}/\partial x_j\ )^t$$

(where $\underset{\sim}{v}^t$ denotes the transpose of $\underset{\sim}{v}$). Further notations used in this introduction and throughout the paper are collected in the next section.

The given coefficients a_{ijkl} are constants satisfying

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad 1 \le i, j, k, l \le 3.$$

The tensor of these coefficients is called the compliance tensor. Note that the compliance tensor is determined by specifying 21 of the coefficients.

We shall consider in this paper the mixed displacement and traction boundary conditions:

$$\underset{\sim}{u} = \underset{\sim}{g}_1 \quad \text{on } \Gamma_1,$$

$$\underset{\approx}{\sigma}\,\underset{\sim}{n} = \underset{\sim}{g}_2 \quad \text{on } \Gamma_2. \tag{1.3}$$

Here Γ_1 and Γ_2 are open subsets of $\partial\Omega$ with $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$. For now we assume that Γ_1 and Γ_2 are nonempty. The case of unmixed boundary conditions is considered in Section 4.

We consider a particular class of anisotropic materials, those admitting three orthogonal planes of symmetry, which are termed orthotropic. Included in this class are hexagonal and cubic crystalline structures [14, page 31]. Orthotropic materials are also used to model woods, plywood and other composites [14, pages 59-60], and some biological substances, such as the basilar membrane of the inner ear [10]. Orthotropic materials are characterized by a compliance matrix of the following form:

$$a_{iiii} = 1/E_i, \quad i = 1, 2, 3,$$

$$a_{iijj} = -\nu_{ij}/E_j, \quad 1 \leq i < j \leq 3,$$

$$a_{iijk} = 0, \quad i = 1, 2, 3, \quad 1 \leq j < k \leq 3,$$

$$a_{ijij} = 1/G_k, \quad \text{where } \{k\} = \{1, 2, 3\} \backslash \{i, j\}, \quad 1 \leq i < j \leq 3,$$

$$a_{ijkl} = 0, \quad 1 \leq i < j \leq 3, \quad 1 \leq k < l \leq 3, \quad (i, j) \neq (k, l).$$

Here the E_i are the Young's moduli of the material, the G_i are the shear moduli of the material, and the ν_{ij} are the Poisson ratios. The relation

$$\nu_{ij} E_i = \nu_{ji} E_j, \quad 1 \leq i < j \leq 3,$$

is satisfied, so an orthotropic material is defined by nine independent constants.

The Young's modulus E_i is the ratio of tension to extension when the body is in a state of pure tension in the ith coordinate direction. The shear modulus G_i is the ratio of shear stress to shear strain when the body is in a state of pure shear orthogonal to the ith coordinate direction. It is thus physically evident that $E_i > 0$ and $G_i > 0$, as we shall henceforth assume. The Poisson ratio ν_{ij} is the ratio of compression in the ith direction to extension in the jth direction for a material in a state of pure tension in the jth direction. Thus it seems physically plausible that $\nu_i \geq 0$, as we shall assume (although apparently there are materials violating this condition [8]).

We introduce the symmetrized Poisson ratios $\nu_i = (\nu_{jk} \nu_{kj})^{1/2}$, where $\{j, k\} = \{1, 2, 3\} \backslash \{i\}$, $i = 1, 2, 3$, and the 3×3 symmetric matrices:

$$D = \begin{pmatrix} E_1^{-1/2} & 0 & 0 \\ 0 & E_2^{-1/2} & 0 \\ 0 & 0 & E_3^{-1/2} \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & -\nu_3 & -\nu_2 \\ -\nu_3 & 1 & -\nu_1 \\ -\nu_2 & -\nu_1 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1/G_1 & 0 & 0 \\ 0 & 1/G_2 & 0 \\ 0 & 0 & 1/G_3 \end{pmatrix},$$

and set $B = DMD$. Then the constitutive law (1.1) for an orthotropic material may be written

$$\begin{aligned} &B \,\underset{\sim}{\mathrm{diag}}\, \underset{\approx}{\sigma} = \underset{\sim}{\mathrm{diag}}\, \underset{\approx}{\epsilon}(\underset{\sim}{u}), \\ &G \,\underset{\sim}{\mathrm{offd}}\, \underset{\approx}{\sigma} = \underset{\sim}{\mathrm{offd}}\, \underset{\approx}{\epsilon}(\underset{\sim}{u}), \end{aligned} \tag{1.4}$$

where

$$\underset{\sim}{\mathrm{diag}}\, \underset{\approx}{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33})^t \quad \text{and} \quad \underset{\sim}{\mathrm{offd}}\, \underset{\approx}{\sigma} = (\sigma_{23}, \sigma_{13}, \sigma_{12})^t.$$

It is often assumed that the compliance tensor is positive definite. In this case, $\underset{\approx}{\sigma}$ can be eliminated and the existence, uniqueness, and continuous dependence of the resulting boundary value problem is well known. Our interest is in the uniform continuous dependence of the solution including cases where the compliance tensor is only semidefinite. When the compliance tensor is singular, the displacement automatically satisfies a linear constraint [17] and we shall speak of a constrained material. As discussed below, this includes the important case of incompressible orthotropic materials which frequently appear in the engineering literature (e.g., [21],[7],[19], and [11]). Existence and uniqueness theorems for certain boundary value problems for incompressible anisotropic materials have been established by Debognie [5]. However, he does not consider constraints other than incompressibility nor continuous dependence of the solution on the moduli.

We shall therefore assume that the compliance tensor is positive semidefinite, i.e., that

$$A \underset{\approx}{\tau} : \underset{\approx}{\tau} \geq 0 \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{\mathbf{R}}_s$$

where $\underset{\approx}{\mathbf{R}}_s$ denotes the space of 3×3 symmetric tensors and

$$\underset{\approx}{\tau} : \underset{\approx}{\sigma} = \sum_{i,j=1}^{3} \tau_{ij} \, \sigma_{ij}.$$

In light of (1.4) and the positivity of the G_i, this is clearly equivalent to the assumption that the matrix B is positive semidefinite. Now

$$\det B = (E_1 E_2 E_3)^{-1} \det M$$

and

$$\det M = 1 - 2\nu_1\nu_2\nu_3 - \nu_1^2 - \nu_2^2 - \nu_3^2.$$

Hence necessarily, $\underset{\sim}{\nu} = (\nu_1, \nu_2, \nu_3)^t$ belongs to

$$P := \{ \underset{\sim}{\nu} : \nu_i \geq 0, \quad i = 1, 2, 3, \quad 1 - 2\nu_1\nu_2\nu_3 - \nu_1^2 - \nu_2^2 - \nu_3^2 \geq 0 \}. \tag{1.5}$$

Note that in particular, $\nu_i \leq 1$ for all i. Moreover, given that $\nu_i \geq 0$, it is easily verified that M is a positive semidefinite matrix if and only if $\underset{\sim}{\nu} \in P$.

In this paper we shall consider the questions of existence, uniqueness, apriori estimates, and continuous dependence of solutions to the system (1.1), (1.2), (1.3) in the orthotropic case. Before stating our main theorem, we recall the known results in the much simpler case of an isotropic material. This is the special case in which

$$E_1 = E_2 = E_3 := E,$$

$$\nu_1 = \nu_2 = \nu_3 := \nu,$$

$$G_1 = G_2 = G_3 = E/(1+\nu).$$

In this case the constitutive law (1.1) reduces to

$$[(1+\nu)/E] \, \underset{\approx}{\sigma} - (\nu/E) \, tr(\underset{\approx}{\sigma}) \, \underset{\approx}{\delta} = \underset{\approx}{\epsilon}(\underset{\sim}{u}) \tag{1.6}$$

where $tr(\underset{\approx}{\sigma})$ denotes the trace of $\underset{\approx}{\sigma}$ and $\underset{\approx}{\delta}$ is the 3×3 identity matrix. The Young's modulus E satisfies $0 < E < \infty$ and the positive semidefiniteness condition $\underset{\sim}{\nu} \in P$ reduces to $0 \leq \nu \leq 1/2$. The compliance tensor is positive definite in this case except when $\nu = 1/2$, which corresponds to an incompressible isotropic material. Hence, if $0 \leq \nu < 1/2$, the constitutive law (1.6) may be inverted and the resulting expression for $\underset{\approx}{\sigma}$ substituted in

(1.2) and (1.3). The resulting system, with unknown $\underset{\sim}{u}$, is coercive and standard variational arguments give existence and uniqueness of the solution and a apriori bound for $\underset{\sim}{u}$ in $\underset{\sim}{H}^1(\Omega)$ which is uniform for $\nu \in [0,1/2)$. Unfortunately, this method cannot be used to imply the existence of a solution for $\nu = 1/2$, nor to obtain a uniform apriori bound on $\underset{\approx}{\sigma}$. However, using other methods, the following theorem may be proved.

Theorem 1.1: Let E and ν be real numbers satisfying

$$E > 0, \quad 0 \leq \nu \leq 1/2.$$

Then for all sufficiently smooth data $\underset{\sim}{f}$, $\underset{\sim}{g}_1$, and $\underset{\sim}{g}_2$, there exists a unique pair $(\underset{\approx}{\sigma}, \underset{\sim}{u}) \in \underset{\approx}{L}^2(\Omega) \times \underset{\sim}{H}^1(\Omega)$ satisfying the system of isotropic elasticity (1.6), (1.2), and (1.3). Moreover,

$$\|\underset{\approx}{\sigma}\|_0 + \|\underset{\sim}{u}\|_1 \leq C(\|\underset{\sim}{f}\|_{-1,D} + |\underset{\sim}{g}_1|_{1/2,\Gamma_1} + |\underset{\sim}{g}_2|_{-1/2,\Gamma_2})$$

where C is a constant depending only on Ω and positive upper and lower bounds for E, and the solution $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ depends continuously on E, ν, $\underset{\sim}{f}$, $\underset{\sim}{g}_1$, and $\underset{\sim}{g}_2$. (The norms appearing in the apriori estimate will be defined in the following section.)

We shall prove a result analogous to Theorem 1.1 for orthotropic elasticity. The set P of possible values of the Poisson ratios, defined in (1.5), is pictured in Figure (1.1).

Figure 1.1

The set P of possible values of the Poisson ratios.

Limiting values of the $\underset{\sim}{\nu} = (\nu_1, \nu_2, \nu_3)^t$, that is, values for which the compliance tensor ceases to be positive definite (or $\det B = 0$) are those points on the curved boundary of P. We shall refer to this curved portion of the boundary of P, a curvilinear triangle, as the constraint surface. For $\underset{\sim}{\nu}$ not on the constraint surface, one can again invert the constitutive equation and so it is relatively straightforward to prove that there exists a unique solution to the equations and establish a uniform apriori estimate on the displacement. We shall show that for $\underset{\sim}{\nu}$ on the constraint surface, with the exception of the three corner points, one also gets existence and uniqueness, and we establish uniform estimates and continuous dependence for both displacement and stress. The three corner points on the constraint surface, where two of the Poisson ratios vanish and the third is equal to unity, must be excluded - as we discuss in Section 6, the elasticity problem degenerates as $\underset{\sim}{\nu}$ approaches one of these points. Our continuous dependence results will be valid for $\underset{\sim}{\nu} \in P_0 := P\backslash\{(1,0,0)^t, (0,1,0)^t, (0,0,1)^t\}$.

Our analysis applies in particular to incompressible orthotropic materials. An anisotropic material is called incompressible if for every $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ satisfying (1.1),

$$\operatorname{div} \underset{\sim}{u} = 0.$$

This holds if and only if $A\underset{\approx}{\delta} = 0$. From (1.4) we see that an orthotropic incompressible material is characterized by the condition

$$M\,(E_1^{-1/2}, E_2^{-1/2}, E_3^{-1/2})^t = 0. \tag{1.7}$$

In particular, $\det M = 0$, so $\underset{\sim}{\nu}$ lies on the constraint surface. Moreover, it is easy to check that (1.7) precludes the possibility that $\underset{\sim}{\nu}$ is one of the three corner points. Conversely, if $\underset{\sim}{\nu}$ is any noncorner point on the constraint surface, we show below that M admits a null vector with strictly positive components (Lemma 3.6) and hence the ν_i are the Poisson ratios for some incompressible material.

The main aim of this paper is to establish the following theorem.

Theorem 1.2: Let $E_i > 0$, $G_i > 0$, and $\underset{\sim}{\nu} \in P_0$.

i) For all $\underset{\sim}{f} \in \underset{\approx}{L}^2(\Omega)$, $\underset{\sim}{g_1} \in \underset{\sim}{H}^{1/2}(\Gamma_1)$, and $\underset{\sim}{g_2} \in \underset{\sim}{L}^2(\Gamma_2)$, there exists a unique pair $(\underset{\approx}{\sigma}, \underset{\sim}{u}) \in \underset{\approx}{L}^2(\Omega) \times \underset{\sim}{H}^1(\Omega)$ satisfying the boundary value problem (1.4), (1.2), and (1.3).

ii) The solution satisfies the apriori estimate

$$\|\underset{\approx}{\sigma}\|_0 + \|\underset{\sim}{u}\|_1 \le C(\|\underset{\sim}{f}\|_{-1,D} + |\underset{\sim}{g_1}|_{1/2,\Gamma_1} + |\underset{\sim}{g_2}|_{-1/2,\Gamma_2})$$

where C is a constant depending only on Ω, positive upper and lower bounds for E_i and G_i, and a positive lower bound for the distance of $\underset{\sim}{\nu}$ from the three corners of P.

iii) The solution depends continuously on the elastic moduli $\underset{\sim}{E}$, $\underset{\sim}{G}$, $\underset{\sim}{\nu}$, and on the data $\underset{\sim}{f}$, $\underset{\sim}{g_1}$, and $\underset{\sim}{g_2}$.

The question of continuous dependence on the elastic moduli near an elastic constraint is of great importance. Without such continuous dependence results, the use of constrained models, which represent an idealization of nearly constrained materials, would be unjustified. Nonetheless this question remains largely unresolved. Theorem 1.2 apparently provides the first proof of convergence of unconstrained materials to a constrained material outside of the simplest case, that of an isotropic incompressible material. The isotropic case was examined by Bramble and Payne [3] who proved continuous dependence results for the pure displacement and traction problems and, in particular, showed that as the Poisson ratio tends to 1/2 the displacement and all of its derivatives converge at interior points to the corresponding quantity for the incompressible problem. Results of the same sort have since been derived by Mikhlin [16], Kobel'kov [12], Lazarev [13], and Rostamian [18]. For nonlinear elastic materials asymptotic expansions have been devised which support the convergence of an almost constrained material to a constrained one. Of course these do not provide proofs of convergence. See Spencer [20] for the constraint of incompressibility of elastic solid and Antman [2] for that of inextensibility of an elastica.

Rostamian [18] has derived abstract conditions on the compliance tensor of an anisotropic linearly elastic material which insure continuous dependence of the solution on the elastic moduli. He applied his theory only to the known case of isotropic elasticity, regaining the results of Bramble and Payne [3] and showing also convergence of the stresses. It seems likely that our analysis could be modified to provide a verification of Rostamian's conditions, although we have preferred to argue more directly. Note, however, that Rostamian's theorem is closely related to the more general theorem of Brezzi on which we have relied.

An outline of the paper is as follows. Section 2 contains additional notation used in the paper along with the statement of a theorem due to Brezzi [4] dealing with abstract saddle point problems. This theorem will play a major role in our subsequent analysis. The proof of Theorem 1.2 is given in Section 3. In Section 4 we consider the cases of pure traction and pure displacement boundary conditions and in Section 5 apply the analysis of Section 3 to prove ellipticity of the elastic system uniformly with respect to the elastic moduli. In Section 6, we illucidate the nature of the exceptional cases when $\underset{\sim}{\nu}$ is a corner point of the constraint surface. Finally, in Section 7, we use the ideas previously developed to derive two alternate formulations of the elasticity equations which may be more convenient for some computational and analytic purposes. In the first of these formulations the stress $\underset{\approx}{\sigma}$ is eliminated and a new scalar variable p is introduced. In the case of an isotropic incompressible material these equations are equivalent to the stationary Stokes equations. Related formulations have been previously introduced by Herrmann [9] for isotropic materials; by Taylor, Pfister, and Herrmann [21] and Key [11] for orthotropic materials; and by Debongie [5] for incompressible anisotropic materials. The second formulation is a further simplification possible in the two dimensional constrained case and results in a single fourth order equation, analogous to reduction of the Stokes system to the biharmonic problem via the introduction of a stream function.

2. Notation and Preliminary Results

We underscore 3×3 symmetric tensors by $\approx$ and 3-vectors by $\sim$. For vector $\underset{\sim}{u} = (u_1, u_2, u_3)^t$, we write $\underset{\sim}{u} \in \underset{\sim}{H}^1(\Omega)$ if $u_i \in H^1(\Omega)$ for $i = 1, 2, 3$, and set $\| \underset{\sim}{u} \|_1 = (\sum_{i=1}^3 \| u_i \|_1^2)^{1/2}$. For 3×3 symmetric tensors $\underset{\approx}{\sigma} = (\sigma_{ij})$, we write $\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega)$ if $\sigma_{ij} \in L^2(\Omega)$ for $i, j = 1, 2, 3$ and set $\| \underset{\approx}{\sigma} \|_0 = (\sum_{i,j=1}^3 \| \sigma_{ij} \|_0^2)^{1/2}$.

We shall require some function spaces defined on a smoothly bounded open subset Γ' of Γ. By $\underset{\sim}{H}^{1/2}(\Gamma')$ we denote the usual Sobolev space [15, Ch.1,Sec.7]. The subspace consisting of functions whose extension to Γ by zero lies in $\underset{\sim}{H}^{1/2}(\Gamma)$ is denoted by $\underset{\sim}{H}_{00}^{1/2}(\Gamma)$. The norm is taken as the graph norm of the extension by zero, which induces a strictly finer topology than the $\underset{\sim}{H}^{1/2}(\Gamma')$ norm, unless $\Gamma' = \Gamma$, in which case $\underset{\sim}{H}_{00}^{1/2}(\Gamma') = \underset{\sim}{H}^{1/2}(\Gamma')$ [15, Ch.1,Sec.11]. By $\underset{\sim}{H}^{-1/2}(\Gamma')$ we mean the normed dual of $\underset{\sim}{H}_{00}^{1/2}(\Gamma')$. The norms in $\underset{\sim}{H}^{1/2}(\Gamma')$ and $\underset{\sim}{H}^{-1/2}(\Gamma')$ are denoted by $|\cdot|_{1/2,\Gamma'}$ and $|\cdot|_{-1/2,\Gamma'}$ respectively, with the subscript being dropped in case $\Gamma' = \Gamma$.

We further define

$$\underset{\sim}{H}_0^1(\Omega) = \{ \underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : \underset{\sim}{v}|_\Gamma = 0 \},$$

and

$$\underset{\sim}{H}_D^1(\Omega) = \{ \underset{\sim}{v} \in \underset{\sim}{H}^1(\Omega) : \underset{\sim}{v}|_{\Gamma_1} = 0 \},$$

and denote by $\| \underset{\sim}{f} \|_{-1,0}$ and $\| \underset{\sim}{f} \|_{-1,D}$ the norms in the dual spaces of $\underset{\sim}{H}_0^1(\Omega)$ and $\underset{\sim}{H}_D^1(\Omega)$, respectively.

It is convenient to describe all the bounds we require on the elastic moduli in terms of a single constant α. We shall make reference to the following hypotheses which are to hold for some $\alpha \in (0,1/4]$:

$$\alpha \le E_i \le \alpha^{-1}, \quad \alpha \le G_i \le \alpha^{-1}, \quad 0 \le \nu_i \le 1 - \alpha, \quad i = 1, 2, 3,$$

$$1 - 2\nu_1\nu_2\nu_3 - {\nu_1}^2 - {\nu_2}^2 - {\nu_3}^2 \ge 0. \tag{2.1}$$

Many of the results in this paper will be derived using a theorem of F. Brezzi [4] dealing with saddle point problems of the following type:

Find $(\sigma,u) \in W \times V$ such that:

$$a(\sigma,\tau) + b(\tau,u) = \langle g,\tau \rangle \quad \text{for all } \tau \in W, \tag{2.2}$$

$$b(\sigma,v) = \langle f,v \rangle \quad \text{for all } v \in V, \tag{2.3}$$

where W and V are real Hilbert spaces, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms on $W \times W$ and $W \times V$ respectively, and g and f are given functions in W^* and V^* (the duals of W and V respectively).

Let $Z = \{\tau \in W : b(\tau,v) = 0 \text{ for all } v \in V\}$. Then one version of Brezzi's theorem is the following:

Theorem 2.1: Suppose there is a constant $\gamma > 0$ such that

a) $a(\tau,\tau) \geq \gamma \, \|\tau\|_W^2$ for all $\tau \in Z$

and

b) $\inf\limits_{0 \neq v \in V} \sup\limits_{0 \neq \tau \in W} \dfrac{b(\tau,v)}{\|\tau\|_W \, \|v\|_V} \geq \gamma.$

Then for all $(f,g) \in V^* \times W^*$, there is a unique solution $(\sigma,u) \in W \times V$ of Problem (2.2) - (2.3). Moreover,

$$\|\sigma\|_W + \|u\|_V \leq C(\|g\|_{W^*} + \|f\|_{V^*}),$$

where C depends only on γ and bounds for the bilinear forms a and b.

We will be applying Brezzi's Theorem in the case

$$a(\underset{\approx}{\sigma}, \underset{\approx}{\tau}) = \int_\Omega A \underset{\approx}{\sigma} : \underset{\approx}{\tau} \, d\underset{\sim}{x}, \qquad b(\underset{\approx}{\tau}, \underset{\sim}{v}) = -\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{v}) : \underset{\approx}{\tau} \, d\underset{\sim}{x}. \tag{2.4}$$

3. Proof of the Main Theorem

As usual, we impose the Dirichlet condition by setting $u^1 = E(g_1)$ with $E: H^{1/2}(\Gamma_1) \to H^1(\Omega)$ a continuous extension operator, and seek a pair (σ, u^2) such that

$$\begin{aligned} A\sigma - \epsilon(u^2) &= \epsilon(u^1), \\ \operatorname{div} \sigma &= f, \\ u^2 &= 0 \quad \text{on } \Gamma_1, \\ \sigma n &= g_2 \quad \text{on } \Gamma_2. \end{aligned} \tag{3.1}$$

We then take $u = u^1 + u^2$, so that the problem (1.1)-(1.3) is satisfied. In terms of the bilinear forms (2.4), a weak form of (3.1) is

Find $\sigma \in L^2(\Omega)$, $u^2 \in H^1_D(\Omega)$ such that

$$a(\sigma, \tau) + b(\tau, u^2) = -b(\tau, u^1) \quad \text{for all } \tau \in L^2(\Omega),$$

$$b(\sigma, v) = \int_\Omega f \cdot v \, dx - \int_{\Gamma_2} g_2 \cdot v \, ds \quad \text{for all } v \in H^1_D(\Omega). \tag{3.2}$$

To prove parts (i) and (ii) of Theorem 1.2, it suffices to prove that (3.2) admits a unique solution and establish the estimate

$$\| \sigma \|_0 + \| u^2 \|_1 \le C (\| \epsilon(u^1) \|_0 + \| f \|_{-1,D} + | g_2 |_{-1/2,\Gamma_2}) \tag{3.3}$$

where C is a constant depending only on Ω and α in (2.1) .

To prove part (iii) of Theorem 1.2, we show continuous dependence at $(\bar{\sigma}, \bar{u})$, the solution of (1.1)-(1.3) with A, f, g_1, and g_2 replaced by $\bar{A}$, $\bar{f}$, $\bar{g}_1$, and $\bar{g}_2$ respectively. Setting $\bar{u} = \bar{u}^1 + \bar{u}^2$ as above, it follows easily that

$(\sigma - \bar{\sigma}, u^2 - \bar{u}^2) \in L^2(\Omega) \times H^1_D(\Omega)$ satisfies:

$$a(\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{u}^2 - \bar{\underset{\sim}{u}}^2)$$

$$= -b(\underset{\approx}{\tau}, \underset{\sim}{u}^1 - \bar{\underset{\sim}{u}}^1) + \int_\Omega (\bar{A} - A)\, \bar{\underset{\approx}{\sigma}} : \underset{\approx}{\tau}\, d\underset{\sim}{x} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega), \tag{3.4}$$

$$b(\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}, \underset{\sim}{v}) = \int_\Omega (\underset{\sim}{f} - \bar{\underset{\sim}{f}}) \cdot \underset{\sim}{v}\, d\underset{\sim}{x} - \int_{\partial\Omega} (\underset{\sim}{g}_2 - \bar{\underset{\sim}{g}}_2) \cdot \underset{\sim}{v}\, ds \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega).$$

Suppose we prove the following lemma.

Lemma 3.1: Let A be the compliance tensor for an orthotropic material whose elastic moduli satisfy (2.1). Let $\underset{\approx}{G} \in \underset{\approx}{L}^2(\Omega)^*$, $\underset{\sim}{F} \in \underset{\sim}{H}^1_D(\Omega)^*$. Then there exist unique functions $\underset{\approx}{\rho} \in \underset{\approx}{L}^2(\Omega)$ and $\underset{\sim}{z} \in \underset{\sim}{H}^1_D(\Omega)$ such that

$$a(\underset{\approx}{\rho}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{z}) = \langle \underset{\approx}{G}, \underset{\approx}{\tau} \rangle \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega),$$

$$b(\underset{\approx}{\rho}, \underset{\sim}{v}) = \langle \underset{\sim}{F}, \underset{\sim}{v} \rangle \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}^1_D(\Omega).$$

Moreover

$$\|\underset{\approx}{\rho}\|_0 + \|\underset{\sim}{z}\|_1 \le C(\|\underset{\approx}{G}\|_0 + \|\underset{\sim}{F}\|_{-1,D})$$

where C depends only on Ω and α.

With this lemma, existence and uniqueness for problem (3.2) and the estimate (3.3) follow easily, giving parts (i) and (ii) of Theorem 1.2. From (3.4), the definition of $\underset{\sim}{u}^i$ and $\bar{\underset{\sim}{u}}^i$, and the lemma, we get

$$\|\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}\|_0 + \|\underset{\sim}{u} - \bar{\underset{\sim}{u}}\|_1$$

$$\le C(\|\underset{\sim}{f} - \bar{\underset{\sim}{f}}\|_{-1,D} + |\underset{\sim}{g}_1 - \bar{\underset{\sim}{g}}_1|_{1/2,\Gamma_1} + |\underset{\sim}{g}_2 - \bar{\underset{\sim}{g}}_2|_{-1/2,\Gamma_2} + \|(\bar{A} - A)\,\bar{\underset{\approx}{\sigma}}\|_0).$$

Now

$$\|(\bar{A} - A)\,\bar{\underset{\approx}{\sigma}}\|_0 \le K\, |\bar{A} - A|,$$

where $|\cdot|$ is any tensor norm and the constant K depends only on $\bar{\underset{\sim}{f}}$, $\bar{\underset{\sim}{g}}_1$, $\bar{\underset{\sim}{g}}_2$, Ω, and α. This implies the continuous dependence result of Theorem 1.2.

It remains to prove Lemma 3.1. We apply Brezzi's theorem (Theorem 2.1) to reduce Lemma 3.1 to the verification of the following two lemmas.

Lemma 3.2: There exists a constant $\gamma > 0$ depending only on α and Ω such that

$$\int_\Omega A\underset{\approx}{\sigma} : \underset{\approx}{\sigma}\, d\underset{\sim}{x} \geq \gamma \, \|\underset{\approx}{\sigma}\|_0^2 \quad \text{for all } \underset{\approx}{\sigma} \in \underset{\approx}{Z},$$

$$\text{where } \underset{\approx}{Z} = \{\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega) : \int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\epsilon}(\underset{\sim}{v})\, d\underset{\sim}{x} = 0 \text{ for all } \underset{\sim}{v} \in \underset{\sim}{H}_D^1(\Omega)\}.$$

Lemma 3.3: There exists $\gamma > 0$ depending only on Ω such that

$$\inf_{0 \neq \underset{\sim}{v} \in \underset{\sim}{H}_D^1(\Omega)} \; \sup_{0 \neq \underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega)} \frac{\int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{v}) : \underset{\approx}{\tau}\, d\underset{\sim}{x}}{\|\underset{\sim}{v}\|_1 \, \|\underset{\approx}{\tau}\|_0} \geq \gamma.$$

The proof of Lemma 3.3 is immediate: given $\underset{\sim}{v}$, we take $\underset{\approx}{\tau} = \underset{\approx}{\epsilon}(\underset{\sim}{v})$ and apply Korn's inequality. Since the tensor A is only positive semidefinite, Lemma 3.2 is not obvious. To prove it, we show that only one eigenvalue of the matrix B can be small and analyze the associated eigenspace. This is the content of Lemmas 3.4-3.6.

Lemma 3.4: Let $\lambda_3 \geq \lambda_2 \geq \lambda_1$ denote the eigenvalues of B. Then for all $\underset{\sim}{\nu} \in P$ and E_i satisfying $0 < \alpha \leq E_i \leq \alpha^{-1}$,

$$\lambda_2 \geq \alpha^3/3.$$

Proof: Expanding the characteristic polynomial of B, we have that the eigenvalues λ_i of B satisfy

$$p(\lambda) = -\lambda^3 + R\lambda^2 - S\lambda + T = 0$$

where

$$R = 1/E_1 + 1/E_2 + 1/E_3$$

$$S = \frac{1-\nu_3{}^2}{E_1E_2} + \frac{1-\nu_2{}^2}{E_1E_3} + \frac{1-\nu_1{}^2}{E_2E_3}$$

and

$$T = (1 - 2\nu_1\nu_2\nu_3 - \nu_1{}^2 - \nu_2{}^2 - \nu_3{}^2)/(E_1E_2E_3).$$

Since B is positive semidefinite for $\underset{\sim}{\nu} \in P$, $\lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$. Now $p'(\lambda) = -3\lambda^2 + 2R\lambda - S$

which vanishes for $\lambda = [R \pm (R^2 - 3S)^{1/2}]/3$. If $\lambda_2 > \lambda_1$, we get from Rolle's theorem that there exists a λ^* satisfying $\lambda_2 > \lambda^* > \lambda_1$ such that $p'(\lambda^*) = 0$. Hence $\lambda_2 \geq [R - (R^2 - 3S)^{1/2}]/3$. If $\lambda_2 = \lambda_1$, then $p'(\lambda_2) = 0$ and the same conclusion holds. Now

$$[R - (R^2 - 3S)^{1/2}]/3 \;=\; S/[R + (R^2 - 3S)^{1/2}] \;\geq\; S/(2R).$$

Since

$$R \;=\; 1/E_1 \;+\; 1/E_2 \;+\; 1/E_3 \;\leq\; 3/\alpha$$

and

$$\begin{aligned} S \;&=\; \frac{1-\nu_3{}^2}{E_1E_2} + \frac{1-\nu_2{}^2}{E_1E_3} + \frac{1-\nu_1{}^2}{E_2E_3} \\ &\geq\; \alpha^2[(1-\nu_3{}^2) \;+\; (1-\nu_2{}^2) \;+\; (1-\nu_1{}^2)] \\ &\geq\; \alpha^2(2 + 2\nu_1\nu_2\nu_3) \;\geq\; 2\alpha^2 \end{aligned}$$

for $\underset{\sim}{\nu} \in P$, we obtain that $\lambda_2 \geq \alpha^3/3$.

Lemma 3.5: Suppose the hypotheses of Lemma 3.4 hold. Then

$$E_1E_2E_3\lambda_2\lambda_3 \;\leq\; 3/\alpha.$$

Proof: Using the expansion of the characteristic polynomial introduced in the proof of Lemma 3.4, we have

$$\lambda_1\lambda_2 \;+\; \lambda_1\lambda_3 \;+\; \lambda_2\lambda_3 = \frac{1-\nu_3{}^2}{E_1E_2} + \frac{1-\nu_2{}^2}{E_1E_3} + \frac{1-\nu_1{}^2}{E_2E_3}.$$

The result follows by multiplying through by $E_1E_2E_3$ and observing that $\lambda_i \geq 0$ and $1 - \nu_i{}^2 \leq 1$.

Lemma 3.6: Let $\underset{\sim}{w}$ denote a unit eigenvector of B with eigenvalue λ_1 and first nonzero component positive. Suppose that $\alpha \leq E_i \leq \alpha^{-1}$, $i = 1, 2, 3$, and that $\underset{\sim}{\nu} \in P$ satisfies

$$1 - \nu_j \;\geq\; \alpha, \qquad j = 1, 2, 3$$

and

$$\nu_1 + \nu_2 + \nu_3 - \max(\nu_1, \nu_2, \nu_3) - \min(\nu_1, \nu_2, \nu_3) \;\geq\; \alpha.$$

Then $w_j \geq \alpha^4/3, \quad j = 1, 2, 3.$

Proof: Let

$$N = (N_{ij}) = \begin{pmatrix} 1-\nu_1^2 & \nu_1\nu_2+\nu_3 & \nu_1\nu_3+\nu_2 \\ \nu_1\nu_2+\nu_3 & 1-\nu_2^2 & \nu_2\nu_3+\nu_1 \\ \nu_1\nu_3+\nu_2 & \nu_2\nu_3+\nu_1 & 1-\nu_3^2 \end{pmatrix}$$

Then $NM = (\det M)\, I$. If $\underset{\sim}{w}^i$ are the eigenvectors of B corresponding to eigenvalues λ_i, normalized as in the statement of the lemma, then $DMD\,\underset{\sim}{w}^i = \lambda_i\,\underset{\sim}{w}^i$ and so $(\det M)\,\underset{\sim}{w}^i = \lambda_i\, D^{-1}ND^{-1}\,\underset{\sim}{w}^i$. If $\det M \neq 0$, this implies that $\underset{\sim}{w}^i$ is an eigenvector of $D^{-1}ND^{-1}$ with eigenvalue $\det M/\lambda_i = E_1E_2E_3\lambda_1\lambda_2\lambda_3/\lambda_i$. Hence $E_1E_2E_3\lambda_2\lambda_3$ is the largest eigenvalue of $D^{-1}ND^{-1}$ with corresponding eigenvector $\underset{\sim}{w} = \underset{\sim}{w}^1$. By continuity, this result also holds for $\det M = 0$. Hence in all cases

$$w_i = \frac{1}{E_1E_2E_3\lambda_2\lambda_3} \sum_{j=1}^{3} E_i^{1/2}E_j^{1/2}N_{ij}\,w_j.$$

Using the hypotheses on $\underset{\sim}{\nu}$, we see that

$$1-\nu_i^2 = (1-\nu_i)(1+\nu_i) \geq \alpha$$

while

$$\nu_i\nu_j+\nu_k \geq \alpha^2 \quad \text{for } \{i,j,k\} = \{1,2,3\}$$

since at least two of ν_1, ν_2, ν_3 exceed α. Hence $N_{ij} \geq \alpha^2$ and $E_i^{1/2}E_j^{1/2}N_{ij} \geq \alpha^3$. By the Perron-Frobenius theorem, it follows that $w_j \geq 0$ for $j = 1,2,3$, whence we obtain

$$w_i \geq \frac{\alpha^3}{E_1E_2E_3\lambda_2\lambda_3} \sum_{j=1}^{3} w_j \geq \frac{\alpha^3}{E_1E_2E_3\lambda_2\lambda_3} \sum_{j=1}^{3} w_j^2 = \frac{\alpha^3}{E_1E_2E_3\lambda_2\lambda_3}.$$

The result follows from the previous lemma.

The final ingredient in the proof of Lemma 3.2 is the following lemma which is well known in the isotropic case.

Lemma 3.7: Let $\underset{\sim}{z} \in \mathbf{R}^3$ be a unit vector satisfying $z_i \geq \alpha_0 > 0$, $i = 1,2,3$ and let $\underset{\approx}{\tau}_0$ be a diagonal tensor with $\operatorname{diag} \underset{\approx}{\tau}_0 = \underset{\sim}{z}$. For $\underset{\approx}{\tau}$ a symmetric 3×3 tensor, define $\underset{\approx}{\tau}_T = (\underset{\approx}{\tau} : \underset{\approx}{\tau}_0)\,\underset{\approx}{\tau}_0$ and $\underset{\approx}{\tau}_D = \underset{\approx}{\tau} - \underset{\approx}{\tau}_T$. Then all $\underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega)$ satisfying

$$\int_\Omega \epsilon(v) : \tau \, dx = 0 \quad \text{for all } v \in H^1_D(\Omega) \tag{3.5}$$

also satisfy $\|\tau_T\|_0 \le C \|\tau_D\|_0$ with C depending only on Ω and α_0.

Proof: There exists $p \in H^1_D(\Omega)$ such that

$$\text{div } p = \tau : \tau_0 \quad \text{and} \quad \|p\|_1 \le C \|\tau : \tau_0\|_0,$$

where C depends only on Ω. Let $q = \tau_0^{-1} p$. Since $z_i \ge \alpha_0 > 0$,

$$\|q\|_1 \le C \|\tau : \tau_0\|_0 / \alpha_0.$$

Now

$$\tau_0 : grad\ q = \sum_{i=1}^{3} z_i \frac{\partial q_i}{\partial x_i} = \text{div } p = \tau : \tau_0.$$

So

$$\|\tau : \tau_0\|_0^2 = \int_\Omega (\tau_0 : grad\ q)(\tau : \tau_0) \, dx$$

$$= \int_\Omega grad\ q : \tau_T \, dx = \int_\Omega grad\ q : (\tau - \tau_D) \, dx = \int_\Omega \epsilon(q) : (\tau - \tau_D) \, dx$$

$$= -\int_\Omega \epsilon(q) : \tau_D \, dx,$$

since $q \in H^1_D(\Omega)$ and τ satisfies (3.5). Thus

$$\|\tau : \tau_0\|_0^2 \le \|q\|_1 \|\tau_D\|_0 \le C \|\tau : \tau_0\|_0 \|\tau_D\|_0 / \alpha_0$$

and the lemma follows easily.

Proof of Lemma 3.2: Let w be as in Lemma 3.6 and let σ_0 be the diagonal tensor with $\text{diag}\, \sigma_0 = w$. Define

$$\sigma_T = (\sigma : \sigma_0) \sigma_0 \quad \text{and} \quad \sigma_D = \sigma - \sigma_T$$

Then

$$\underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}_T = 0, \quad \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_T = (\underset{\approx}{\sigma} : \underset{\approx}{\sigma}_0)\, \underset{\sim}{w}, \quad \text{and} \quad \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{w} = 0.$$

Therefore

$$\begin{aligned} A \underset{\approx}{\sigma} : \underset{\approx}{\sigma} &= A \underset{\approx}{\sigma}_D : \underset{\approx}{\sigma}_D + 2A \underset{\approx}{\sigma}_T : \underset{\approx}{\sigma}_D + A \underset{\approx}{\sigma}_T : \underset{\approx}{\sigma}_T \\ &= B\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D + G\, \underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}_D \\ &+ 2B\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_T \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D + B\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_T \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_T \\ &= B\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D + \lambda_1\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_T \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_T + G\, \underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}_D. \end{aligned}$$

Let $\underset{\sim}{w}^i$ form an orthonormal basis of eigenvectors of B with $\underset{\sim}{w}^1 = \underset{\sim}{w}$. Then since $\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{w}^1 = 0$,

$$\begin{aligned} B\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D &= \lambda_2 (\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{w}^2)^2 + \lambda_3 (\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{w}^3)^2 \\ &\geq \lambda_2\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D. \end{aligned}$$

Hence

$$\begin{aligned} A \underset{\approx}{\sigma} : \underset{\approx}{\sigma} &\geq \lambda_1 |\underset{\approx}{\sigma}_T|^2 + \alpha\, |\underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}_D|^2 + \lambda_2\, |\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}_D|^2 \\ &\geq \lambda_1 |\underset{\approx}{\sigma}_T|_0^2 + \min(\alpha/2,\ \lambda_2)\, |\underset{\approx}{\sigma}_D|^2. \end{aligned}$$

Now $\lambda_2 \geq \alpha^3/3$ by Lemma 3.4 and $1/4 \geq \alpha > 0$ implies that

$$A \underset{\approx}{\sigma} : \underset{\approx}{\sigma} \geq \lambda_1 |\underset{\approx}{\sigma}_T|^2 + \alpha^3 |\underset{\approx}{\sigma}_D|^2/3. \tag{3.6}$$

We now distinguish two cases. To simplify the presentation, we assume (without loss of generality) that $\nu_3 \geq \nu_2 \geq \nu_1$.

Case 1: $\nu_2 < \alpha$.

Since by hypothesis (2.1), $1 - \nu_3 \geq \alpha$ and $0 < \alpha \leq 1/4$,

$$\begin{aligned} \det M &= 1 - {\nu_1}^2 - {\nu_2}^2 - {\nu_3}^2 - 2\nu_1\nu_2\nu_3 \\ &\geq 1 - \alpha^2 - \alpha^2 - (1-\alpha)^2 - 2\alpha^2(1-\alpha) \geq (1-2\alpha)(2-\alpha)\alpha \geq 7\alpha/8. \end{aligned}$$

But

$$\det M = E_1E_2E_3 \det B = E_1E_2E_3\lambda_1\lambda_2\lambda_3 \leq 3\lambda_1/\alpha$$

by Lemma 3.5, so $\lambda_1 \geq 7\alpha^2/24$, which clearly exceeds $\alpha^3/3$. Hence (3.6) gives

$$A \underset{\approx}{\sigma} : \underset{\approx}{\sigma} \geq \alpha^3(|\underset{\approx}{\sigma}_T|^2 + |\underset{\approx}{\sigma}_D|^2)/3 = \alpha^3 |\underset{\approx}{\sigma}|^2/3$$

and the proof is completed in this case by integrating over Ω and taking $\gamma = \alpha^3/3$.

Case 2: $\nu_2 \geq \alpha$.

In this case the hypotheses of Lemma 3.6 are satisfied. Hence $w_j \geq \alpha^4/3$, $j = 1, 2, 3$. We can now apply Lemma 3.7 with $\underset{\sim}{z} = \underset{\sim}{w}$ and $\alpha_0 = \alpha^4/3$. Then $\underset{\approx}{\tau}_0 = \underset{\approx}{\sigma}_0$ so the lemma implies that $\| \underset{\approx}{\sigma}_D \|_0 \geq K \| \underset{\approx}{\sigma}_T \|_0$ where $K > 0$ depends only on Ω and α. Combining this result with (3.6), we obtain

$$\int_\Omega A \underset{\approx}{\sigma} : \underset{\approx}{\sigma} \, d\underset{\sim}{x} \geq \alpha^3 \| \underset{\approx}{\sigma}_D \|_0^2/3 = \alpha^3 \left(\frac{K}{K+1} \| \underset{\approx}{\sigma}_D \|_0^2 + \frac{1}{K+1} \| \underset{\approx}{\sigma}_D \|_0^2\right)/3$$

$$\geq \frac{\alpha^3 K}{3(K+1)} \left(\| \underset{\approx}{\sigma}_D \|_0^2 + \| \underset{\approx}{\sigma}_T \|_0^2 \right) = \gamma \| \underset{\approx}{\sigma} \|_0^2.$$

Hence the lemma is proved.

4. Pure Traction and Pure Displacement Boundary Conditions

In this section we briefly indicate the changes necessary to analyze the system of orthotropic elasticity, (1.4), (1.2), when the mixed boundary conditions (1.3) are replaced by either the displacement boundary condition

$$\underset{\sim}{u} = \underset{\sim}{g} \quad \text{on } \Gamma = \partial\Omega, \tag{4.1}$$

or the traction boundary condition

$$\underset{\approx}{\sigma}\,\underset{\sim}{n} = \underset{\sim}{g} \quad \text{on } \Gamma. \tag{4.2}$$

The latter case is entirely straightforward and we dispose of it immediately. A necessary and sufficient condition for the existence of a solution is the compatibility condition

$$\int_\Gamma \underset{\sim}{g}\cdot\underset{\sim}{v}\,ds = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,d\underset{\sim}{x} \quad \text{for all } \underset{\sim}{v}\in \underset{\sim}{RM}, \tag{4.3}$$

where

$$\underset{\sim}{RM} = \{\underset{\sim}{v}\in \underset{\sim}{L}^2(\Omega) : \underset{\sim}{v} = \underset{\sim}{c} + \underset{\approx}{Q}\,\underset{\sim}{x},\ \underset{\sim}{c}\in \underset{\sim}{\mathbf{R}},\ \underset{\approx}{Q}\in\underset{\approx}{\mathbf{R}},\ \underset{\approx}{Q}+\underset{\approx}{Q}^t = 0\}$$

is the space of rigid motions. When (4.3) holds, the solution is determined up to the addition of a rigid motion and uniqueness may be obtained by requiring $\underset{\sim}{u}\in \underset{\sim}{H}^1_\perp(\Omega)$, the orthogonal complement of $\underset{\sim}{RM}$ in $\underset{\sim}{H}^1(\Omega)$.

A weak formulation of the traction problem seeks $\underset{\approx}{\sigma}\in\underset{\approx}{L}^2(\Omega)$, $\underset{\sim}{u}\in\underset{\sim}{H}^1_\perp(\Omega)$ such that

$$a(\underset{\approx}{\sigma},\underset{\approx}{\tau}) + b(\underset{\approx}{\tau},\underset{\sim}{u}) = 0 \quad \text{for all } \underset{\approx}{\tau}\in\underset{\approx}{L}^2(\Omega),$$

$$b(\underset{\approx}{\sigma},\underset{\sim}{v}) = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,d\underset{\sim}{x} - \int_\Gamma \underset{\sim}{g}\cdot\underset{\sim}{v}\,ds \quad \text{for all } \underset{\sim}{v}\in\underset{\sim}{H}^1_\perp(\Omega).$$

Note that the latter equation actually holds for all $\underset{\sim}{v}\in\underset{\sim}{H}^1(\Omega)$ when the compatibility condition (4.3) is satisfied, so this weak formulation is justified. Proceeding as in Section 3, we may apply Brezzi's Theorem to the analysis of this formulation to obtain the direct analogue of Theorem 1.2.

The case of displacement boundary conditions is considerably more complicated, due to the existence of a compatibility condition only for constrained materials, the condition depending moreover, on the compliance tensor. As remarked earlier, for $\underset{\sim}{\nu}$ bounded away from the constraint surface, the constitutive equation can be inverted and existence, uniqueness, apriori estimates, and continuous dependence easily established by standard variational arguements. We therefore henceforth restrict our attention to $\underset{\sim}{\nu}$ in a neighborhood of the constraint surface excluding a small region about each corner. By Lemmas 3.4 and 3.6 we may choose this neighborhood so that if hypothesis (2.1) is satisfied then

the least eigenvalue λ_1 of B is simple, (4.4)

and

the least eigenvector $\underset{\sim}{w}$ (normalized to be of unit length with first non-zero component positive) has all components bounded stricly above zero. (4.5)

We shall use the notation $\underset{\approx}{\sigma}_0(A)$ to denote the diagonal tensor with diagonal equal to $\underset{\sim}{w}$.

For a constrained material there is a compatibility condition which is necessary for the existence of a solution to the displacement boundary value problem. From (1.1), (4.1) and the fact that the material is homogeneous (specifically that $\underset{\approx}{\sigma}_0 = \underset{\approx}{\sigma}_0(A)$ is independent of $\underset{\sim}{x} \in \Omega$), we see that

$$\begin{aligned} \lambda_1 \int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_0 \, d\underset{\sim}{x} &= \int_\Omega \underset{\approx}{\sigma} : A \underset{\approx}{\sigma}_0 \, d\underset{\sim}{x} \\ &= \int_\Omega A \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_0 \, d\underset{\sim}{x} = \int_\Omega \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\sigma}_0 \, d\underset{\sim}{x} \\ &= -\int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{\mathrm{div}} \, \underset{\approx}{\sigma}_0 \, d\underset{\sim}{x} + \int_\Gamma \underset{\sim}{u} \cdot \underset{\approx}{\sigma}_0 \, \underset{\sim}{n} \, ds = \int_\Gamma \underset{\sim}{g} \cdot \underset{\approx}{\sigma}_0 \, \underset{\sim}{n} \, ds. \end{aligned} \tag{4.6}$$

When A is singular, $\lambda_1 = 0$, implying the necessary condition

$$\int_\Gamma \underset{\sim}{g} \cdot \underset{\approx}{\sigma}_0(A) \, \underset{\sim}{n} \, ds = 0. \tag{4.7}$$

When (4.7) does hold, uniqueness fails in that $(0, \underset{\approx}{\sigma}_0(A))$ satisfies the homogeneous system. Uniqueness is restored by adding the side condition

$$\int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_0(A)\, d\underset{\sim}{x} \;=\; 0. \tag{4.8}$$

Note that for $\lambda_1 \neq 0$, (4.8) follows from (4.7) by (4.6).

We remark that the only constrained isotropic materials (with finite positive Young's modulus) are incompressible, so have Poisson ratio 1/2. In this case $\underset{\approx}{\sigma}_0 = \underset{\approx}{\delta}$, the identity tensor. Thus the compatibility condition (4.7) reduces to

$$\int_\Gamma \underset{\sim}{g} \cdot \underset{\sim}{n}\, ds \;=\; 0$$

and the side condition (4.8) to

$$\int_\Omega \mathrm{tr}(\underset{\approx}{\sigma})\, d\underset{\sim}{x} \;=\; 0.$$

We now establish the analogues of parts (i) and (ii) of Theorem 1.2 for displacement boundary conditions. For a weak formulation of the problem, we define the space

$$\underset{\approx}{W}_A \;=\; \{\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega) : \int_\Omega \underset{\approx}{\sigma} : \underset{\approx}{\sigma}_0(A)\, d\underset{\sim}{x} = 0.$$

The proof of the following lemma, which differs only slightly from that of Lemma 3.1, will be discussed at the end of the section.

Lemma 4.1: Let $\underset{\approx}{G} \in \underset{\approx}{W}_A^*$, $\underset{\sim}{F} \in \underset{\sim}{H}_0^1(\Omega)^*$. Then there is a unique pair $(\underset{\approx}{\rho}, \underset{\sim}{z}) \in \underset{\approx}{W}_A \times \underset{\sim}{H}_0^1(\Omega)$ such that

$$a(\underset{\approx}{\rho}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{z}) \;=\; <\underset{\approx}{G}, \underset{\approx}{\tau}> \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{W}_A, \tag{4.9}$$

$$b(\underset{\approx}{\rho}, \underset{\sim}{v}) \;=\; <\underset{\sim}{F}, \underset{\sim}{v}> \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega).$$

Moreover

$$\| \underset{\approx}{\rho} \|_0 + \| \underset{\sim}{z} \|_1 \;\leq\; C(\| \underset{\approx}{G} \|_{\underset{\approx}{W}_A^*} + \| \underset{\sim}{F} \|_{-1,0})$$

where C depends only on Ω and α in (2.1). Note that if

$$<\underset{\approx}{G}, \underset{\approx}{\sigma}_0(A)> \;=\; 0, \tag{4.10}$$

which will be the case for a Dirichlet problem with compatible data, then the solution of (4.9) satisfies the first equation also for $\tau = \sigma_0(A)$ and hence for all $\tau \in L^2(\Omega)$, not just $\tau \in W_A$. Therefore (4.9) is a valid weak formulation of the Dirichlet problem.

First we suppose that the displacement boundary data g satisfies (4.7). Then the solution to the boundary value problem (1.1), (1.2), (4.1) may be writen as $(\sigma, u^1 + u^2)$ where $u^1 = \mathcal{E}(g)$ with $\mathcal{E}: H^{1/2}(\Gamma) \to H^1(\Omega)$ a bounded extension operator, and the pair (σ, u^2) satisfies (4.8) with $<F, v> = \int_\Omega f \cdot v \, dx$, $<G, \tau> = -b(\tau, u^1)$. The compatibility condition (4.7) insures (4.10), and so Lemma 4.1 implies first, that the displacement problem admits a unique solution (σ, u); and second, that

$$\| \sigma \|_0 + \| u \|_1 \le C(\| f \|_{-1,0} + | g |_{1/2}) \tag{4.11}$$

with C depending only on Ω and α.

If the displacement boundary data violates (4.7) both these conclusions are false. Existence and uniqueness do not hold for a constrained material. Even for an unconstrained material the apriori estimate (4.11) does not hold uniformly. More precisely, $\int_\Omega \sigma : \sigma_0(A) \, dx$ cannot be bounded independently of the material constants. However we can derive a uniform apriori bound on u and on the orthogonal projection $\hat{\sigma}$ of σ on the complement of the one dimensional space spanned by $\sigma_0 = \sigma_0(A)$. To this end we decompose the solution as

$$(\sigma, u) = (\check{\sigma}, \check{u}) + (\hat{\sigma}, \hat{u})$$

where

$$\check{\sigma} = \theta \sigma_0 / \lambda_1, \qquad \check{u} = \theta \sigma_0 x,$$

$$\theta = \int_{\partial\Omega} g \cdot \sigma_0 n \, ds \, / \, \text{measure}(\Omega).$$

Then $\hat{\sigma}$ is indeed the projection of σ orthogonal to σ_0, as follows from (4.6) and the pair $(\hat{\sigma}, \hat{u})$ solves the boundary value problem

$$A \hat{\sigma} = \epsilon(\hat{u}) \quad \text{in } \Omega,$$

$$\operatorname{div} \hat{\sigma} = f \quad \text{in } \Omega,$$

$$\hat{u} = g - \theta \sigma_0 x \quad \text{on } \partial\Omega.$$

The boundary data for this problem is compatible since

$$\int_\Gamma \underset{\approx}{\sigma}_0\, \underset{\sim}{x} \cdot \underset{\approx}{\sigma}_0\, \underset{\sim}{n}\, ds \;=\; \int_\Omega \underset{\approx}{\epsilon}(\underset{\approx}{\sigma}_0\, \underset{\sim}{x}) : \underset{\approx}{\sigma}_0\, d\underset{\sim}{x}$$

$$= \int_\Omega |\underset{\approx}{\sigma}_0|^2\, d\underset{\sim}{x} \;=\; \text{measure}(\Omega).$$

Thus Lemma 4.1 implies

$$\begin{aligned} \|\hat{\underset{\approx}{\sigma}}\|_0 + \|\hat{\underset{\sim}{u}}\|_1 &\le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g} - \theta\, \underset{\approx}{\sigma}_0\, \underset{\sim}{x}|_{1/2}) \\ &\le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g}|_{1/2}). \end{aligned}$$

Clearly also $\|\check{\underset{\sim}{u}}\|_1 + |\theta| \le C|\underset{\sim}{g}|_{1/2}$, so

$$\|\hat{\underset{\approx}{\sigma}}\|_0 + \|\underset{\sim}{u}\|_1 \le C(\|\underset{\sim}{f}\|_{-1,0} + |\underset{\sim}{g}|_{1/2}), \tag{4.12}$$

which gives the desired apriori bound.

Finally we consider the continuous dependence of the solution on the elastic moduli. Thus we fix a value $\bar{A}$ of the compliance tensor and data $\bar{\underset{\sim}{f}}$ and $\bar{\underset{\sim}{g}}$, and denote by $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$ the corresponding solution. We wish to show that if $(A, \underset{\sim}{f}, \underset{\sim}{g})$ is sufficiently close to $(\bar{A}, \bar{\underset{\sim}{f}}, \bar{\underset{\sim}{g}})$ then the solution $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ determined by $(A, \underset{\sim}{f}, \underset{\sim}{g})$ is arbitrarily near $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$, i.e. that

$$\lim\, (\underset{\approx}{\sigma}, \underset{\sim}{u}) = (\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}}) \quad \text{in } \underset{\approx}{L}^2(\Omega) \times \underset{\sim}{H}^1(\Omega). \tag{4.13}$$

Of course the elastic moduli for both $\bar{A}$ and A are assumed to satisfy (2.1). Moreover we may assume that the limiting material is constrained, i.e., that $\bar{A}$ is singular, since otherwise the result is obvious. Now for $\bar{A}$ singular we must suppose that

$$\int_\Gamma \underset{\sim}{g} \cdot \bar{\underset{\approx}{\sigma}}_0\, \underset{\sim}{n}\, ds = 0, \tag{4.14}$$

where $\bar{\underset{\approx}{\sigma}}_0 = \underset{\approx}{\sigma}_0(\bar{A})$, in order that the solution $(\bar{\underset{\approx}{\sigma}}, \bar{\underset{\sim}{u}})$ exist and (4.13) make sense. This condition is not, however, sufficient to make sense of (4.13) since even if (4.14) holds there may exist singular tensors A arbitrarily near $\bar{A}$ for which $\underset{\sim}{g}$ is not compatible and hence for which $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ is undefined. We may circumvent this difficulty in two ways. First, we may consider only $\underset{\sim}{g} = 0$. In this case there is no problem of incompatibility and (4.13) follows from (4.11) by a straightforward argument, similar to that at the beginning of Section 3. Second, to derive a result valid for nonzero $\underset{\sim}{g}$ satisfying (4.14), we consider the singular compliance tensor $\bar{A}$ as the limit of positive definite tensors A, i.e., we restrict A in (4.13) to be nonsingular. Even with this restriction, however, it is not hard to see that (4.13) is not valid, as $\underset{\approx}{\sigma}$ may have a large component in the direction given by $\underset{\approx}{\sigma}_0(A)$ which may become unbounded as A tends to $\bar{A}$. However we shall show that

$$\lim \left(\| \underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}} \|_{\underset{\approx}{L}^2(\Omega)/\underset{\approx}{\sigma}_0(A)} + \| \underset{\sim}{u} - \bar{\underset{\sim}{u}} \|_1 \right) = 0 \tag{4.15}$$

where the quotient seminorm in (4.15) is defined by

$$\| \underset{\approx}{\rho} \|_{\underset{\approx}{L}^2(\Omega)/\underset{\approx}{\sigma}_0(A)} = \inf_{c \in \mathbb{R}} \| \underset{\approx}{\rho} + c\, \underset{\approx}{\sigma}_0(A) \|_{\underset{\approx}{L}^2(\Omega)},$$

and the limit is taken as $(A, \underset{\sim}{f}, \underset{\sim}{g})$ tends to $(\bar{A}, \bar{\underset{\sim}{f}}, \bar{\underset{\sim}{g}})$ with A nonsingular. Note that this seminorm depends on A, but for all A exceeds the quotient seminorm on $\underset{\approx}{L}^2(\Omega)$ induced by the three dimensional subspace of constant diagonal tensors.

To prove (4.15) we note that

$$a(\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{u} - \bar{\underset{\sim}{u}}) = \int_\Omega (\bar{A} - A)\, \bar{\underset{\approx}{\sigma}} : \underset{\approx}{\tau}\, d\underset{\sim}{x} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{W}_A,$$

$$b(\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}, \underset{\sim}{v}) = \int_\Omega (\underset{\sim}{f} - \bar{\underset{\sim}{f}}) \cdot \underset{\sim}{v}\, d\underset{\sim}{x} \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega).$$

Now let $\underset{\approx}{\rho}$ denote the projection of $\underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}}$ on the orthogonal complement of $\underset{\approx}{\sigma}_0(A)$ in $\underset{\approx}{L}^2(\Omega)$, and let $\underset{\sim}{z} = \underset{\sim}{u} - \bar{\underset{\sim}{u}} - \underset{\sim}{\xi}(\underset{\sim}{g} - \bar{\underset{\sim}{g}})$. Then $(\underset{\approx}{\rho}, \underset{\sim}{z}) \in \underset{\approx}{W}_A \times \underset{\sim}{H}_0^1(\Omega)$ and

$$a(\underset{\approx}{\rho}, \underset{\approx}{\tau}) + b(\underset{\approx}{\tau}, \underset{\sim}{z}) = \int_\Omega (\bar{A} - A)\, \bar{\underset{\approx}{\sigma}} : \underset{\approx}{\tau}\, d\underset{\sim}{x}$$
$$- b(\underset{\approx}{\tau}, \underset{\sim}{\xi}(\underset{\sim}{g} - \bar{\underset{\sim}{g}})) \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{W}_A,$$

$$b(\underset{\approx}{\rho}, \underset{\sim}{v}) = \int_\Omega (\underset{\sim}{f} - \bar{\underset{\sim}{f}}) \cdot \underset{\sim}{v}\, d\underset{\sim}{x} \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{H}_0^1(\Omega).$$

By Lemma 4.1

$$\| \underset{\approx}{\rho} \|_0 + \| \underset{\sim}{z} \|_1 \le C \left(|\bar{A} - A|\, \| \bar{\underset{\approx}{\sigma}} \|_0 + \| \underset{\sim}{\xi}(\underset{\sim}{g} - \bar{\underset{\sim}{g}}) \|_1 + \| \underset{\sim}{f} - \bar{\underset{\sim}{f}} \|_{-1,0} \right)$$
$$\le C \left(|\bar{A} - A| + | \underset{\sim}{g} - \bar{\underset{\sim}{g}} |_{1/2,\Gamma} + \| \underset{\sim}{f} - \bar{\underset{\sim}{f}} \|_{-1,0} \right).$$

Further

$$\| \underset{\sim}{u} - \bar{\underset{\sim}{u}} \|_1 \leq \| \underset{\sim}{z} \|_1 + C | \underset{\sim}{g} - \bar{\underset{\sim}{g}} |_{1/2,\Gamma}$$

and

$$\| \underset{\approx}{\sigma} - \bar{\underset{\approx}{\sigma}} \|_{\underset{\approx}{L}^2(\Omega)/\underset{\approx}{\sigma}_0(A)} = \| \underset{\approx}{\rho} \|_0,$$

and so (4.15) is established.

We close this section with a brief discussion of the proof of Lemma 4.1. The proof follows very closely that of Lemma 3.1 and differs significantly in only one detail. In the statement of Lemma 3.7, which was used in the proof of Lemma 3.1, we must of course replace the space $\underset{\sim}{H}^1_D(\Omega)$ with $\underset{\sim}{H}^1_0(\Omega)$. We must also replace the space $\underset{\approx}{L}^2(\Omega)$ with $\{\underset{\approx}{\tau} \in \underset{\approx}{L}^2(\Omega) : \int_\Omega \underset{\approx}{\tau} : \underset{\approx}{\tau}_0 \, d\underset{\sim}{x} = 0\}$. Only when $\underset{\approx}{\tau}$ lies in this space does the differential equation

$$\operatorname{div} \underset{\sim}{p} = \underset{\approx}{\tau} : \underset{\approx}{\tau}_0$$

have a solution in $\underset{\sim}{H}^1_0(\Omega)$, and so the proof of Lemma 3.7 can be carried out as before. The additional hypothesis that $\underset{\approx}{\tau}$ be orthogonal to $\underset{\approx}{\tau}_0$ causes no problem, since in the application to the proof of Lemma 4.1 this hypothesis follows from the membership of $\underset{\approx}{\rho}$ in $\underset{\approx}{W}_A$.

5. Ellipticity

The system (1.1)-(1.2) of anisotropic elasticity is elliptic in the sense of Agmon, Douglis, and Nirenberg [1], at least when the compliance tensor is positive definite. In this section we show ellipticity of the system for any orthotropic material whose compliances satisfy (2.1), and, more importantly, that the ellipticity is uniform with respect to the compliances in the sense that the symbolic determinant whose nonvanishing defines ellipticity may be bounded above and below by positive constants depending only on the constant α in (2.1).

The property of ellipticity has numerous consequences. For example, it implies interior regularity estimates on the solution of the equations, and the uniformity of the bounds on the symbolic determinant imply uniformity of the interior estimates [6]. Ellipticity of the differential equations is also a fundamental condition for regularity of solutions up to the boundary, but for this it is not sufficient. To derive uniform regularity results valid up to the boundary one must also verify the complementing condition [1] for the boundary conditions of interest and uniformly bound the "minor constant" appearing therein. This appears to be a quite formidable task.

For the verification of ellipticity we write the system (1.4),(1.2) in the form:

$$\begin{pmatrix} B & 0 & -R(\underset{\sim}{\nabla}) \\ 0 & 2G & -S(\underset{\sim}{\nabla}) \\ -R(\underset{\sim}{\nabla}) & -S(\underset{\sim}{\nabla}) & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{\operatorname{diag}}\, \underset{\approx}{\sigma} \\ \underset{\sim}{\operatorname{offd}}\, \underset{\approx}{\sigma} \\ \underset{\sim}{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\underset{\sim}{f} \end{pmatrix} \tag{5.1}$$

where B and G are the 3×3 matrices defined in the introduction, $\underset{\sim}{\nabla} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^t$, and for any $\underset{\sim}{\theta} = (\theta_1, \theta_2, \theta_3)^t$, $R(\underset{\sim}{\theta})$ is a 3×3 diagonal matrix with $\underset{\sim}{\operatorname{diag}}\, R(\underset{\sim}{\theta}) = \underset{\sim}{\theta}$, and $S(\underset{\sim}{\theta})$ is a 3×3 symmetric matrix with 0 diagonal and $\underset{\sim}{\operatorname{offd}}\, S(\underset{\sim}{\theta}) = \underset{\sim}{\theta}$. Note that for any $\underset{\approx}{\sigma} \in \underset{\approx}{\mathbf{R}}_s$,

$$R(\underset{\sim}{\theta})\, \underset{\sim}{\operatorname{diag}}\, \underset{\approx}{\sigma} + S(\underset{\sim}{\theta})\, \underset{\sim}{\operatorname{offd}}\, \underset{\approx}{\sigma} = \underset{\approx}{\sigma}\, \underset{\sim}{\theta}.$$

Let $l(\underset{\sim}{\nabla})$ denote the 9×9 matrix given in (5.1). Defining $s_i = -1$, $t_i = 1$ for $1 \le i \le 6$, and $s_i = 0$, $t_i = 2$, for $7 \le i \le 9$, we have that $\deg l_{ij} \le s_i + t_j$, with equality when $l_{ij} \ne 0$. The following theorem asserts the uniform ellipticity of the system (5.1) in the sense of [1], [6].

Theorem 5.1: Suppose that (2.1) is satisfied for some positive constant α. Then there exists a positive constant β depending only on α such that

$$\beta|\underset{\sim}{\theta}|^2 \leq \det(l(\underset{\sim}{\theta})) \leq \beta^{-1}|\underset{\sim}{\theta}|^2 \quad \text{for all vectors } \underset{\sim}{\theta} \in \underset{\sim}{\mathbf{R}}. \tag{5.2}$$

Since $\det l(\underset{\sim}{\theta}) = 0$ when $\underset{\sim}{\theta} = 0$ and since $\det l(\underset{\sim}{\theta})$ is a homogeneous polynomial of degree 2 in $\underset{\sim}{\theta}$, (5.2) is equivalent to the condition

$$\beta \leq \det(l(\underset{\sim}{\theta})) \leq \beta^{-1} \quad \text{for all unit vectors } \underset{\sim}{\theta} \in \underset{\sim}{\mathbf{R}}. \tag{5.3}$$

The asserted upper bound is obvious, and we discuss only the lower bound. Let $l^{-1}(\underset{\sim}{\theta})$ denote the inverse of the matrix $l(\underset{\sim}{\theta}) = [l_{ij}(\underset{\sim}{\theta})]$. We shall bound the spectral norm $\| l(\underset{\sim}{\theta})^{-1} \|$ by a constant C depending only on α. This will imply that the eigenvalues of $l(\underset{\sim}{\theta})$ are all bounded below by $1/C$, so that $\det l(\underset{\sim}{\theta}) \geq 1/C^9$ as desired.

To prove the invertibility of $l(\underset{\sim}{\theta})$ and establish the uniform bound on $l(\underset{\sim}{\theta})^{-1}$, we apply Brezzi's Theorem (Theorem 2.1) to the finite dimensional problem:

Given $(\underset{\approx}{G}, \underset{\sim}{F}) \in \underset{\approx}{\mathbf{R}}_s \times \underset{\sim}{\mathbf{R}}$, find $(\underset{\approx}{\sigma}, \underset{\sim}{u}) \in \underset{\approx}{\mathbf{R}}_s \times \underset{\sim}{\mathbf{R}}$ such that

$$B\,\underset{\sim}{\text{diag}}\,\underset{\approx}{\sigma}\cdot\underset{\sim}{\text{diag}}\,\underset{\approx}{\tau} + 2G\,\underset{\sim}{\text{offd}}\,\underset{\approx}{\sigma}\cdot\underset{\sim}{\text{offd}}\,\underset{\approx}{\tau} - \underset{\sim}{u}\cdot\underset{\approx}{\tau}\,\underset{\sim}{\theta} = \underset{\approx}{G}:\underset{\approx}{\tau} \quad \text{for all } \underset{\approx}{\tau} \in \underset{\approx}{\mathbf{R}}_s, \tag{5.4}$$

and

$$\underset{\approx}{\sigma}\,\underset{\sim}{\theta}\cdot\underset{\sim}{v} = \underset{\sim}{F}\cdot\underset{\sim}{v} \quad \text{for all } \underset{\sim}{v} \in \underset{\sim}{\mathbf{R}}. \tag{5.5}$$

It is easily checked that $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ solves this problem if and only if

$$l(\underset{\sim}{\theta}) \begin{pmatrix} \underset{\sim}{\text{diag}}\,\underset{\approx}{\sigma} \\ \underset{\sim}{\text{offd}}\,\underset{\approx}{\sigma} \\ \underset{\sim}{u} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{\text{diag}}\,\underset{\approx}{G} \\ \underset{\sim}{\text{offd}}\,\underset{\approx}{G} \\ \underset{\sim}{F} \end{pmatrix}$$

Hence it suffices to show that this problem has a unique solution and that

$$|\underset{\approx}{\sigma}| + |\underset{\sim}{u}| \leq C(|\underset{\approx}{G}| + |\underset{\sim}{F}|).$$

By Brezzi's Theorem, it suffices to prove that there exists $\gamma > 0$ such that

$$B\,\underset{\sim}{\text{diag}}\,\underset{\approx}{\sigma}\cdot\underset{\sim}{\text{diag}}\,\underset{\approx}{\sigma} + 2G\,\underset{\sim}{\text{offd}}\,\underset{\approx}{\sigma}\cdot\underset{\sim}{\text{offd}}\,\underset{\approx}{\sigma} \geq \gamma|\underset{\approx}{\sigma}|^2 \tag{5.6}$$

$$\text{for all } \underset{\approx}{\sigma} \in \underset{\approx}{\mathbf{R}}_s, \text{ satisfying } \underset{\approx}{\sigma}\,\underset{\sim}{\theta} = 0,$$

and

$$\inf_{0 \neq \underset{\sim}{v} \in \underset{\sim}{\mathbf{R}}} \; \sup_{0 \neq \underset{\approx}{\sigma} \in \underset{\approx}{\mathbf{R}},} \frac{\underset{\approx}{\sigma}\, \underset{\sim}{\theta} \cdot \underset{\sim}{v}}{|\underset{\approx}{\sigma}|\, |\underset{\sim}{v}|} \geq \gamma. \tag{5.7}$$

The proof of (5.7) is direct. If $\underset{\approx}{\sigma} = \sqrt{3}\, U^t R(U \underset{\sim}{v}) U$, where U is an orthogonal matrix chosen so that $\sqrt{3}\, U \underset{\sim}{\theta} = (1,1,1)^t$, then $|\underset{\approx}{\sigma}| \leq C |\underset{\sim}{v}|$ and $\underset{\approx}{\sigma}\underset{\sim}{\theta} = \underset{\sim}{v}$.

The proof of (5.6) is analogous to that of Lemma 3.2. In place of Lemma 3.7 we use the following result.

Lemma 5.1: Let $\underset{\sim}{z}, \underset{\sim}{\theta} \in \underset{\sim}{\mathbf{R}}$ be unit vectors and suppose that $z_i \geq \alpha_0 > 0$, $i = 1,2,3$. Let $\underset{\approx}{\tau}_0$ be a diagonal tensor with $\underset{\sim}{\text{diag}}\, \underset{\approx}{\tau}_0 = \underset{\sim}{z}$ and define $\underset{\approx}{\tau}_T = (\underset{\approx}{\tau} : \underset{\approx}{\tau}_0)\, \underset{\approx}{\tau}_0$ and $\underset{\approx}{\tau}_D = \underset{\approx}{\tau} - \underset{\approx}{\tau}_T$, $\underset{\approx}{\tau} \in \underset{\approx}{\mathbf{R}}_s$. Then

$$|\underset{\approx}{\tau}_T| \leq |\underset{\approx}{\tau}_D| / \alpha_0 \quad \text{for all } \underset{\approx}{\tau} \text{ satisfying } \underset{\approx}{\tau}\, \underset{\sim}{\theta} = 0.$$

Proof: Since

$$\underset{\approx}{\tau}_T\, \underset{\sim}{\theta} \cdot \underset{\approx}{\tau}_0^{-1}\, \underset{\sim}{\theta} = (\underset{\approx}{\tau} : \underset{\approx}{\tau}_0) \underset{\approx}{\tau}_0\, \underset{\sim}{\theta} \cdot \underset{\approx}{\tau}_0^{-1}\, \underset{\sim}{\theta} = \underset{\approx}{\tau} : \underset{\approx}{\tau}_0$$

and

$$\underset{\approx}{\tau}_D\, \underset{\sim}{\theta} = (\underset{\approx}{\tau} - \underset{\approx}{\tau}_T)\, \underset{\sim}{\theta} = -\underset{\approx}{\tau}_T\, \underset{\sim}{\theta},$$

$$|\underset{\approx}{\tau}_T| = |\underset{\approx}{\tau} : \underset{\approx}{\tau}_0| = |\underset{\approx}{\tau}_D\, \underset{\sim}{\theta} \cdot \underset{\approx}{\tau}_0^{-1}\, \underset{\sim}{\theta}|$$

$$\leq |\underset{\approx}{\tau}_D\, \underset{\sim}{\theta}|\, |\underset{\approx}{\tau}_0^{-1}\, \underset{\sim}{\theta}| \leq |\underset{\approx}{\tau}_D| / \alpha_0.$$

The estimate (5.6) follows easily from Lemmas 3.4-3.6 and Lemma 5.1.

6. The Case of a Symmetrized Poisson Ratio Equal to Unity

If one of the symmetrized Poisson ratios ν_i is equal to unity, then the condition of semidefiniteness of the compliance tensor requires that the other two symmetrized Poisson ratios vanish. Thus $\underset{\sim}{\nu}$ is a corner point of the constraint surface, a case we have systematically excluded from consideration. In fact, we shall show in this section that the elasticity system is not elliptic in this case, and that the Dirichlet problem admits no solution unless the boundary data satisfies infinitely many independent constraints.

Without loss of generality we consider the case $\nu_1 = \nu_2 = 0, \ \ \nu_3 = 1$. It is easy to verify that the determinant of the matrix $l(\underset{\sim}{\theta})$ defined in the previous section vanishes for $\underset{\sim}{\theta} = \underset{\sim}{\nu}$. In fact the first two rows are linearly dependent. Thus the system is not elliptic in this case.

To achieve an understanding of the nature of the degeneracy in this case we consider the internal constraint implied by the constitutive equation. The vector $(\sqrt{E_1},\sqrt{E_2},0)^t$ is a null vector of the matrix B, so (1.4) implies that

$$\sqrt{E_1}\,\partial u_1/\partial x_1 + \sqrt{E_2}\,\partial u_2/\partial x_2 = 0 \tag{6.1}$$

for every possible displacement of the material. If we integrate the equation over the cross-section $\Omega_q = \bar{\Omega} \cap \{ \underset{\sim}{x} : x_3 = q\}$ we find that

$$\int_{\partial \Omega_q} (\sqrt{E_1}\, u_1\, n_1{}^q + \sqrt{E_2}\, u_2\, n_2{}^q)\, ds = 0 \tag{6.2}$$

where $\partial \Omega_q$ is the boundary of Ω_q in the plane $x_3 = q$ and $(n_1{}^q, n_2{}^q, 0)$ is its unit normal there. Equation (6.2) is a constraint that the boundary values of $\underset{\sim}{u}$ must satisfy. By varying q we achieve an infinite family of such constraints. Note moreover that the planes $x_3 = q$ are characteristic surfaces for the equation (6.1), and so $\underset{\sim}{u}$ can not be specified arbitrarily on an open subset of such a plane. The case $E_1 = E_2$ admits a particularly clear interpretation. Then (6.1) is a plane incompressibility constraint, and the material may be viewed as a composite of plane incompressible lamina.

The special nature of the present case is also clearly indicated by the classification of constraints in linearly elastic materials due to Pipkin [17]. A constrained material admits a nonzero tensor $\underset{\approx}{\tau} \in \underset{\approx}{\mathbf{R}}$ which is in the null space of the compliance tensor. Pipkin defines the dimension of the constraint to be the rank of $\underset{\approx}{r}$. The simplest constraints, as he points out, are the three dimensional constraints, which admit no characteristic surfaces [17]. From Lemma 3.6 it follows that if $\underset{\sim}{\nu}$ lies on the curved boundary of P but is not a corner point, the constraint is three dimensional. However when $\underset{\sim}{\nu}$ is a corner point the constraint is two dimensional.

7. The Displacement - Pressure Formulation of Orthotropic Elasticity

The system (1.1), (1.2) of three dimensional elasticity involves nine independent scalar unknowns. This is often considered too many for computational purposes and other formulations are preferred. When the compliance tensor is invertible, the simplest possibility is to solve (1.1) for $\underset{\approx}{\sigma}$ and substitute in (1.2) to obtain the displacement equations of elasticity, which involve only the three displacements as unknowns. However, when the compliance tensor is singular this procedure is not possible and when it is nearly singular it is usually not advisable. For isotropic materials, incompressible or not, another formulation, which involves only the displacement and one stress quantity (a pressure) as unknowns, is widely used. In the incompressible limit, this formulation reduces to the Stokes equations.

For orthotropic elasticity, there is an analogous formulation which may be simply derived in light of the preceding considerations. Taylor, Pfister, and Herrmann [21] and Key [11] have also presented formulations of orthotropic elasticity involving fewer unknowns than (1.1), (1.2). Key's formulation in particular is very close to the one we consider here.

The idea of our derivation is as follows. The constitutive equations

$$B \ \underset{\sim}{\mathrm{diag}} \, \underset{\approx}{\sigma} = \underset{\sim}{\mathrm{diag}} \, \underset{\approx}{\epsilon}(\underset{\sim}{u}),$$

$$G \ \underset{\sim}{\mathrm{offd}} \, \underset{\approx}{\sigma} = \underset{\sim}{\mathrm{offd}} \, \underset{\approx}{\epsilon}(\underset{\sim}{u}),$$

may not be solvable for $\underset{\approx}{\sigma}$ since B may vanish on a one dimensional space spanned by $\underset{\sim}{w}$, an eigenvector of B with least eigenvalue. Thus we decompose $\underset{\sim}{\mathrm{diag}} \, \underset{\approx}{\sigma}$ as $p \, \underset{\sim}{w}$ plus a vector orthogonal to $\underset{\sim}{w}$ and take as fundamental unknowns $\underset{\sim}{u}$ and p. The above constitutive equations may then be solved for $\underset{\approx}{\sigma}$ in terms of $\underset{\sim}{u}$ and p and the result substituted into the equilibrium equation (1.2).

Before proceeding, we introduce some notation. For any vector-valued function $\underset{\sim}{v}$, define vector-valued functions $\underset{\sim}{L} \, \underset{\sim}{v}$ and $\underset{\sim}{K} \, \underset{\sim}{v}$ with components

$$L_i \, \underset{\sim}{v} = \partial v_i / \partial x_i,$$

$$K_i \, \underset{\sim}{v} = (\partial v_j / \partial x_k + \partial v_k / \partial x_j)/2, \quad \{i, j, k\} = \{1, 2, 3\}.$$

Thus

$$\underset{\sim}{L}\ \underset{\sim}{v} = \underset{\sim}{\text{diag}}\ \underset{\approx}{\epsilon}(\underset{\sim}{v}) \quad \text{and} \quad \underset{\sim}{K}\ \underset{\sim}{v} = \underset{\sim}{\text{offd}}\ \underset{\approx}{\epsilon}(\underset{\sim}{v}).$$

In this notation the system of orthotropic elasticity reads

$$B\, \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma} = \underset{\sim}{L}\ \underset{\sim}{u}, \tag{7.1}$$

$$G\, \underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma} = \underset{\sim}{K}\ \underset{\sim}{u}, \tag{7.2}$$

$$\underset{\sim}{L}(\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}) + 2\, \underset{\sim}{K}(\underset{\sim}{\text{offd}}\, \underset{\approx}{\sigma}) = \underset{\sim}{f}. \tag{7.3}$$

Now recall that $\lambda_3 \geq \lambda_2 \geq \lambda_1$ denote the eigenvalues of B and $\underset{\sim}{w}^1 = \underset{\sim}{w}$, $\underset{\sim}{w}^2$, $\underset{\sim}{w}^3$ associated unit eigenvectors. Assuming the hypotheses of Lemma 3.3, we have that $\lambda_2, \lambda_3 > 0$, and hence we may define

$$F = \lambda_2^{-1}\, \underset{\sim}{w}^2 (\underset{\sim}{w}^2)^t + \lambda_3^{-1}\, \underset{\sim}{w}^3 (\underset{\sim}{w}^3)^t.$$

Now

$$\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma} = \underset{\sim}{z} + p\, \underset{\sim}{w} \tag{7.4}$$

where

$$\underset{\sim}{z} = \sum_{i=2}^{3} [(\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}) \cdot \underset{\sim}{w}^i]\, \underset{\sim}{w}^i$$

and

$$p = \underset{\sim}{w} \cdot \underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma}. \tag{7.5}$$

Applying FB to (7.4) and using (7.1), we get $\underset{\sim}{z} = F\, \underset{\sim}{L}\, \underset{\sim}{u}$, and so

$$\underset{\sim}{\text{diag}}\, \underset{\approx}{\sigma} = F\, \underset{\sim}{L}\, \underset{\sim}{u} + p\, \underset{\sim}{w}. \tag{7.6}$$

Inverting (7.2) and substituting the result together with (7.6) in (7.3) yields

$$\underset{\sim}{L}(F\, \underset{\sim}{L}\, \underset{\sim}{u}) + 2\, \underset{\sim}{K}(G^{-1}\, \underset{\sim}{K}\, \underset{\sim}{u}) + \underset{\sim}{L}(p\, \underset{\sim}{w}) = \underset{\sim}{f}. \tag{7.7}$$

Next multiply (7.5) by λ_1 and use the symmetry of B together with (7.1) to get

$$\underset{\sim}{w} \cdot \underset{\sim}{L}\, \underset{\sim}{u} - \lambda_1 p = 0. \tag{7.8}$$

Equations (7.7) and (7.8) give the desired formulation of the equations of orthotropic elasticity.

For a two dimensional constrained orthotropic material it is possible to reduce the elastic system further, to a fourth order elliptic equation for a single scalar unknown. In the incompressible isotropic case this is the biharmonic equation.

The equations of plane strain orthotropic elasticity are derived by assuming that $\underset{\approx}{\sigma}$ and $\underset{\sim}{u}$ are independent of x_3 and that $u_3 = 0$. Then $\epsilon_{i3}(\underset{\sim}{u}) = 0$, $i = 1, 2, 3$, and from (1.4) we obtain the constitutive equations

$$H \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \end{pmatrix} = \begin{pmatrix} \epsilon_{11}(\underset{\sim}{u}) \\ \epsilon_{22}(\underset{\sim}{u}) \end{pmatrix}, \qquad \sigma_{12}/G_3 = \epsilon_{12}(\underset{\sim}{u}),$$

where

$$H = \begin{pmatrix} (1-\nu_2^2)/E_1 & (-\nu_3 - \nu_1\nu_2)/(E_1E_2)^{1/2} \\ (-\nu_3 - \nu_1\nu_2)/(E_1E_2)^{1/2} & (1-\nu_1^2)/E_2 \end{pmatrix}$$

If the material is constrained, then the eigenvalues of H are $\lambda_1 = 0$ and $\lambda_2 = \mathrm{tr}(H)$ with corresponding unit eigenvectors $\underset{\sim}{w}^1 = (\beta, \gamma)^t$ and $\underset{\sim}{w}^2 = (\gamma, -\beta)^t$, where

$$\beta = [H_{22}/\mathrm{tr}(H)]^{1/2} \quad \text{and} \quad \gamma = [H_{11}/\mathrm{tr}(H)]^{1/2}.$$

Defining

$$F = H/\mathrm{tr}(H)^2 \quad \text{and} \quad p = \underset{\sim}{w}^1 \cdot (\sigma_{11}, \sigma_{22})^t,$$

the analogue of (7.6) in this case is

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \end{pmatrix} = F \begin{pmatrix} \partial u_1/\partial x_1 \\ \partial u_2/\partial x_2 \end{pmatrix} + p \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.$$

Since $\sigma_{12} = \sigma_{21} = G_3\,\epsilon_{12}(\underset{\sim}{u})$, and $\sigma_{13} = \sigma_{23} = 0$, the analogue of (7.7) with unknowns u_1, u_2, and p is easily obtained by using the above identities to eliminate $\underset{\approx}{\sigma}$ from the first two equations comprising (1.2). In this case, equation (7.8) becomes $\mathrm{div}\,(\beta u_1, \gamma u_2) = 0$. Hence, there is a scalar ϕ such that $\beta u_1 = \partial\phi/\partial x_2$ and $\gamma u_2 = -\partial\phi/\partial x_1$. Applying $\gamma\,\partial/\partial x_2$ to the analogue of the first coordinate equation of (7.7), $\beta\,\partial/\partial x_1$ to the second, and subtracting we find

$$\begin{aligned} \mathcal{L}\phi &:= [\beta G_3/(2\gamma)]\,\partial^4\phi/\partial x_1{}^4 + [\gamma G_3/(2\beta)]\,\partial^4\phi/\partial x_2{}^4 \\ &\quad - (G_3 + 1/H_{12})\,\partial^4\phi/\partial x_1{}^2\,\partial x_2{}^2 \\ &= \gamma\,\partial f_1/\partial x_2 - \beta\,\partial f_2/\partial x_1 \end{aligned}$$

It is easy to show that $\mathcal{L}$ is a uniformly coercive operator in the sense that

$$\int_{\Omega} (\mathcal{L}\,\phi)\,\phi \; d\underset{\sim}{x} \;\geq\; C \,\|\, \phi \,\|_2^2$$

for all $\phi \in H_0^2(\Omega)$, with $C > 0$ depending only on Ω and α in (2.1).

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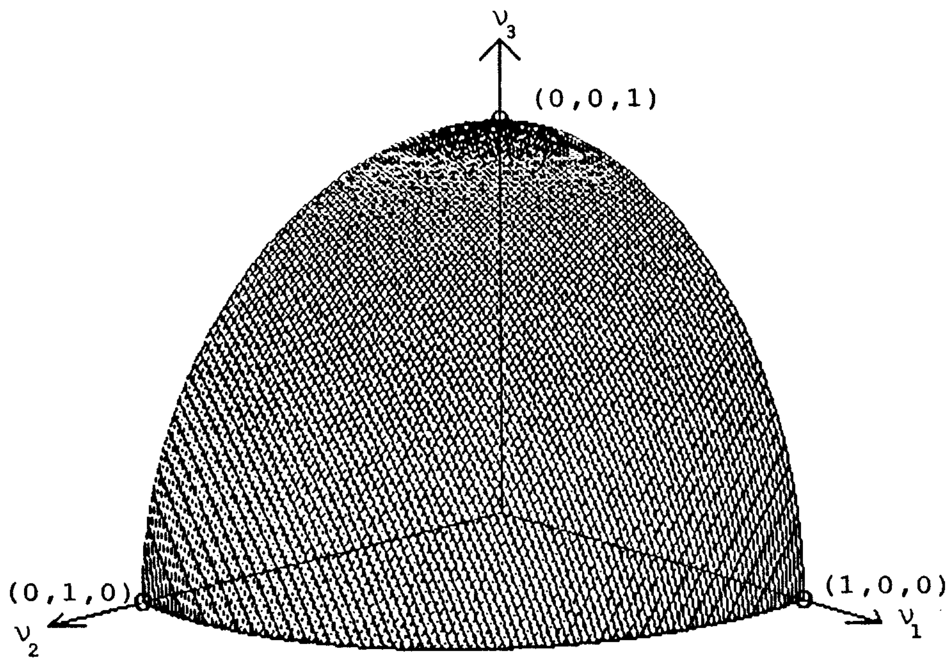
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