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ANALYSIS OF A LINEAR-LINEAR FINITE ELEMENT FOR THE REISSNER-MINDLIN PLATE MODEL

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An analysis is presented for a recently proposed finite element method for the Reissner–Mindlin plate problem. The method is based on the standard variational principle, uses nonconforming linear elements to approximate the rotations and conforming linear elements to approximate the transverse displacements, and avoids the usual "locking problem" by interpolating the shear stress into a rotated space of lowest order Raviart-Thomas elements. When the plate thickness $t = O(h)$, it is proved that the method gives optimal order error estimates uniform in t. However, the analysis suggests and numerical calculations confirm that the method can produce poor approximations for moderate sized values of the plate thickness. Indeed, for t fixed, the method does not converge as the mesh size h tends to zero.

1. Introduction

The purpose of this paper is to study a low order finite element scheme proposed by Oñate, Zarate, and Flores[15] for the approximation of the Reissner–Mindlin plate equations. The main difficulty in the finite element approximation of these equations is the problem of "locking", which results in poor approximations for thin plates, and the scheme proposed in Ref. 15 is one of several which have been proposed to overcome locking. An attractive feature of this method is that it uses only linear finite elements. In this paper we prove that when $t = O(h)$, h being the finite element mesh size, the method attains optimal order accuracy, giving good approximations and avoiding the locking problem. However, as our analysis suggests and we confirm by means of numerical calculations, when t is large compared to h (the case of moderately thick plates), the method does not work well. Indeed, the

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method is not convergent in the classical sense, i.e., when h tends to zero with t fixed.

The Reissner–Mindlin models determine functions ϕ and ω, which are defined on the middle surface Ω of the plate and approximate the rotation vector and transverse displacement, respectively, as the minimizers of the energy functional

$$J(\boldsymbol{\phi},\omega) = \frac{1}{2}\int_\Omega \mathcal{C}\,\mathcal{E}\,\boldsymbol{\phi} : \mathcal{E}\,\boldsymbol{\phi} + \frac{\lambda t^{-2}}{2}\int_\Omega |\boldsymbol{\phi} - \mathbf{grad}\,\omega|^2 - \int_\Omega g\omega$$

over a subspace of $\boldsymbol{H}^1(\Omega) \times H^1(\Omega)$ incorporating essential boundary conditions. Here $\mathcal{E}\boldsymbol{\phi}$ denotes the symmetric part of the gradient of $\boldsymbol{\phi}$, g the scaled transverse loading function, t the plate thickness, and $\lambda = Ek/2(1+\nu)$ where E is Young's modulus, ν the Poisson ratio, and k the shear correction factor. For all 2×2 symmetric matrices $\boldsymbol{\tau}$, $\mathcal{C}\boldsymbol{\tau}$ is defined by

$$\mathcal{C}\boldsymbol{\tau} = \frac{E}{12(1-\nu^2)}[(1-\nu)\boldsymbol{\tau} + \nu\,\mathrm{tr}(\boldsymbol{\tau})I]\,.$$

As is now well understood, standard finite element methods for the Reissner–Mindlin plate, which approximate $\boldsymbol{\phi}$ and ω by the minimizer of the above energy functional over a finite element subspace of $\boldsymbol{H}^1(\Omega) \times H^1(\Omega)$, usually do not converge uniformly with respect to the plate thickness. Rather they are plagued by a deterioration of accuracy as t tends to zero, known as locking. Many of the methods which have been proposed to overcome locking take the following form. The approximate solution $(\boldsymbol{\phi}_h, \omega_h)$ is determined in a finite element space $\boldsymbol{V}_h \times W_h$ as the minimizer of a modified energy functional

$$J_h(\boldsymbol{\phi},\omega) = \frac{1}{2}\int_\Omega \mathcal{C}\mathcal{E}\boldsymbol{\phi} : \mathcal{E}\boldsymbol{\phi} + \frac{\lambda t^{-2}}{2}\int_\Omega |\boldsymbol{R}_h\boldsymbol{\phi} - \mathbf{grad}\,\omega|^2 - \int_\Omega g\omega\,. \tag{1.1}$$

The modification consists of incorporating the "reduction operator" $\boldsymbol{R}_h : \boldsymbol{V}_h \to \boldsymbol{\Gamma}_h$ where $\boldsymbol{\Gamma}_h$ is an auxiliary finite element space and $\boldsymbol{R}_h$ is typically either an interpolation operator or an $\boldsymbol{L}^2$-projection operator. The finite element spaces $\boldsymbol{V}_h$ and W_h may either be conforming or nonconforming. If they are nonconforming, the differential operators in (1.1) are of course applied element-by-element. Table 1 exhibits four such methods which use triangular finite elements of relatively low order.

In the element diagrams in the table, the filled circle, open circle, and arrow are used to denote degrees of freedom. The filled circle, open circle, denote the value of both components of a vector quantity at the node, the value of a scalar quantity at the node, and the value of the tangential component of a vector quantity at an edge node respectively. Thus three different spaces W_h are depicted: standard Lagrange elements of degree one for the methods of Durán–Liberman and Oñate–Zarate–Flores, Lagrange elements of degree two for the method of Brezzi–Fortin–Stenberg, and nonconforming piecewise linear elements for the method of Arnold–Falk. For $\boldsymbol{V}_h$ the spaces are less standard (although they are all recognizable as finite element spaces which have been used to approximate the velocity for Stokes flow). For the

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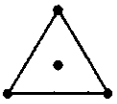
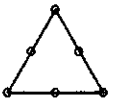
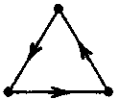
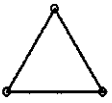
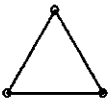
Table. 1. Finite element schemes for the Reissner–Mindlin plate based on the reduced energy functional (1.1).

V_h	W_h	Γ_h	R_h	Reference
			P_0	Arnold, Falk (Ref. 3)
			Π_1	Brezzi, Fortin, Stenberg (Ref. 8)
			Π_0	Durán, Liberman (Ref. 10)
			Π_0	Oñate, Zarate, Flores (Ref. 15)

Arnold–Falk method this is the space of continuous piecewise linear vector fields augmented by cubic bubbles, for the Brezzi–Fortin–Stenberg method the space of continuous piecewise quadratic vector fields augmented by cubic bubbles, for the Durán–Liberman method the subspace of continuous piecewise quadratic vector fields for which the normal component is linear on each edge, and for the Oñate–Zarate–Flores method, the space of nonconforming piecewise linear vector fields. The third and fourth columns in Table 1 show the range $\boldsymbol{\Gamma}_h$ of the reduction operator $\boldsymbol{R}_h$, and the operator itself. For the Arnold–Falk method, $\boldsymbol{\Gamma}_h$ is the space of piecewise constant vector fields and the reduction operator is the L^2-projection, that is, the elementwise averaging operator. For the Durán–Liberman method and the Oñate–Zarate–Flores method, $\boldsymbol{\Gamma}_h$ is the lowest order Raviart–Thomas subspace of $\boldsymbol{H}(\mathrm{rot})$. That is, it is the space of piecewise linear vector fields for which the tangential component is constant on each element edge and continuous from element to element. For these methods, the reduction operator is the natural interpolation operator associated with this space. For the method of Brezzi, Fortin and Stenberg, $\boldsymbol{\Gamma}_h$ is the Raviart–Thomas approximation to $\boldsymbol{H}(\mathrm{rot})$ of one order higher, and $\boldsymbol{R}_h$ the corresponding interpolant.

The Arnold–Falk method was the first Reissner–Mindlin element computable in the primitive variables $\boldsymbol{\phi}$ and ω which was proved to converge with optimal order, uniformly with respect to the plate thickness. Brezzi, Fortin, and Stenberg,[8] following up on the work in Ref. 6, presented an approach for devising and verifying locking-free Reissner–Mindlin elements, and as an application devised several families of such elements. Assuming uniform regularity of the solution they proved that these elements converge with optimal order uniformly in t. The second method depicted in Table 1 is the simplest treated there. The Durán–Liberman method is a simpler method which also fits within the framework of Ref. 8 and so converges

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with optimal order uniformly in t. The Oñate–Zarate–Flores method was introduced in Ref. 15 and good performance shown through numerical tests. It is appealing because of its simplicity, but, in contrast to the other elements depicted in the table, it has not been proven to be locking-free. In this paper we analyze the convergence of this method. Our approach is strongly influenced by that of Ref. 8, but we must also consider the effect of the additional consistency error owing to the nonconformity of the approximation for $\boldsymbol{\phi}$. We shall show, among other estimates, that

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 \leq C \max(h^2, t^2)\|g\|_0 .$$

Thus, we have optimal order convergence in L^2 for both variables if the plate thickness t tends to zero at least as quickly as the mesh size h. However this estimate does not even establish convergence of the method in the classical sense, that is, when the mesh size tends to zero while the plate thickness is held fixed. In fact, we show by means of a numerical example that such convergence does not hold.

Before closing this section, we recall another important approach to the development of low order locking-free finite element schemes: the use of stabilization techniques. These are not based on the reduced energy functional (1.1), but rather on a modification of it in which the coefficient t^{-2} is relaced by $(t^2 + \alpha h^2)^{-1}$ for a suitable constant α. A method proposed by Duran, Ghioldi, and Wolanksi[9] and simultaneously by Franca and Stenberg[13] uses the same elements as the method of Arnold and Falk, except that the bubble functions are not included in the space $\boldsymbol{V}_h$. In both works it is shown that this choice of spaces results in a uniformly optimal order method when used with the stabilized energy functional. In fact, as discussed in Ref. 1, essentially the same scheme results by using static condensation to eliminate the bubble function in the method of Ref. 3. A similar stabilized method was proposed by Pitkäranta,[16] except that the transverse displacement was approximated by conforming quadratic instead of nonconforming linear, and no reduction operator is needed. A simpler stabilized method has been recently proposed and analyzed by Brezzi, Fortin, and Stenberg.[8] This uses continuous piecewise linear elements for both $\mathbf{V}_h$ and W_h, and the reduction operator is $\boldsymbol{\Pi}_0$, the interpolation into the lowest order Raviart–Thomas space. Finally, a variety of methods have been proposed and analyzed using more involved modifications of the energy function to achieve stabilization. Cf. Refs. 14 and 17.

In the next section, we present a mixed formulation of the Reissner–Mindlin problem which will facilitate the analysis of the Oñate–Zarate–Flores method. In Sec. 3 we prove optimal order error estimates for the case when $t \leq Ch$. An interesting relationship between the Oñate–Zarate–Flores method and the Morley method for the approximation of the biharmonic problem is established in Sec. 4. Finally, in Sec. 5, we present the results of some numerical computations which show that for t fixed, the method does not converge as the mesh size h tends to zero.

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2. Variational Formulations

For simplicity we henceforth assume that the domain Ω is a convex polygon and restrict our attention to the case of a (hard) clamped plate. The rotation vector and transverse displacement may then be determined as the unique solution to the following weak formulation.

Find $(\boldsymbol{\phi}, \omega) \in \overset{\circ}{\boldsymbol{H}}{}^1 \times \overset{\circ}{H}{}^1$ satisfying:

$$(\mathcal{C}\mathcal{E}\boldsymbol{\phi}, \mathcal{E}\boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \mathbf{grad}\,\omega, \boldsymbol{\psi} - \mathbf{grad}\,\mu) = (g, \mu) \quad \text{for all } (\boldsymbol{\psi}, \mu) \in \overset{\circ}{\boldsymbol{H}}{}^1 \times \overset{\circ}{H}{}^1 .$$

The space $\overset{\circ}{H}{}^1$ denotes the usual Sobolev space of square integrable functions on Ω which vanish on $\partial\Omega$ and which possess square integrable first derivatives, and $\overset{\circ}{\boldsymbol{H}}{}^1$ denotes the corresponding space of two-vector-valued functions. (We use boldface to denote vector-valued analog of spaces and operators generally.) The parentheses denote the L^2 (or $\mathbf{L}^2$) inner product. In view of the analysis to follow we also recall the definition of the differential operators

$$\mathbf{curl}\, q = \begin{pmatrix} -\partial q/\partial y \\ \partial q/\partial x \end{pmatrix}, \quad \mathrm{rot}\,\boldsymbol{\psi} = (\,\partial\psi_1/\partial y - \partial\psi_2/\partial x\,)\ ,$$

and the space

$$\overset{\circ}{\boldsymbol{H}}(\mathrm{rot}) \equiv \{\, \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega) \mid \mathrm{rot}\,\boldsymbol{\psi} \in L^2(\Omega), \boldsymbol{\psi}\cdot\mathbf{s} = 0 \text{ on } \partial\Omega \,\} .$$

For the precise description and analysis of the approximation scheme we introduce several finite element spaces. We assume that a quasiuniform shape-regular family of triangulations of Ω is given with the characteristic mesh size h tending to zero. For each triangulation, we define

$\overset{\circ}{M}{}^1_0$, the usual conforming piecewise linear approximation of $\overset{\circ}{H}{}^1$, consisting of continuous piecewise linear functions vanishing on the boundary;

$\overset{\circ}{M}{}^1_*$, the usual nonconforming piecewise linear approximation of $\overset{\circ}{H}{}^1$, consisting of piecewise linear functions which are continuous at the midpoints of element edges and vanish at the midpoints of boundary edges;

M^0, the space of piecewise constant functions; and

$\boldsymbol{\Gamma}_h$, the lowest order Raviart–Thomas subspace of $\overset{\circ}{\boldsymbol{H}}(\mathrm{rot})$, consisting of vector-valued functions which on each finite element are of the form $(a - by, c + bx)^{\mathrm{T}}$ for some a, b, $c \in \mathbb{R}$, and for which the tangential component on each element edge is continuous from element to element and vanishes on boundary edges.

We shall use $\overset{\circ}{\boldsymbol{M}}{}^1_*$, the vector analog of the nonconforming space to approximate $\boldsymbol{\phi}$. Differential operators such as $\mathcal{E}$ and rot may be applied to functions in $\overset{\circ}{\boldsymbol{M}}{}^1_*$ element by element; we shall write $\mathcal{E}_h$ and rot_h in this case.

For each of these spaces we define a projection operator mapping into the space as follows:

$$\Pi_h^1 : \mathring{H}^1 \cap C(\Omega) \to \mathring{M}_0^1,\ (\psi - \Pi_h^1\psi)(v) = 0 \text{ for all vertices } v,$$

$$\Pi_h^* : \mathring{H}^1 \to \mathring{M}_*^1,\ \int_e (\psi - \Pi_h^*\psi) = 0 \text{ for all edges } e,$$

$$\Pi_h^0 : L^2 \to M^0,\ \int_T (\psi - \Pi_h^0\psi) = 0 \text{ for all triangles } T,$$

$$\boldsymbol{\Pi}_h^\Gamma : \mathring{\boldsymbol{H}}(\mathrm{rot}) \to \boldsymbol{\Gamma}_h,\ \int_e (\boldsymbol{\psi} - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\psi}) \cdot \boldsymbol{s} = 0 \text{ for all edges } e.$$

Since $\int_e \boldsymbol{\psi} \cdot \boldsymbol{s}$ is well-defined for $\boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1$, the Raviart–Thomas projection $\boldsymbol{\Pi}_h^\Gamma\boldsymbol{\psi}$ is also well-defined for such $\boldsymbol{\psi}$. These operators have a number of properties which will enter the analysis below and which we now collect. First they give optimal order approximation:

$$\|\psi - \Pi_h^1\psi\|_0 + h\|\psi - \Pi_h^1\psi\|_1 \le Ch^2\|\psi\|_2\,, \quad \text{for all } \psi \in \mathring{H}^1 \cap H^2\,, \tag{2.1}$$

$$\|\psi - \Pi_h^*\psi\|_0 + h\|\psi - \Pi_h^*\psi\|_{1,h} \le Ch^2\|\psi\|_2\,, \quad \text{for all } \psi \in \mathring{H}^1 \cap H^2\,, \tag{2.2}$$

$$\|\psi - \Pi_h^0\psi\|_0 \le Ch\|\psi\|_1\,, \quad \text{for all } \psi \in H^1\,, \tag{2.3}$$

$$\|\boldsymbol{\psi} - \boldsymbol{\Pi}_h^\Gamma\psi\|_0 \le Ch\|\psi\|_1\,, \quad \text{for all } \psi \in \boldsymbol{H}^1 \cap \mathring{\boldsymbol{H}}(\mathrm{rot})\,, \tag{2.4}$$

$$\|\boldsymbol{\psi} - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\psi}\|_0 \le Ch\|\boldsymbol{\psi}\|_{1,h}\,, \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1\,, \tag{2.5}$$

where we use the definition

$$\|\boldsymbol{\psi}\|_{1,h}^2 = \|\psi\|_0^2 + \|\,\mathbf{grad}_h\,\boldsymbol{\psi}\|_0^2\,.$$

Next, we recall the well-known commutativity property of the Raviart–Thomas projection,

$$\mathrm{rot}\,\boldsymbol{\Pi}_h^\Gamma\boldsymbol{\psi} = \Pi_h^0\,\mathrm{rot}\,\boldsymbol{\psi} \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1\,, \tag{2.6}$$

and its analog for $\boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1$, which takes the form

$$\mathrm{rot}\,\boldsymbol{\Pi}_h^\Gamma\boldsymbol{\psi} = \mathrm{rot}_h\,\boldsymbol{\psi} \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1\,. \tag{2.7}$$

The analogous property also holds for the projection onto the nonconforming space $\mathring{\boldsymbol{M}}_*^1$:

$$\mathrm{rot}_h\,\boldsymbol{\Pi}_h^*\boldsymbol{\psi} = \Pi_h^0\,\mathrm{rot}\,\boldsymbol{\psi} \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1\,. \tag{2.8}$$

The operator $\mathbf{curl}_h : M^0 \to \boldsymbol{\Gamma}_h$ is defined by the equation

$$(\mathbf{curl}_h\,q, \boldsymbol{\psi}) = (q, \mathrm{rot}\,\boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{\Gamma}_h\,. \tag{2.9}$$

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Using it we may state the following discrete analog of the Helmholtz decomposition (cf. Lemma 3.1 of Ref. 8).

Lemma 2.1.

$$\mathbf{\Gamma}_h = \mathbf{grad}\, \mathring{M}_0^1 \oplus \mathbf{curl}_h\, M^0 .$$

This is an $\boldsymbol{L}^2$ orthogonal decomposition.

The approximation scheme of Oñate, Zarate and Flores uses $\mathring{\boldsymbol{M}}_*^1$ as an approximation of the space $\mathring{\boldsymbol{H}}^1$ for the rotations and $\mathring{M}_0^1$ as an approximation of the space $\mathring{H}^1$ for the displacements. In addition it makes use of the interpolation operator $\mathbf{\Pi}_h^\Gamma$ as a reduction operator. Thus the discrete solution $(\boldsymbol{\phi}_h, \omega_h) \in \mathring{\boldsymbol{M}}_*^1 \times \mathring{M}_0^1$ is determined by the equations

$$\begin{aligned}(\mathcal{C}\mathcal{E}_h\boldsymbol{\phi}_h, \mathcal{E}_h\boldsymbol{\psi}) + \lambda t^{-2}(\mathbf{\Pi}_h^\Gamma\boldsymbol{\phi}_h - \mathbf{grad}\,\omega_h, \mathbf{\Pi}_h^\Gamma\boldsymbol{\psi} - \mathbf{grad}\,\mu)\\ = (g, \mu) \quad \text{for all } (\boldsymbol{\psi}, \mu) \in \mathring{\boldsymbol{M}}_*^1 \times \mathring{M}_0^1 .\end{aligned} \tag{2.10}$$

To analyze the scheme, we follow Ref. 8 to obtain an alternate weak formulation of the continuous Reissner–Mindlin problem. First we use the Helmholtz decomposition to write

$$\lambda t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = \mathbf{grad}\, r + \mathbf{curl}\, p$$

with r vanishing on $\partial\Omega$ and p normalized to have mean value zero. Next we introduce the auxiliary variable $\boldsymbol{\alpha} = \mathbf{curl}\, p$. It is then easy to check that $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in \mathring{H}^1 \times \mathring{\boldsymbol{H}}^1 \times \hat{L}^2 \times \mathring{\boldsymbol{H}}(\mathrm{rot}) \times \mathring{H}^1$ satisfies the following equations:

$$(\mathbf{grad}\, r, \mathbf{grad}\,\mu) = (g, \mu) \qquad \text{for all } \mu \in \mathring{H}^1 , \tag{2.11}$$

$$(\mathcal{C}\mathcal{E}\boldsymbol{\phi}, \mathcal{E}\boldsymbol{\psi}) - (p, \mathrm{rot}\,\boldsymbol{\psi}) = (\mathbf{grad}\, r, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1 , \tag{2.12}$$

$$-(\mathrm{rot}\,\boldsymbol{\phi}, q) - \lambda^{-1}t^2(\mathrm{rot}\,\alpha, q) = 0 \quad \text{for all } q \in \hat{L}^2 , \tag{2.13}$$

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) - (p, \mathrm{rot}\,\boldsymbol{\delta}) = 0 \quad \text{for all } \boldsymbol{\delta} \in \mathring{\boldsymbol{H}}(\mathrm{rot}) , \tag{2.14}$$

$$(\mathbf{grad}\,\omega, \mathbf{grad}\, s) = (\boldsymbol{\phi} + \lambda^{-1}t^2\,\mathbf{grad}\, r, \mathbf{grad}\, s) \quad \text{for all } s \in \mathring{H}^1 . \tag{2.15}$$

Note that we use a circumflex over a space to denote the subspace consisting of functions of mean value zero.

As observed in many papers on the subject, the two Poisson problems (2.11) and (2.15) decouple from this system. To study the remaining Eqs. (2.12)–(2.14), we follow Brezzi, Fortin and Stenberg[8] and define $A : [\mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})] \times [\mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})] \to \mathbb{R}$ and $B : [\mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})] \times \hat{L}^2 \to \mathbb{R}$ by

$$A(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) = (\mathcal{C}\mathcal{E}\boldsymbol{\phi}, \mathcal{E}\boldsymbol{\psi}) + \lambda^{-1}t^2(\boldsymbol{\alpha}, \boldsymbol{\delta}) ,$$

$$B(\boldsymbol{\psi}, \boldsymbol{\delta}; q) = -(\mathrm{rot}\,\boldsymbol{\psi}, q) - \lambda^{-1}t^2(\mathrm{rot}\,\boldsymbol{\delta}, q) .$$

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With respect to the t-dependent norm

$$|||\boldsymbol{\psi}, \boldsymbol{\delta}|||^2 = \|\boldsymbol{\psi}\|_1^2 + t^2\|\boldsymbol{\delta}\|_0^2 + t^4\|\operatorname{rot}\boldsymbol{\delta}\|_0^2$$

on $\mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})$ and the usual norm on $\hat{L}^2$, the forms A and B are bounded uniformly in t. Note that for the exact solution we have $\boldsymbol{\alpha} = \mathbf{curl}\, p$ and $t^2 \operatorname{rot}\boldsymbol{\alpha} = -\lambda \operatorname{rot}\boldsymbol{\phi}$, so $|||\boldsymbol{\phi}, \boldsymbol{\alpha}|||^2 = \|\boldsymbol{\phi}\|_1^2 + \lambda^2\|\operatorname{rot}\boldsymbol{\phi}\|_0^2 + t^2\|\mathbf{curl}\, p\|_0^2$. The continuous problem (2.12)–(2.14) can then be cast into the form:

Find $(\boldsymbol{\phi}, \boldsymbol{\alpha}) \in \mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})$ and $p \in \hat{L}^2$ such that

$$A(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) + B(\boldsymbol{\psi}, \boldsymbol{\delta}; p) = (\mathbf{grad}\, r, \boldsymbol{\psi}) \quad \text{for all } (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})\,, \tag{2.16}$$

$$B(\boldsymbol{\phi}, \boldsymbol{\alpha}; q) = 0 \quad \text{for all } q \in \hat{L}^2\,. \tag{2.17}$$

This problem is a saddle point problem of the form considered by Brezzi in Ref. 7. The two hypotheses of Brezzi's theorem, which we now state, are easily verified using Korn's inequality for the first and simply choosing $\boldsymbol{\delta} = 0$ in the second.

(B1) There exists $\gamma > 0$ independent of t such that

$$A(\boldsymbol{\psi}, \boldsymbol{\delta}; \boldsymbol{\psi}, \boldsymbol{\delta}) \geq \gamma |||\boldsymbol{\psi}, \boldsymbol{\delta}|||^2$$

for all

$$\begin{aligned}(\boldsymbol{\psi}, \boldsymbol{\delta}) \in Z =: &\left\{ (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot}) \,|\, B(\boldsymbol{\psi}, \boldsymbol{\delta}; q) = 0 \quad \text{for all } q \in \hat{L}^2 \right\} \\ = &\left\{ (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot}) \,|\, \lambda^{-1} t^2 \operatorname{rot}\boldsymbol{\delta} = -\operatorname{rot}\boldsymbol{\psi} \right\}.\end{aligned}$$

(B2) There exists $\gamma > 0$ independent of t such that

$$\inf_{q \in \hat{L}^2} \sup_{(\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{H}}^1 \times \mathring{\boldsymbol{H}}(\mathrm{rot})} \frac{B(\boldsymbol{\psi}, \boldsymbol{\delta}; q)}{|||\boldsymbol{\psi}, \boldsymbol{\delta}|||\ \|q\|_0} \geq \gamma\,.$$

We note for future reference the following regularity results for (2.11)–(2.15). The bounds for r, ω, $\boldsymbol{\phi}$ and p were proved in Ref. 3 and those for $\boldsymbol{\alpha}$ follow from the equations $\boldsymbol{\alpha} = \mathbf{curl}\, p$ and $t^2 \operatorname{rot}\boldsymbol{\alpha} = -\lambda \operatorname{rot}\boldsymbol{\phi}$. There exists a constant C independent of t such that

$$\|r\|_1 + \|\omega\|_1 + \|\boldsymbol{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|\boldsymbol{\alpha}\|_0 + t\|\boldsymbol{\alpha}\|_1 + t^2\|\operatorname{rot}\boldsymbol{\alpha}\|_1 \leq C\|g\|_{-1}\,, \tag{2.18}$$

$$\|r\|_2 + \|\omega\|_2 \leq C\|g\|_0\,. \tag{2.19}$$

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We now derive the discrete version of (2.11)–(2.15). Let $(\boldsymbol{\phi}_h, \omega_h)$ solve (2.10). Then $\mathbf{grad}\,\omega_h$ and $\boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h$ belong to $\boldsymbol{\Gamma}_h$, so we invoke Lemma 2.1 to write

$$\lambda t^{-2}(\mathbf{grad}\,\omega_h - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h) = \mathbf{grad}\, r_h + \mathbf{curl}_h\, p_h$$

with $r_h \in \overset{\circ}{M}{}_0^1$ and $p_h \in \hat{M}^0$. Set $\boldsymbol{\alpha}_h = \mathbf{curl}_h\, p_h \in \boldsymbol{\Gamma}_h$. Using (2.9) and (2.7), it is then easy to see that the quintuple $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in \overset{\circ}{M}{}_0^1 \times \overset{\circ}{\boldsymbol{M}}{}_*^1 \times \hat{M}^0 \times \boldsymbol{\Gamma}_h \times \overset{\circ}{M}{}_0^1$ satisfies

$$(\mathbf{grad}\, r_h, \mathbf{grad}\,\mu) = (g, \mu) \quad \text{for all } \mu \in \overset{\circ}{M}{}_0^1\,, \tag{2.20}$$

$$(\mathcal{C}\mathcal{E}_h \boldsymbol{\phi}_h, \mathcal{E}_h \boldsymbol{\psi}) - (p_h, \mathrm{rot}_h\, \boldsymbol{\psi}) = (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \overset{\circ}{\boldsymbol{M}}{}_*^1\,, \tag{2.21}$$

$$-(\mathrm{rot}_h\, \boldsymbol{\phi}_h, q) - \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\alpha}_h, q) = 0 \quad \text{for all } q \in \hat{M}^0\,, \tag{2.22}$$

$$(\boldsymbol{\alpha}_h, \boldsymbol{\delta}) - (p_h, \mathrm{rot}\, \boldsymbol{\delta}) = 0 \quad \text{for all } \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h\,, \tag{2.23}$$

$$(\mathbf{grad}\,\omega_h, \mathbf{grad}\, s) = (\boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h + \lambda^{-1} t^2\, \mathbf{grad}\, r_h, \mathbf{grad}\, s) \quad \text{for all } s \in \overset{\circ}{M}{}_0^1\,. \tag{2.24}$$

Once again the main part of the analysis deals with Eqs. (2.21)–(2.23), which we may rewrite in standard form:

Find $(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h) \in \overset{\circ}{\boldsymbol{M}}{}_*^1 \times \boldsymbol{\Gamma}_h$, $p_h \in \hat{M}^0$ such that

$$A_h(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h; \boldsymbol{\psi}, \boldsymbol{\delta}) + B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; p_h) = (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\psi}) \quad \text{for all } (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \overset{\circ}{\boldsymbol{M}}{}_*^1 \times \boldsymbol{\Gamma}_h\,, \tag{2.25}$$

$$B_h(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h; q) = 0 \quad \text{for all } q \in \hat{M}^0\,, \tag{2.26}$$

where

$$A_h(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) = (\mathcal{C}\mathcal{E}_h \boldsymbol{\phi}, \mathcal{E}_h \boldsymbol{\psi}) + \lambda^{-1} t^2 (\boldsymbol{\alpha}, \boldsymbol{\delta})\,,$$
$$B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; q) = -(\mathrm{rot}_h\, \boldsymbol{\psi}, q) - \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\delta}, q)\,.$$

Note that A_h and B_h are defined just as A and B except that $\mathcal{E}$ and rot are replaced by the piecewise defined operators $\mathcal{E}_h$ and rot_h when applied to functions in the nonconforming space $\overset{\circ}{\boldsymbol{M}}{}_*^1$.

3. Error Estimates

In the preceding section, we reduced the continuous Reissner–Mindlin system to the saddle point problem (2.16)–(2.17), and the Oñate–Zarate–Flores method to the discrete analog of this problem, (2.25)–(2.26). The well-posedness of (2.16)–(2.17) followed easily from the continuous Brezzi conditions (B1) and (B2). A seemingly natural way to proceed with the error analysis would be to establish the discrete analogs of the Brezzi conditions for the discrete problem (2.25)–(2.26), and

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then to apply standard arguments from the theory of mixed methods. However this approach fails, because the discrete analog of condition (B1) does not hold. To understand why the condition fails in the discrete case even though it holds in the continuous case, recall that the continuous result was a simple consequence of Korn's inequality. However the discrete analog of Korn's inequality for nonconforming linear elements, that is, the statement that

$$\|\mathbf{grad}_h\,\boldsymbol{\psi}\|_0 \le C\|\mathcal{E}_h\boldsymbol{\psi}\|_0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{M}}{}^1_*\,,$$

is known *not* to be true (cf. Refs. 11 and 12). All is not lost, however. The discrete analog of condition (B1) requires coercivity of the form A_h only over the subspace Z_h of $\mathring{\boldsymbol{M}}{}^1_* \times \boldsymbol{\Gamma}_h$ which is defined in Eq. (3.3) below. But, as we show in Lemma 3.2 below, such coercivity does hold with a constant γ of the form $c\min(1, h^2/t^2)$. Thus, if $t = O(h)$, γ is $O(1)$, while for fixed t, γ tends to zero as h tends to zero. This observation is important in understanding both the good performance of the method when $t = O(h)$ and its failure to converge for fixed t as h tends to zero.

From the strong form of Eq. (2.16), we obtain, using integration by parts, that

$$A_h(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) + B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; p) = (\mathbf{grad}\, r, \boldsymbol{\psi}) + \sum_T \int_{\partial T} [(\mathcal{C}\mathcal{E}\boldsymbol{\phi})\boldsymbol{n} + p\boldsymbol{s}] \cdot \boldsymbol{\psi}$$

$$\text{for all } (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{M}}{}^1_* \times \boldsymbol{\Gamma}_h\,.$$

Combining this equation with Eq. (2.25) gives

$$\begin{aligned} A_h(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h; \boldsymbol{\psi}, \boldsymbol{\delta}) + B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; p_h) &= A_h(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) + B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; p) \\ &\quad + (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\psi}) - (\mathbf{grad}\, r, \boldsymbol{\psi}) - \sum_T \int_{\partial T} [(\mathcal{C}\mathcal{E}\boldsymbol{\phi})\boldsymbol{n} + p\boldsymbol{s}] \cdot \boldsymbol{\psi} \\ &\qquad \text{for all } (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{M}}{}^1_* \times \boldsymbol{\Gamma}_h\,, \end{aligned} \tag{3.1}$$

whence

$$\begin{aligned} A_h(\boldsymbol{\Pi}_h^*\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\alpha} - \boldsymbol{\alpha}_h; \boldsymbol{\psi}, \boldsymbol{\delta}) + B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; p - p_h) &= A_h(\boldsymbol{\Pi}_h^*\boldsymbol{\phi} - \boldsymbol{\phi}, \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\alpha} - \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) \\ &\quad - (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\psi}) + (\mathbf{grad}\, r, \boldsymbol{\psi}) + \sum_T \int_{\partial T} [(\mathcal{C}\mathcal{E}\boldsymbol{\phi})\boldsymbol{n} + p\boldsymbol{s}] \cdot \boldsymbol{\psi} \\ &\qquad \text{for all } (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{M}}{}^1_* \times \boldsymbol{\Gamma}_h\,. \end{aligned} \tag{3.2}$$

We now prove a sequence of lemmas which allow us to derive error estimates from (3.2). For the first two we define

$$\begin{aligned} Z_h &= \left\{ (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{M}}{}^1_* \times \boldsymbol{\Gamma}_h \,|\, B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; q) = 0 \quad \text{for all } q \in \hat{M}^0 \right\} \\ &= \left\{ (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \mathring{\boldsymbol{M}}{}^1_* \times \boldsymbol{\Gamma}_h \,|\, \lambda^{-1} t^2 \operatorname{rot} \boldsymbol{\delta} = \operatorname{rot}_h \boldsymbol{\psi} \quad \text{for all } q \in \hat{M}^0 \right\}. \end{aligned} \tag{3.3}$$

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We note that if $(\boldsymbol{\psi}, \boldsymbol{\delta}) \in Z_h$, then

$$B_h(\boldsymbol{\psi}, \boldsymbol{\delta}; q) = 0 \quad \text{for all } q \in L^2 . \tag{3.4}$$

Lemma 3.1. *The solutions of* (2.11)–(2.15) *satisfy* $(\boldsymbol{\Pi}_h^* \boldsymbol{\phi}, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\alpha}) \in Z_h$. *The solutions of* (2.20)–(2.24) *satisfy* $(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h) \in Z_h$.

Proof. The second statement is immediate from (2.22). To verify the first, we use (2.8), (2.13), and (2.6) to obtain

$$\operatorname{rot}_h \boldsymbol{\Pi}_h^* \boldsymbol{\phi} = \Pi_h^0 \operatorname{rot} \boldsymbol{\phi} = -\lambda^{-1} t^2 \Pi_h^0 \operatorname{rot} \boldsymbol{\alpha} = -\lambda^{-1} t^2 \operatorname{rot} \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\alpha}. \qquad \square$$

Lemma 3.2. *There exists* $\gamma > 0$ *independent of* h *and* t *such that*

$$A_h(\boldsymbol{\psi}, \boldsymbol{\delta}; \boldsymbol{\psi}, \boldsymbol{\delta}) \geq \gamma \Big[\min(1, h^2/t^2) \|\boldsymbol{\psi}\|_{1,h}^2 + \|\mathcal{E}_h \boldsymbol{\psi}\|_0^2 + t^2 \|\boldsymbol{\delta}\|_0^2 + h^2 t^2 \|\operatorname{rot} \boldsymbol{\delta}\|_0^2 \Big]$$
$$\text{for all } (\boldsymbol{\psi}, \boldsymbol{\delta}) \in Z_h .$$

Proof. First we recall the discrete Poincaré inequality (see, e.g., Ref. 3),

$$\|\boldsymbol{\psi}\|_0 \leq C \|\mathbf{grad}_h \boldsymbol{\psi}\|_0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1 ,$$

which, together with the differential identity

$$\mathbf{grad}_h \boldsymbol{\psi} = \mathcal{E}_h \boldsymbol{\psi} + \frac{1}{2} \operatorname{rot}_h \boldsymbol{\psi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

implies that

$$\|\boldsymbol{\psi}\|_{1,h} \leq C(\|\mathcal{E}_h \boldsymbol{\psi}\|_0 + \|\operatorname{rot}_h \boldsymbol{\psi}\|_0) \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1 . \tag{3.5}$$

Using an inverse inequality and then (3.3), we get that

$$\begin{aligned} A_h(\boldsymbol{\psi}, \boldsymbol{\delta}; \boldsymbol{\psi}, \boldsymbol{\delta}) &\geq \gamma_1 (\|\mathcal{E}_h \boldsymbol{\psi}\|_0^2 + t^2 \|\boldsymbol{\delta}\|_0^2) \\ &\geq \gamma_2 (\|\mathcal{E}_h \boldsymbol{\psi}\|_0^2 + t^2 \|\boldsymbol{\delta}\|_0^2 + h^2 t^2 \|\operatorname{rot} \boldsymbol{\delta}\|_0^2) \\ &\geq \gamma_3 (\|\mathcal{E}_h \boldsymbol{\psi}\|_0^2 + t^2 \|\boldsymbol{\delta}\|_0^2 + h^2 t^2 \|\operatorname{rot} \boldsymbol{\delta}\|_0^2 \| + h^2/t^2 \|\operatorname{rot}_h \boldsymbol{\psi}\|_0^2) , \end{aligned}$$

where the γ_i are positive constants independent of h and t. The desired estimate then follows from (3.5). $\square$

Our next lemma is a standard bound on the consistency error due to the nonconformity of $\mathring{\boldsymbol{M}}_*^1$ as an approximation of $\mathring{\boldsymbol{H}}^1$. For a proof see, e.g., Ref. 3.

Lemma 3.3. *There exists a constant* C *independent of* h *such that*

$$\sum_T \int_{\partial T} [(\mathcal{C}\mathcal{E}\boldsymbol{\phi})\boldsymbol{n} + p\boldsymbol{s}] \cdot \boldsymbol{\psi}| \leq Ch(\|\boldsymbol{\phi}\|_2 + \|p\|_1) \|\boldsymbol{\psi}\|_{1,h}$$

for all $\boldsymbol{\phi} \in \boldsymbol{H}^2$, $p \in H^1$, *and* $\boldsymbol{\psi} \in \mathring{\boldsymbol{M}}_*^1$.

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Next we bound the consistency error attributable to the reduction operator and the approximate forcing function in (2.21).

Lemma 3.4. *There exists a constant C independent of h such that if for some $g \in L^2$, $r \in \mathring{H}^1$ satisfies* (2.11), *$r_h \in \mathring{M}_0^1$ satisfies* (2.20), *and $\psi \in \mathring{\boldsymbol{M}}_*^1$, then*

$$|(\mathbf{grad}\, r, \psi) - (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \psi)| \le Ch\|r\|_2 \|\psi\|_{1,h}\,.$$

Proof. Clearly

$$\begin{aligned}|(\mathbf{grad}\, r, \psi) - (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \psi)| &= |(\mathbf{grad}[r - r_h], \psi) + (\mathbf{grad}\, r_h, \psi - \boldsymbol{\Pi}_h^\Gamma \psi)| \\ &\le \|r - r_h\|_1 \|\psi\|_0 + \|r_h\|_1 \|\psi - \boldsymbol{\Pi}_h^\Gamma \psi\|_0\,.\end{aligned}$$

In view of (2.5) and the obvious estimates

$$\|r_h\|_1 \le C\|r\|_1\,, \quad \|r - r_h\|_1 \le Ch\|r\|_2\,,$$

the lemma follows. □

We are now ready to prove the basic energy estimate for the method.

Theorem 3.5. *There exists a constant C independent of h and t such that*

$$\begin{aligned}\|\phi - \phi_h\|_{1,h} + t^2 \|\operatorname{rot}(\alpha - \alpha_h)\|_0 &\le Ch \max(1, t^2/h^2)\|g\|_0\,, \\ \|\mathcal{E}_h(\phi - \phi_h)\|_0 + t\|\alpha - \alpha_h\|_0 &\le Ch \max(1, t/h)\|g\|_0\,.\end{aligned}$$

Proof. Choosing $(\psi, \delta) = (\boldsymbol{\Pi}_h^* \phi - \phi_h, \boldsymbol{\Pi}_h^\Gamma \alpha - \alpha_h)$ in (3.2) and using (3.4) and Lemma 3.1 we get that

$$\begin{aligned}&A_h(\boldsymbol{\Pi}_h^* \phi - \phi_h, \boldsymbol{\Pi}_h^\Gamma \alpha - \alpha_h; \boldsymbol{\Pi}_h^* \phi - \phi_h, \boldsymbol{\Pi}_h^\Gamma \alpha - \alpha_h) \\ &\quad = A_h(\boldsymbol{\Pi}_h^* \phi - \phi, \boldsymbol{\Pi}_h^\Gamma \alpha - \alpha; \boldsymbol{\Pi}_h^* \phi - \phi_h, \boldsymbol{\Pi}_h^\Gamma \alpha - \alpha_h) \\ &\qquad - (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma[\boldsymbol{\Pi}_h^* \phi - \phi_h]) + (\mathbf{grad}\, r, \boldsymbol{\Pi}_h^* \phi - \phi_h) \\ &\qquad + \sum_T \int_{\partial T} [(\mathcal{C}\mathcal{E}\phi)\boldsymbol{n} + ps] \cdot (\boldsymbol{\Pi}_h^* \phi - \phi_h)\,. \qquad (3.6)\end{aligned}$$

Applying Lemma 3.3, Lemma 3.4, Schwarz's inequality, and the approximation error bounds (2.2)–(2.4), the terms on the right-hand side may be bounded above by

$$\begin{aligned}&Ch(\|\phi\|_2 + \|p\|_1 + \|r\|_2 + t\|\alpha\|_1)(\|\boldsymbol{\Pi}_h^* \phi - \phi_h\|_{1,h} + t\|\boldsymbol{\Pi}_h^\Gamma \alpha - \alpha_h\|_0) \\ &\quad \le Ch\|g\|_0(\|\boldsymbol{\Pi}_h^* \phi - \phi_h\|_{1,h} + t\|\boldsymbol{\Pi}_h^\Gamma \alpha - \alpha_h\|_0)\,,\end{aligned}$$

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where C is independent of h and t and we have invoked the regularity estimates (2.18)–(2.19). On the other hand, Lemma 3.2 furnishes a lower bound for the left-hand side of (3.6):

$$\gamma\Big[\min(1,h^2/t^2)\|\mathbf{\Pi}_h^*\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,h}^2+\|\mathcal{E}_h(\mathbf{\Pi}_h^*\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0^2+t^2\|\mathbf{\Pi}_h^\Gamma\boldsymbol{\alpha}-\boldsymbol{\alpha}_h\|_0^2 \\ +h^2t^2\|\operatorname{rot}(\mathbf{\Pi}_h^\Gamma\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)\|_0^2\Big].$$

Combining these two bounds and performing some simple manipulations we get

$$\|\mathbf{\Pi}_h^*\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,h}+t^2\|\operatorname{rot}(\mathbf{\Pi}_h^\Gamma\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)\|_0\le Ch\max(1,t^2/h^2)\|g\|_0\,,$$
$$\|\mathcal{E}_h(\mathbf{\Pi}_h^*\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0+t\|\mathbf{\Pi}_h^\Gamma\boldsymbol{\alpha}-\boldsymbol{\alpha}_h\|_0\le Ch\max(1,t/h)\|g\|_0\,.$$

Now,

$$\|\boldsymbol{\phi}-\mathbf{\Pi}_h^*\boldsymbol{\phi}\|_{1,h}+t\|\boldsymbol{\alpha}-\mathbf{\Pi}_h^\Gamma\boldsymbol{\alpha}\|_0+t^2\|\operatorname{rot}(\boldsymbol{\alpha}-\mathbf{\Pi}_h^\Gamma\boldsymbol{\alpha})\|_0\le Ch\|g\|_0\,,$$

as follows from the (2.2)–(2.4), (2.6) and (2.18). The theorem then follows using the triangle inequality. □

We now consider L^2 estimates. As usual, we first define an appropriate dual problem. For $\boldsymbol{d}\in\boldsymbol{L}^2$, let $(\boldsymbol{\phi}^d,p^d,\boldsymbol{\alpha}^d)\in\mathring{\boldsymbol{H}}^1\times\hat{L}^2\times\mathring{\boldsymbol{H}}(\operatorname{rot})$ satisfy

$$(\mathcal{C}\mathcal{E}\boldsymbol{\phi}^d,\mathcal{E}\boldsymbol{\psi})-(p^d,\operatorname{rot}\boldsymbol{\psi})=(\boldsymbol{d},\boldsymbol{\psi})\quad\text{for all }\boldsymbol{\psi}\in\mathring{\boldsymbol{H}}^1\,,\tag{3.7}$$
$$-(\operatorname{rot}\boldsymbol{\phi}^d,q)-t^2(\operatorname{rot}\boldsymbol{\alpha}^d,q)=0\quad\text{for all }q\in\hat{L}^2\,,\tag{3.8}$$
$$(\boldsymbol{\alpha}^d,\boldsymbol{\delta})-(p^d,\operatorname{rot}\boldsymbol{\delta})=0\quad\text{for all }\boldsymbol{\delta}\in\mathring{\boldsymbol{H}}(\operatorname{rot})\,,\tag{3.9}$$

and let $\omega^d\in\mathring{H}^1$ be determined by

$$(\mathbf{grad}\,\omega^d,\mathbf{grad}\,s)=(\boldsymbol{\phi}^d,\mathbf{grad}\,s)\quad\text{for all }s\in\mathring{H}^1\,.\tag{3.10}$$

Then, from Theorem 7.1 of Ref. 3, we have the following regularity result.

$$\|\omega^d\|_2+\|\boldsymbol{\phi}^d\|_2+\|p^d\|_1+t\|p^d\|_2+\|\boldsymbol{\alpha}^d\|_0+t\|\operatorname{rot}\boldsymbol{\alpha}^d\|_0\le C\|\mathbf{d}\|_0\,.\tag{3.11}$$

Before using this dual problem to obtain L^2 estimates, it will be convenient to first establish two approximation results. The first deals with an approximation property of $\mathbf{\Pi}_h^\Gamma$, when applied to functions of a special form and the second bounds the consistency error attributable to the reduction operator and the approximate forcing function in a manner different from Lemma 3.4.

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Lemma 3.6. *Assume that* $\boldsymbol{\psi} \in \mathring{\boldsymbol{H}}^1$, $\mu \in \mathring{H}^1 \cap H^2$, $z \in H^2$ *and*

$$\lambda t^{-2}(\mathbf{grad}\,\mu - \boldsymbol{\psi}) = \mathbf{grad}\,s + \mathbf{curl}\,q$$

for some $s \in \mathring{H}^1 \cap H^2$ *and* $q \in \hat{H}^2$. *Then*

$$|(\mathbf{grad}\,z, \boldsymbol{\psi} - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\psi})| \le Ch\max(h,t)\|z\|_2\Big[\|\mu\|_2 + t(\|s\|_2 + \|q\|_2)\Big]\,.$$

Proof. First observe that for each edge e joining vertices a and b, $\boldsymbol{\Pi}_h^\Gamma\,\mathbf{grad}\,\mu$ is an element of $\boldsymbol{\Gamma}_h$ satisfying

$$\int_e (\boldsymbol{\Pi}_h^\Gamma\,\mathbf{grad}\,\mu)\cdot \boldsymbol{s} = \int_e \mathbf{grad}\,\mu\cdot\boldsymbol{s} = \int_e d\mu/ds = \mu(b) - \mu(a)$$
$$= \Pi_h^1\mu(b) - \Pi_h^1\mu(a) = \int_e d(\Pi_h^1\mu)/ds = \int_e \mathbf{grad}\,\Pi_h^1\mu\cdot\mathbf{s}\,.$$

Since $\mathbf{grad}\,\Pi_h^1\mu \in \boldsymbol{\Gamma}_h$ and from the above, has the same values at the degrees of freedom as $\boldsymbol{\Pi}_h^\Gamma\,\mathbf{grad}\,\mu$, we get

$$\boldsymbol{\Pi}_h^\Gamma\,\mathbf{grad}\,\mu = \mathbf{grad}\,\Pi_h^1\mu\,. \tag{3.12}$$

Using this fact, we have

$$\begin{aligned}(\mathbf{grad}\,z, \boldsymbol{\psi} - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\psi}) &= (\mathbf{grad}\,z, \mathbf{grad}\,\mu - \boldsymbol{\Pi}_h^\Gamma[\mathbf{grad}\,\mu])\\ &\quad - \lambda^{-1}t^2(\mathbf{grad}\,z, [\boldsymbol{I} - \boldsymbol{\Pi}_h^\Gamma][\mathbf{grad}\,s + \mathbf{curl}\,q])\\ &= -(\Delta\,z, \mu - \Pi_h^1\mu) - \lambda^{-1}t^2(\mathbf{grad}\,z, [\boldsymbol{I} - \boldsymbol{\Pi}_h^\Gamma][\mathbf{grad}\,s + \mathbf{curl}\,q])\\ &\le \|z\|_2\|\mu - \Pi_h^1\mu\|_0 + Ct^2\|z\|_1\|(\boldsymbol{I} - \boldsymbol{\Pi}_h^\Gamma)(\mathbf{grad}\,s + \mathbf{curl}\,q)\|_0\\ &\le C[h^2\|\mu\|_2\|z\|_2 + t^2h\|z\|_1(\|s\|_2 + \|q\|_2)]\\ &\le Ch\max(h,t)\|z\|_2\Big[\|\mu\|_2 + t(\|s\|_2 + \|q\|_2)\Big]\,. \qquad \square\end{aligned}$$

Lemma 3.7. *Assume that for some* $g \in L^2$, $r \in \mathring{H}^1$ *satisfies* (2.11) *and* $r_h \in \mathring{M}_0^1$ *satisfies* (2.20) *and that* $(\boldsymbol{\phi}^d, p^d, \boldsymbol{\alpha}^d, \omega^d)$ *satisfies the above dual problem. Then there exists a constant* C *independent of* h *such that*

$$|(\mathbf{grad}\,r, \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d) - (\mathbf{grad}\,r_h, \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d)| \le Ch\max(h,t)\|g\|_0\|\boldsymbol{d}\|_0\,.$$

Proof. We first observe that $\boldsymbol{\Pi}_h^\Gamma\boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d = \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\phi}^d$. Hence,

$$\begin{aligned}&(\mathbf{grad}\,r, \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d) - (\mathbf{grad}\,r_h, \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d)\\ &\quad = (\mathbf{grad}[r - r_h], \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d) + (\mathbf{grad}\,r_h, \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d)\\ &\quad = (\mathbf{grad}[r - r_h], \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d - \boldsymbol{\phi}^d) - (r - r_h, \operatorname{div}\boldsymbol{\phi}^d)\\ &\qquad + (\mathbf{grad}\,r_h, \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d)\\ &\quad = (\mathbf{grad}[r - r_h], \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d - \boldsymbol{\phi}^d) - (r - r_h, \operatorname{div}\boldsymbol{\phi}^d) + (\mathbf{grad}\,r_h, \boldsymbol{\Pi}_h^*\boldsymbol{\phi}^d - \boldsymbol{\phi}^d)\\ &\qquad + (\mathbf{grad}[r_h - r], \boldsymbol{\phi}^d - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\phi}^d) + (\mathbf{grad}\,r, \boldsymbol{\phi}^d - \boldsymbol{\Pi}_h^\Gamma\boldsymbol{\phi}^d)\,.\end{aligned}$$

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Applying standard estimates, we obtain

$$\begin{aligned}|(\mathbf{grad}\, r, \mathbf{\Pi}_h^* \boldsymbol{\phi}^d) - (\mathbf{grad}\, r_h, \mathbf{\Pi}_h^\Gamma \mathbf{\Pi}_h^* \boldsymbol{\phi}^d)| &\le Ch^2 \|r\|_2 \|\boldsymbol{\phi}^d\|_2 + |(\mathbf{grad}\, r, \boldsymbol{\phi}^d - \mathbf{\Pi}_h^\Gamma \boldsymbol{\phi}^d)| \\ &\le Ch^2 \|g\|_0 \|\boldsymbol{d}\|_0 + |(\mathbf{grad}\, r, \boldsymbol{\phi}^d - \mathbf{\Pi}_h^\Gamma \boldsymbol{\phi}^d)| .\end{aligned}$$

To estimate the last term, we observe that from (3.7)–(3.10), it follows that $\boldsymbol{\phi}^d$ has the Helmholtz decomposition

$$\mathbf{grad}\, \omega^d - \boldsymbol{\phi}^d = \lambda^{-1} t^2 \,\mathbf{curl}\, p^d .$$

Applying Lemma 3.6 with $\boldsymbol{\psi} = \boldsymbol{\phi}^d$, $\mu = \omega^d$, $z = r$, $s = 0$, $q = p^d$ and the *a priori* estimates (2.19) and (3.11), we get

$$|(\mathbf{grad}\, r, \boldsymbol{\phi}^d - \mathbf{\Pi}_h^\Gamma \boldsymbol{\phi}^d)| \le Ch \max(h,t) \|r\|_2 [\|\omega^d\|_2 + t\|p^d\|_2] \le Ch \max(h,t) \|g\|_0 \|\boldsymbol{d}\|_0 .$$

Combining these results establishes the lemma. □

Using this result, we now prove the first of our L^2 estimates.

Theorem 3.8.

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \le C \max(h^2, t^2) \|g\|_0 .$$

Proof. Integrating the strong form of (3.7) by parts, we obtain

$$\begin{aligned}(\boldsymbol{d}, \boldsymbol{\phi} - \boldsymbol{\phi}_h) = {} & (\mathcal{C}\mathcal{E}\boldsymbol{\phi}^d, \mathcal{E}_h[\boldsymbol{\phi} - \boldsymbol{\phi}_h]) \\ & - (p^d, \mathrm{rot}_h[\boldsymbol{\phi} - \boldsymbol{\phi}_h]) - \sum_T \int_{\partial T} \Big[\mathcal{C}\mathcal{E}\boldsymbol{\phi}^d \boldsymbol{n} + p^d \boldsymbol{s}\Big] \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) .\end{aligned}$$

Now observe that using (2.13), (2.22), and (3.9), we get

$$(p^d, \mathrm{rot}_h[\boldsymbol{\phi} - \boldsymbol{\phi}_h]) = -\lambda^{-1} t^2 (p^d, \mathrm{rot}[\boldsymbol{\alpha} - \boldsymbol{\alpha}_h]) = -\lambda^{-1} t^2 (\boldsymbol{\alpha}^d, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h)$$

so that we may rewrite the equation above as

$$(\boldsymbol{d}, \boldsymbol{\phi} - \boldsymbol{\phi}_h) = A_h(\boldsymbol{\phi}^d, \boldsymbol{\alpha}^d; \boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h) - \sum_T \int_{\partial T} [\mathcal{C}\mathcal{E}\boldsymbol{\phi}^d \boldsymbol{n} + p^d \boldsymbol{s}] \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) .$$

By Lemma 3.1, $(\mathbf{\Pi}_h^* \boldsymbol{\phi}^d, \mathbf{\Pi}_h^\Gamma \boldsymbol{\alpha}^d) \in Z_h$ and hence from (3.4), $B_h(\mathbf{\Pi}_h^* \boldsymbol{\phi}^d, \mathbf{\Pi}_h^\Gamma \boldsymbol{\alpha}^d; q) = 0$ for all $q \in L^2(\Omega)$. From (3.1), we then obtain

$$\begin{aligned}& A_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h; \mathbf{\Pi}_h^* \boldsymbol{\phi}^d, \mathbf{\Pi}_h^\Gamma \boldsymbol{\alpha}^d) \\ & \quad = -(\mathbf{grad}\, r_h, \mathbf{\Pi}_h^\Gamma \mathbf{\Pi}_h^* \boldsymbol{\phi}^d) + (\mathbf{grad}\, r, \mathbf{\Pi}_h^* \boldsymbol{\phi}^d) + \sum_T \int_{\partial T} [(\mathcal{C}\mathcal{E}\boldsymbol{\phi})\boldsymbol{n} + p\boldsymbol{s}] \cdot \mathbf{\Pi}_h^* \boldsymbol{\phi}^d .\end{aligned}$$

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Combining these results, we obtain

$$
\begin{aligned}
(\boldsymbol{d}, \boldsymbol{\phi} - \boldsymbol{\phi}_h) = {} & A_h(\boldsymbol{\phi}^d - \boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d, \boldsymbol{\alpha}^d - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\alpha}^d; \boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h) \\
& + \sum_T \int_{\partial T} (\mathcal{C}\mathcal{E}\boldsymbol{\phi}\boldsymbol{n} + p\boldsymbol{s}) \cdot (\boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d - \boldsymbol{\phi}^d) \\
& - \sum_T \int_{\partial T} (\mathcal{C}\mathcal{E}\boldsymbol{\phi}^d \boldsymbol{n} + p^d \boldsymbol{s}) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}_h) \\
& + (\mathbf{grad}\, r, \boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d) - (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d) .
\end{aligned}
$$

Setting $\boldsymbol{d} = \boldsymbol{\phi} - \boldsymbol{\phi}_h$, and applying standard results for nonconforming methods, Theorem 3.5, and Lemma 3.7, we obtain:

$$
\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0^2 \le C \, \big[& \|\mathcal{E}_h(\boldsymbol{\phi}^d - \boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d)\|_0 \|\mathcal{E}_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0 + t^2 \|\boldsymbol{\alpha}^d - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\alpha}^d\|_0 \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0 \\
& + h(\|\boldsymbol{\phi}\|_2 + \|p\|_1) \|\mathbf{grad}_h(\boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d - \boldsymbol{\phi}^d)\|_0 \\
& + h(\|\boldsymbol{\phi}^d\|_2 + \|p^d\|_1) \|\mathbf{grad}_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0 \\
& + |(\mathbf{grad}\, r, \boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d) - (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\Pi}_h^* \boldsymbol{\phi}^d)| \big] \\
\le C \, \big[& h \max(h, t)(\|\boldsymbol{\phi}^d\|_2 + t\|\boldsymbol{\alpha}^d\|_1)\|g\|_0 + h^2(\|\boldsymbol{\phi}\|_2 + \|p\|_1)\|\boldsymbol{\phi}^d\|_2 \\
& + \max(h^2, t^2)(\|\boldsymbol{\phi}^d\|_2 + \|p^d\|_1)\|g\|_0 + h \max(h, t)\|g\|_0 \|\boldsymbol{d}\|_0 \big] .
\end{aligned}
$$

Applying (2.18) and (3.11) completes the proof. □

From these results, we easily obtain the following error estimate for the approximation of ω.

Theorem 3.9.

$$\|\omega - \omega_h\|_1 \le C[h + \max(h^2, t^2)]\|g\|_0 .$$

Proof. Using (2.15) and (2.24), then (2.11) and (2.20), we get that

$$
\begin{aligned}
(\mathbf{grad}[\Pi_h^1 \omega - \omega_h], \mathbf{grad}[\Pi_h^1 \omega - \omega_h]) &= (\mathbf{grad}[\Pi_h^1 \omega - \omega], \mathbf{grad}[\Pi_h^1 \omega - \omega_h]) \\
&\quad + (\boldsymbol{\phi} - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h + \lambda^{-1} t^2 \, \mathbf{grad}[r - r_h], \mathbf{grad}[\Pi_h^1 \omega - \omega_h]). \\
&= (\mathbf{grad}[\Pi_h^1 \omega - \omega], \mathbf{grad}[\Pi_h^1 \omega - \omega_h]) \\
&\quad + (\boldsymbol{\phi} - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi} + [\boldsymbol{\phi} - \boldsymbol{\phi}_h] + [\boldsymbol{\Pi}_h^\Gamma - \boldsymbol{I}][\boldsymbol{\phi} - \boldsymbol{\phi}_h], \mathbf{grad}[\Pi_h^1 \omega - \omega_h]) .
\end{aligned}
$$

It easily follows by standard estimates that

$$
\begin{aligned}
\|\omega - \omega_h\|_1 &\le C(\|\omega - \Pi_h^1 \omega\|_1 + \|\boldsymbol{\phi} - \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + h\|\mathbf{grad}_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0) \\
&\le C(h\|\omega\|_2 + h\|\boldsymbol{\phi}\|_1 + \max(t^2, h^2)\|g\|_0) \le C[h + \max(t^2, h^2)]\|g\|_0 . \quad \square
\end{aligned}
$$

Finally, we apply another standard duality argument to obtain an L^2 error estimate for $\omega - \omega_h$.

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Theorem 3.10.

$$\|\omega - \omega_h\|_0 \le C \max(h^2, t^2)\|g\|_0 .$$

Proof. Let $z \in \overset{\circ}{H}{}^1$ and $z_h \in \overset{\circ}{M}{}^1_0$ be the respective solutions of

$$(\mathbf{grad}\, z, \mathbf{grad}\, s) = (\omega - \omega_h, s) \quad \text{for all } s \in \overset{\circ}{H}{}^1 ,$$

$$(\mathbf{grad}\, z_h, \mathbf{grad}\, s) = (\omega - \omega_h, s) \quad \text{for all } s \in \overset{\circ}{M}{}^1_0 .$$

Then, using (2.11) and (2.20), we get

$$\begin{aligned}
(\omega - \omega_h, \omega - \omega_h) &= (\mathbf{grad}[\omega - \omega_h], \mathbf{grad}\, z) \\
&= (\mathbf{grad}[\omega - \omega_h], \mathbf{grad}[z - z_h])(\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}_h + \lambda^{-1} t^2\, \mathbf{grad}[r - r_h], \mathbf{grad}\, z_h) \\
&= (\mathbf{grad}[\omega - \omega_h], \mathbf{grad}[z - z_h]) + (\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}_h, \mathbf{grad}\, z_h) \\
&= (\mathbf{grad}[\omega - \omega_h], \mathbf{grad}[z - z_h]) + (\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}, \mathbf{grad}[z_h - z]) \\
&\quad + (\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}, \mathbf{grad}\, z) + ([\boldsymbol{\Pi}^\Gamma_h - \boldsymbol{I}][\boldsymbol{\phi} - \boldsymbol{\phi}_h], \mathbf{grad}\, z_h) + (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \mathbf{grad}\, z_h) .
\end{aligned}$$

Hence, applying Theorems 3.5–3.9, Lemma 3.6, (2.4), (2.18) and (2.19), we obtain

$$\begin{aligned}
&\|\omega - \omega_h\|_0^2 \\
&\quad \le (\|\omega - \omega_h\|_1 + \|\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}\|_0)\|z - z_h\|_1 + \|[\boldsymbol{\Pi}^\Gamma_h - I][\boldsymbol{\phi} - \boldsymbol{\phi}_h]\|_0 \|\mathbf{grad}\, z_h\|_0 \\
&\qquad + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \|\mathbf{grad}\, z_h\|_0 + |(\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}, \mathbf{grad}\, z)| \\
&\quad \le C[h\|\omega - \omega_h\|_1 + h\|\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}\|_0 + h\|\mathbf{grad}_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0]\|z\|_2 \\
&\qquad + |(\boldsymbol{\phi} - \boldsymbol{\Pi}^\Gamma_h \boldsymbol{\phi}, \mathbf{grad}\, z)| \\
&\quad \le C \max(h^2, t^2)\|g\|_0 \|z\|_2 .
\end{aligned}$$

Using the fact that $\|z\|_2 \le C\|\omega - \omega_h\|_0$, we obtain the result of the lemma. □

4. Connection to the Morley Method for the Biharmonic Equation

The method analyzed in this paper has another interesting property. For fixed h, the method approaches a modified Morley method for the biharmonic as t tends to zero. More precisely, we shall establish the following result. Let $\overset{\circ}{M}{}^2_*$ denote the subspace of Morley elements, consisting of piecewise quadratics which are continuous at element vertices and vanish at boundary vertices, and whose normal derivatives are continuous at midpoints of edges and vanish at midpoints of boundary edges.

Theorem 4.1. *Assume that* $(\boldsymbol{\phi}_h, \omega_h) \in \overset{\circ}{\boldsymbol{M}}{}^1_* \times \overset{\circ}{M}{}^1_0$ *is the solution of* (2.10). *Then*

$$\lim_{t \to 0} \boldsymbol{\phi}_h = \mathbf{grad}_h\, z_h , \quad \lim_{t \to 0} \mathbf{grad}\, \omega_h = \boldsymbol{\Pi}^\Gamma_h\, \mathbf{grad}_h\, z_h ,$$

where $z_h \in \mathring{M}^2_*$ *satisfies the modified Morley equations*

$$(\mathcal{C}\mathcal{E}_h \,\mathbf{grad}_h\, z_h, \mathcal{E}_h \,\mathbf{grad}_h\, v_h) = (g, \Pi^1_h v_h) \quad \text{for all } v_h \in \mathring{M}^2_* . \tag{4.1}$$

We note that a connection to the Morley method is observed in Ref. 15 and that the modified Morley method given above has been previously discussed by Arnold and Brezzi in Ref. 2 where it is shown to be essentially equivalent to the mixed method of Hellan–Hermann–Johnson.

To establish this theorem, we will use the following relationship between the Morley space and the space of nonconforming piecewise linear finite elements. The proof is analogous to that of Theorem 4.1 of Ref. 12.

Lemma 4.2.

$$\mathbf{grad}_h\, \mathring{M}^2_* = \{\psi \in \mathring{\boldsymbol{M}}^1_* : \ \mathrm{rot}_h\, \psi = 0\} .$$

Proof of Theorem 4.1. Let $(\boldsymbol{\phi}^0_h, p^0_h, \boldsymbol{\alpha}^0_h, \omega^0_h)$ denote the solution of (2.21)–(2.24) with $t = 0$. We first show that

$$\lim_{t\to 0} \boldsymbol{\phi}_h = \boldsymbol{\phi}^0_h, \quad \lim_{t\to 0} \mathbf{grad}\, \omega_h = \mathbf{grad}\, \omega^0_h .$$

Subtracting the limit version of (2.21)–(2.24) from the original version, we obtain

$$(\mathcal{C}\mathcal{E}_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h], \mathcal{E}_h \boldsymbol{\psi}) - (p_h - p^0_h, \mathrm{rot}_h\, \boldsymbol{\psi}) = 0 \quad \text{for all } \boldsymbol{\psi} \in \mathring{\boldsymbol{M}}^1_* , \tag{4.2}$$

$$-(\mathrm{rot}_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h], q) - \lambda^{-1} t^2 (\mathrm{rot}[\boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h], q) = \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\alpha}^0_h, q) \text{ for all } q \in \hat{M}^0 , \tag{4.3}$$

$$(\boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h, \boldsymbol{\delta}) - (p_h - p^0_h, \mathrm{rot}\, \boldsymbol{\delta}) = 0 \quad \text{for all } \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h , \tag{4.4}$$

$$(\mathbf{grad}[\omega_h - \omega^0_h], \mathbf{grad}\, s) = (\boldsymbol{\Pi}^\Gamma_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h] + \lambda^{-1} t^2\, \mathbf{grad}\, r_h, \mathbf{grad}\, s) \text{ for all } s \in \mathring{M}^1_0 . \tag{4.5}$$

Choosing $\boldsymbol{\psi} = \boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h$, $q = p^0_h - p_h$, and $\boldsymbol{\delta} = \lambda^{-1} t^2 (\boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h)$ and adding the first three equations, we obtain

$$\begin{aligned}(\mathcal{C}\mathcal{E}_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h], \mathcal{E}_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h]) + \lambda^{-1} t^2 (\boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h, \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h) \\ = \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\alpha}^0_h, p^0_h - p_h) = \lambda^{-1} t^2 (\boldsymbol{\alpha}^0_h, \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h) .\end{aligned}$$

It follows easily that $\lim_{t\to 0} \mathcal{E}_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h] = 0$. Using (4.2) and the properties of the $\mathring{\boldsymbol{M}}^1_* - \hat{M}^0$ Stokes element, it is easy to show that $\lim_{t\to 0} p^0_h - p_h = 0$. It then follows from (4.4) that $\lim_{t\to 0} \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^0_h = 0$. From (4.3), we can then conclude that $\lim_{t\to 0} \mathrm{rot}_h[\boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h] = 0$ and hence that $\lim_{t\to 0} \boldsymbol{\phi}_h - \boldsymbol{\phi}^0_h = 0$. Finally, we conclude from (4.5) that $\lim_{t\to 0} \omega_h - \omega^0_h = 0$.

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To complete the proof, we show that $\boldsymbol{\phi}_h^0 = \mathbf{grad}_h z_h$ and $\mathbf{grad}\,\omega_h^0 = \boldsymbol{\Pi}_h^\Gamma \mathbf{grad}_h z_h$, where $z_h \in \overset{\circ}{M}{}_*^2$ satisfies the modified Morley equation (4.1). Now using (2.22) with $t = 0$, (2.7) and (2.9), we have for all piecewise constant q_h that

$$0 = (\mathrm{rot}_h\, \boldsymbol{\phi}_h^0, q_h) = (\mathrm{rot}\, \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h^0, q_h) = (\boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h^0, \mathbf{curl}_h\, q_h)\,.$$

Using the above, Lemma 2.1, and (2.24) with $t = 0$ it easily follows that $\mathbf{grad}\,\omega_h^0 = \boldsymbol{\Pi}_h^\Gamma \boldsymbol{\phi}_h^0$. Now since $\mathrm{rot}_h\, \boldsymbol{\phi}_h^0 = 0$, we may use Lemma 4.2 to write $\boldsymbol{\phi}_h^0 = \mathbf{grad}_h\, z_h$, where $z_h \in \overset{\circ}{M}{}_*^2$. Thus it only remains to show that z_h satisfies (4.1). Choosing $\boldsymbol{\psi}_h = \mathbf{grad}_h\, v_h$ for $v_h \in \overset{\circ}{M}{}_*^2$ and noting that $\boldsymbol{\psi}_h \in \overset{\circ}{\boldsymbol{M}}{}_*^1$ and $\mathrm{rot}_h\, \boldsymbol{\psi}_h = 0$, we get from (2.21) with $t = 0$, that

$$(\mathcal{C}\mathcal{E}_h \boldsymbol{\phi}_h^0, \mathcal{E}_h\, \mathbf{grad}_h\, v_h) = (\mathbf{grad}\, r_h, \boldsymbol{\Pi}_h^\Gamma\, \mathbf{grad}_h\, v_h)\,.$$

Now observing that (3.12) is also valid when $\mathbf{grad}\,\mu$ is replaced by $\mathbf{grad}_h\, v_h$ (the proof is unchanged), we have

$$\boldsymbol{\Pi}_h^\Gamma\, \mathbf{grad}_h\, v_h = \mathbf{grad}\, \Pi_h^1 v_h\,.$$

Then using (2.20), we get

$$(\mathcal{C}\mathcal{E}_h \boldsymbol{\phi}_h^0, \mathcal{E}_h\, \mathbf{grad}_h\, v_h) = (\mathbf{grad}\, r_h, \mathbf{grad}\, \Pi_h^1 v_h) = (g, \Pi_h^1 v_h)\,.$$

Hence, $z_h \in \overset{\circ}{M}{}_*^2$ satisfies (4.1). □

5. Numerical Results

The error estimates derived in the previous section show that when $t = O(h)$, the method gives optimal order error estimates for all the variables, independent of t. Thus, we do not have a locking problem in the usual sense. However, when h is small compared with t, the error estimates deteriorate and do not show the convergence of the method for fixed t as $h \to 0$. In this section, we present the results of some numerical computations to show that this failure of convergence is not a problem with the proof, but a problem with the method.

The example we consider is a circular plate which is clamped on its edge and loaded by $g = \cos\theta$ (we use polar coordinates r and θ to describe the problem and its solution, but compute in Cartesian coordinates). Exploiting symmetry we need only discretize one quarter of the domain, and thus the computational domain is the quarter of the unit circle contained in the first quadrant. Essential boundary conditions $\phi_1 = \phi_2 = \omega = 0$ are imposed on the curved portion of the boundary, while on the vertical segment of the boundary the only essential boundary conditions imposed are $\phi_2 = \omega = 0$ and on the horizontal segment of the boundary only $\phi_2 = 0$ is imposed. For the Young modulus, Poisson ratio, and shear correction factor we take $E = 1$, $\nu = 0.3$, and $k = 5/6$, and for the thickness we take the three values

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$t = 1$, 0.1 and 0.01. For each value of the thickness we compute on a sequence of six meshes with 4, 16, 64, 256, 1024 and 4048 elements. The fourth mesh, with 256 elements, is shown in Fig. 1.

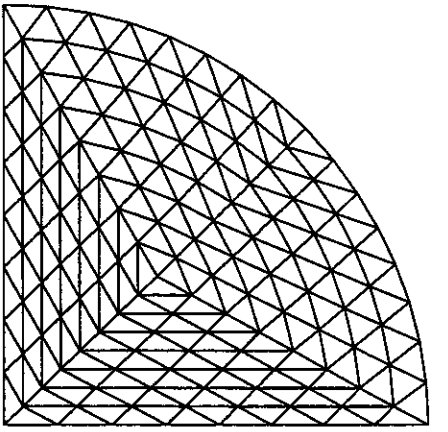

Fig. 1. An intermediate mesh with 256 elements.

The exact solution of this problem can be expressed in terms of a modified Bessel function,[4] and thus we can compute the exact error in the numerical solution. The solution has a boundary layer, but it is too weak to interfere with the convergence of linear elements (all three components are bounded in H^2 uniformly with respect to[5] t). The first graph in Fig. 1 is a log–log plot of the L^2-errors as a function of mesh size for the transverse displacement and the first component of rotation (the error for the second component is very similar). Note the apparent optimal second-order convergence of all three components when $t = 0.01$, but it is evidently divergence when $t = 1$. In the case $t = 0.1$ there is reasonable convergence for h not too small, but the convergence slows significantly when h decreases, especially for the rotation.

For sake of comparison, the performance of two other elements for the same problem are shown. The second plot in Fig. 2 depicts the performance of a straightforward discretization using conforming piecewise linear elements for both the rotation and the displacement. This method suffers from locking. Thus the apparent convergence is good for t large, but poor for t small (precisely the opposite as for the Oñate–Zarate–Flores method). The final plot in Fig. 2 depicts the performance of the Arnold–Falk element. This is a truly locking-free element, in the sense that optimal order convergence can be proven to hold uniformly for $t \in (0, 1]$. The robust performance of this element is clear in the plot.

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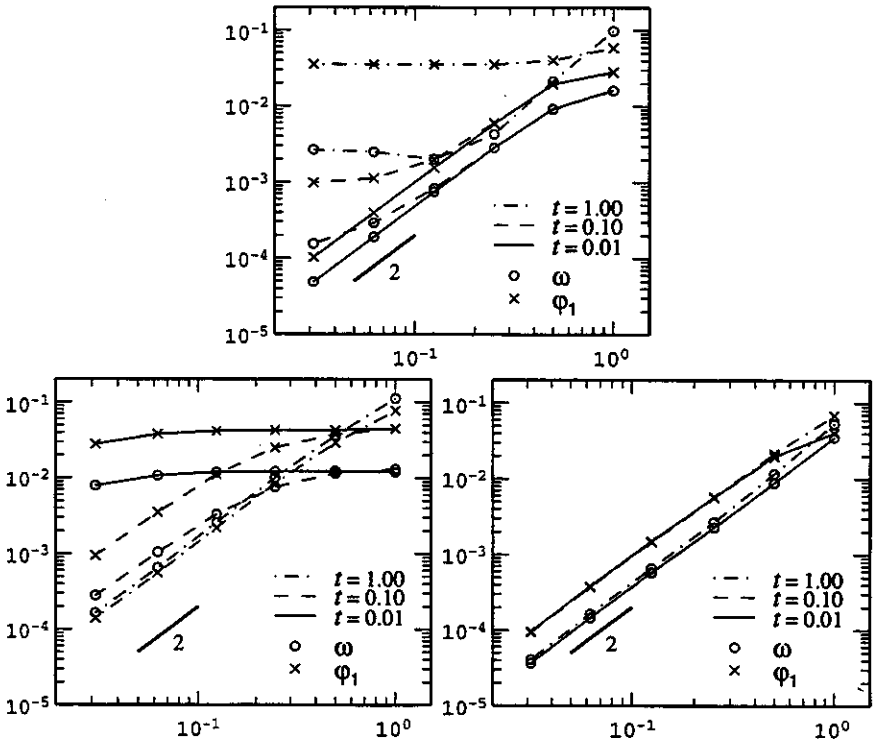

Fig. 2. Plots of L^2-errors verses mesh size for the Oñate–Zarate–Flores method, a standard method using conforming linear elements, and the Arnold–Falk method.

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