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PEERS: A New Mixed Finite Element for Plane Elasticity

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A mixed finite element procedure for plane elasticity is introduced and analyzed. The symmetry of the stress tensor is enforced through the introduction of a Lagrange multiplier. An additional Lagrange multiplier is introduced to simplify the linear algebraic system. Applications are made to incompressible elastic problems and to plasticity problems.

Key words: finite element methods, plane elasticity

1. Introduction

In this paper we develop a new mixed finite element method for plane elasticity problems. A mixed formulation of the elasticity problem characterizes the solution as a saddle point of a Lagrangian functional involving both displacements and stresses, in contrast to a displacement formulation in which the solution is characterized as a minimum of a Lagrangian functional of the displacements alone. It is well known that finite elements for discretization of a mixed formulation must be chosen carefully if accurate results are to be achieved, and an objection to such methods is that acceptable elements often involve many degrees of freedom. A second objection is that mixed methods often lead to the solution of indefinite linear algebraic systems, while displacement methods generally require the solution of a positive definite system. Nonetheless, there are important practical problems for which a mixed method appears preferable. One such is the modelling of nearly incompressible or incompressible materials, for which standard displacement methods furnish notoriously inaccurate results. A second is the modelling of plastic materials. Generally, the elimination of the stresses from the equilibrium and constitutive equations of a material exhibiting plastic behaviour is difficult; consequently, only a mixed formulation is feasible.

Our goal here is to develop a mixed finite element for plane elasticity that involves a relatively small number of degrees of freedom, is stable and accurate in a

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mathematically demonstrable sense, and is applicable to incompressible and plastic problems. The element we propose does not have any vertex degrees of freedom for the stress; as we show in section 5, it can be implemented in such a way as to lead to a positive definite matrix problem.

In developing the element, we had in mind finding an appropriate analogue of the lowest order triangular mixed finite element in the family of elements proposed by Raviart and Thomas for the Laplace equation [18]. This element approximates the solution to Laplace's equation by a piecewise constant function and its gradient field by a restricted piecewise linear field which is determined by the value of its normal component at the midpoint of each element edge. Thus the element involves only a small number of degrees of freedom and moreover it possesses a number of desirable properties which permit a simple and convincing mathematical analysis [8, 9, 11, 18]. A natural idea for the elasticity problem is to approximate the two components of the displacement vector with a pair of piecewise constant functions and the four components of the stress tensor with the Cartesian product of the Raviart-Thomas gradient space with itself. However, this choice of elements violates the symmetry of the stress tensor and cannot be used without modification. In the present element we start from such a Cartesian product element, but then we impose a weakened symmetry condition through the use of a Lagrange multiplier, which enters the system as a new variable which can be interpreted as the rotation of the displacement field. The requirement of mathematical stability leads us further to augment the space of approximate stresses. The resulting element, which we call PEERS (Plane Elasticity Element with Reduced Symmetry), is described in detail below. Note that we have attacked the problem of developing an element with a small number of degrees of freedom in a rather curious fashion, namely by introducing two additional variables to the five original variables of the elasticity system, a fourth independent stress component and the rotation. Nonetheless, PEERS is to our knowledge the smallest stable mixed elasticity element. In an appropriate implementation and after elimination of the degrees of freedom internal to a triangle PEERS has two degrees of freedom located at the midpoint of each element edge (namely the value of the displacement vector) and one located at each vertex (the value of the rotation).

Other mixed elements have been proposed for elasticity. In [17] Johnson and Mercier present two elements, one composite triangular and one composite rectangular, with rather similar applications in mind. Their element is of higher order than ours, but it involves significantly more degrees of freedom and moreover lacks some of the properties which PEERS shares with the Raviart-Thomas element and which allow a particularly simple and complete mathematical analysis (see the discussion of this point in [3]). Arnold, Douglas, and Gupta [3] have presented a family of mixed elasticity elements which do possess these properties, but their main goal was high order approximation, and so the elements have many more degrees of freedom.

The approximate imposition of symmetry through a Lagrange multiplier was suggested by Fraeijs de Veubeke [13]. A family of elements based on this idea, including an element of Fraeijs de Veubeke, has been presented by Amara and Thomas [1]. They consider dual hybrid equilibrium methods, so a comparison with our method is difficult. It appears that all the elements to which their analysis applies are more complex than PEERS. An essential difference between their approach and ours is that they consider approximation of the rotation by discontinuous piecewise polynomials while we approximate it by a continuous piecewise linear function.

The remainder of the paper is organized as follows. After a preliminary section we construct PEERS in section 3 and establish some basic properties which enable the asymptotic error analysis to be carried out in section 4. The main results of this analysis are a quasioptimal estimate in an appropriate norm for the triple consisting of the stress tensor, the displacement vector, and the rotation, first order L^2-estimates for the errors in the displacement and the stress, and second order H^{-1}-estimates for the same quantities. We also derive a simple L^∞ estimate for the displacement. In section 5 we discuss the implementation of the method and in particular sketch the application of the idea of [2] to give an improved implementation which, at the same time, gives a new approximation to the displacement field with higher order accuracy. Finally, in two short concluding paragraphs we discuss the application to problems of incompressible materials and to plasticity problems.

2. Notations and Preliminaries

For convenience we shall consider the elasticity problem to be posed on a convex polygonal domain Ω. For T a subdomain of Ω we denote by $H^s(T)$, $s=0, 1, 2, \cdots$, the usual Sobolev space with norm

$$\|\varphi\|_{s,T}=\left(\sum_{\alpha+\beta\leq s}\left\|\frac{\partial^{\alpha+\beta}\varphi}{\partial x^\alpha\partial y^\beta}\right\|^2_{L^2(T)}\right)^{1/2}.$$

For $T=\Omega$ we write simply H^s and $\|\cdot\|_s$. The subspace of H^1 consisting of functions vanishing on $\partial\Omega$ is denoted by $\mathring{H}^1$.

For any space X we denote by $\underset{\sim}{X}$ [respectively, $\underset{\approx}{X}$] the space of 2-vectors [2×2-tensors] with components in X. Note that we do not restrict to symmetric tensors. If X is normed, associated norms are defined by

$$\|\underset{\sim}{v}\|_{\underset{\sim}{X}}=\left(\sum_{i=1}^{2}\|v_i\|_X^2\right)^{1/2},\qquad \|\underset{\approx}{\tau}\|_{\underset{\approx}{X}}=\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\|\tau_{ij}\|_X^2\right)^{1/2}.$$

We use the same notation $\|\cdot\|_{s,T}$ (or $\|\cdot\|_s$) to denote the norms in $H^s(T)$, $\underset{\sim}{H}^s(T)$, and $\underset{\approx}{H}^s(T)$. We also use the wavy underline to distinguish between scalars, vectors, and tensors.

For functions η, $\underset{\sim}{v}$, and $\underset{\approx}{\tau}$ on Ω we define the differential operators

$$\underset{\sim}{\operatorname{rot}}\,\eta=\left(\frac{\partial\eta}{\partial y}, -\frac{\partial\eta}{\partial x}\right),$$

$$\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{v}=\begin{pmatrix}\dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y}\\[2mm] \dfrac{\partial v_2}{\partial x} & \dfrac{\partial v_2}{\partial y}\end{pmatrix}$$

$$\operatorname{rot}\underset{\sim}{v}=-\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x},$$

$$\underset{\approx}{\varepsilon}(\underset{\sim}{v})=\begin{pmatrix}\dfrac{\partial v_1}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x}+\dfrac{\partial v_1}{\partial y}\right)\\[2mm] \dfrac{1}{2}\left(\dfrac{\partial v_2}{\partial x}+\dfrac{\partial v_1}{\partial y}\right) & \dfrac{\partial v_2}{\partial y}\end{pmatrix}$$

$$\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}=\left(\frac{\partial \tau_{11}}{\partial x}+\frac{\partial \tau_{12}}{\partial y},\ \frac{\partial \tau_{21}}{\partial x}+\frac{\partial \tau_{22}}{\partial y}\right).$$

We also define two constant tensors

$$\underset{\approx}{\delta}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \underset{\approx}{\chi}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

and associate with any tensor $\underset{\approx}{\tau}$ its trace and asymmetry:

$$\operatorname{tr}(\underset{\approx}{\tau})=\underset{\approx}{\tau}:\underset{\approx}{\delta},\qquad \operatorname{as}(\underset{\approx}{\tau})=\underset{\approx}{\tau}:\underset{\approx}{\chi},$$

where the colon indicates the scalar product of tensors given by

$$\underset{\approx}{\tau}:\underset{\approx}{\sigma}=\sum_{i=1}^{2}\sum_{j=1}^{2}\tau_{ij}\sigma_{ij}.$$

We now formulate the elasticity problem. For simplicity of exposition we restrict ourselves to a homogeneous isotropic body in a state of plane strain fixed at the boundary. The classical theory of linear elasticity then requires that

(2.1a) $\qquad \underset{\approx}{\sigma}=2\mu\underset{\approx}{\varepsilon}(\underset{\sim}{u})+\lambda\operatorname{tr}(\underset{\approx}{\varepsilon}(\underset{\sim}{u}))\underset{\approx}{\delta}\qquad$ on Ω,

(2.1b) $\qquad \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}=\underset{\sim}{f}\qquad$ on Ω,

(2.1c) $\qquad \underset{\sim}{u}=0\qquad$ on $\partial\Omega$.

Here $\underset{\sim}{u}$ and $\underset{\approx}{\sigma}$ denote the desired displacements and stresses, $\underset{\sim}{f}$ the imposed load, and μ and λ the positive Lamé constants. Inverting the stress-strain law (2.1a), setting $\gamma=(\operatorname{rot}\underset{\sim}{u})/2$, and noting that

$$\underset{\approx}{\varepsilon}(\underset{\sim}{u})=\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{u}-\gamma\underset{\approx}{\chi},$$

we see that

(2.2a) $\qquad \dfrac{1}{2\mu}\underset{\approx}{\sigma}-\dfrac{\lambda}{4\mu(\mu+\lambda)}\operatorname{tr}(\underset{\approx}{\sigma})\underset{\approx}{\delta}-\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{u}+\gamma\underset{\approx}{\chi}=0\qquad$ on Ω.

We supplement this equation with the equilibrium condition (2.1b), the condition of symmetry of $\underset{\approx}{\sigma}$ [implied by (2.1a)], and the fixed boundary condition (2.1c):

$$\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}=\underset{\sim}{f} \quad \text{on} \quad \Omega, \tag{2.2b}$$

$$\operatorname{as}(\underset{\approx}{\sigma})=0 \quad \text{on} \quad \Omega, \tag{2.2c}$$

$$u=0 \quad \text{on} \quad \partial\Omega. \tag{2.2d}$$

The systems (2.1) and (2.2) are equivalent in the sense that the triple $(\underset{\approx}{\sigma}, \underset{\sim}{u}, \gamma)$ solves (2.2) if and only if $\gamma=(\operatorname{rot}\underset{\sim}{u})/2$ and the pair $(\underset{\approx}{\sigma}, \underset{\sim}{u})$ solves (2.1). The following theorem can be deduced from the method of analysis employed by C. Kenig in some recent, as yet unpublished, work on regularity for the solution of (2.1) on Lipschitz domains.

THEOREM 2.1. *Let $0<\mu_0<\mu_1$, $\mu\in[\mu_0, \mu_1]$, $\lambda\in[0, \infty)$, and $\underset{\sim}{f}\in \underset{\sim}{L}_2$. Then, there exists a unique triple $(\underset{\approx}{\sigma}, \underset{\sim}{u}, \gamma)\in \underset{\approx}{H}^1\times(\underset{\sim}{H}^2\cap \mathring{\underset{\sim}{H}}^1)\times H^1$ satisfying (2.2). Moreover, there exists a constant C depending only on Ω, μ_0 and μ_1 such that*

$$\|\underset{\approx}{\sigma}\|_1+\|\underset{\sim}{u}\|_2+\|\gamma\|_1\le C\|\underset{\sim}{f}\|_0\,.$$

Note that the constant C in the above theorem is independent of λ. The case of λ very large corresponds to a nearly incompressible material. For a proof of this aspect of the theorem see [3, 19].

To define a weak formulation of the problem, we introduce the Hilbert space

$$\underset{\approx}{H}=\{\underset{\approx}{\tau}\in \underset{\approx}{H}^0\,|\,\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\in \underset{\sim}{H}^0\}\,,$$
$$\|\underset{\approx}{\tau}\|_{\underset{\approx}{H}}=(\|\underset{\approx}{\tau}\|_0^2+\|\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\|_0^2)^{1/2}\,.$$

The weak formulation of (2.2) is to find a triple $(\underset{\approx}{\sigma}, \underset{\sim}{u}, \gamma)\in \underset{\approx}{H}\times \underset{\sim}{H}^0\times H^0$ such that (with $\underset{\sim}{u}\cdot\underset{\sim}{v}$ denoting the scalar product of the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$)

$$a(\underset{\approx}{\sigma}, \underset{\approx}{\tau})+\int \underset{\sim}{u}\cdot\operatorname{div}\underset{\approx}{\tau}\,dx+\int\gamma\,\operatorname{as}(\underset{\approx}{\tau})dx=0\,, \qquad \underset{\approx}{\tau}\in\underset{\approx}{H}\,, \tag{2.3a}$$

$$\int \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}\cdot\underset{\sim}{v}\,dx=\int\underset{\sim}{f}\cdot\underset{\sim}{v}\,dx\,, \qquad \underset{\sim}{v}\in\underset{\sim}{H}^0\,, \tag{2.3b}$$

$$\int \operatorname{as}(\underset{\approx}{\sigma})\eta\,dx=0\,, \qquad \eta\in H^0\,, \tag{2.3c}$$

where

$$a(\underset{\approx}{\sigma}, \underset{\approx}{\tau})=\int\left[\frac{1}{2\mu}\underset{\approx}{\sigma}:\underset{\approx}{\tau}-\frac{\lambda}{4\mu(\mu+\lambda)}\operatorname{tr}(\underset{\approx}{\sigma})\operatorname{tr}(\underset{\approx}{\tau})\right]dx\,, \tag{2.4}$$

and the integrals are over Ω. This problem has a unique solution, namely the solution to (2.2).

3. Definition of PEERS

Let $\mathscr{T}$ be a triangulation of Ω with angle bound θ; i.e., $\mathscr{T}$ is a set of closed triangles with union $\bar{\Omega}$ such that any two nondisjoint elements of $\mathscr{T}$ meet in a common vertex or edge, and $\theta>0$ is a lower bound for all angles of the triangulation.

For $T \in \mathcal{T}$ set $h_T = \operatorname{diam}(T)$, and put $h = \sup\{h_T \mid T \in \mathcal{T}\}$.

Let $\mathcal{P}_r$ denote the space of polynomials in two variables of total degree at most r, and let $\underset{\sim}{\mathcal{S}} \subset \underset{\sim}{\mathcal{P}}_1$ be the space of polynomial vector fields of the form $(a+bx, c+by)$, where $a, b, c \in \mathbf{R}$. We set

$$M^r = \{g \in H^0 \mid g|_T \in \mathcal{P}_r,\ T \in \mathcal{T}\}\,,$$

$$\underset{\sim}{S} = \{\underset{\sim}{v} \in \underset{\sim}{H}^0 \mid \underset{\sim}{v}|_T \in \underset{\sim}{\mathcal{S}},\ T \in \mathcal{T}\}\,,$$

$$M_0^r = M^r \cap H^1\,,$$

$$\underset{\sim}{S}_0 = \underset{\sim}{S} \cap \underset{\sim}{H}(\operatorname{div})\,,$$

$$\underset{\approx}{S}_0 = \{\underset{\approx}{\tau} \mid (\tau_{i1}, \tau_{i2}) \in \underset{\sim}{S}_0,\ i = 1, 2\}\,.$$

Here $\underset{\sim}{H}(\operatorname{div})$ denotes the space of $\underset{\sim}{u} \in \underset{\sim}{H}^0$ with $\operatorname{div} \underset{\sim}{u} \in H^0$. The space $\underset{\sim}{S}_0$ is one of the spaces constructed by Raviart and Thomas [18]; it can be defined equivalently as the set of vectors in $\underset{\sim}{S}$ having continuous normal components across interelement boundaries. We also define for each $T \in \mathcal{T}$ the bubble function b_T. This is the unique cubic polynomial on T vanishing on ∂T and normalized by

$$\int_T b_T\, dx = 1\,.$$

We consider b_T to be extended by zero, so that it is an element of M_0^3. Define $\underset{\sim}{B}$ to be the span of $\{\underset{\sim}{\operatorname{rot}}\, b_T\colon T \in \mathcal{T}\}$ and $\underset{\approx}{B} = \{\underset{\approx}{\tau} \mid (\tau_{i1}, \tau_{i2}) \in \underset{\sim}{B},\ i = 1, 2\}$.

The PEERS space for the approximation of the stress is given by

$$\underset{\approx}{V} = \underset{\approx}{S}_0 + \underset{\approx}{B}\,.$$

The displacement and rotation will be approximated in $\underset{\sim}{M}^0$ and M_0^1, respectively. Hence we define PEERS by the relation

$$\text{PEERS} = \underset{\approx}{V} \times \underset{\sim}{M}^0 \times M_0^1\,.$$

Thus, the discrete solution is defined as the triple $(\underset{\approx}{\sigma}_h, \underset{\sim}{u}_h, \gamma_h) \in \text{PEERS}$ such that

$$a(\underset{\approx}{\sigma}_h, \underset{\approx}{\tau}) + \int \underset{\sim}{u}_h \cdot \underset{\sim}{\operatorname{div}}\, \underset{\approx}{\tau}\, dx + \int \gamma_h \operatorname{as}(\underset{\approx}{\tau})\, dx = 0\,, \qquad \underset{\approx}{\tau} \in \underset{\approx}{V}\,, \tag{3.1a}$$

$$\int \underset{\sim}{\operatorname{div}}\, \underset{\approx}{\sigma}_h \cdot \underset{\sim}{v}\, dx = \int \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx\,, \qquad \underset{\sim}{v} \in \underset{\sim}{M}^0\,, \tag{3.1b}$$

$$\int \operatorname{as}(\underset{\approx}{\sigma}_h)\eta\, dx = 0\,, \qquad \eta \in M_0^1\,. \tag{3.1c}$$

The error analysis of the method of Raviart and Thomas associated with their lowest order element is greatly simplified [12] by the existence of a linear operator $\Pi\colon \underset{\sim}{H}^1 \to \underset{\sim}{S}_0$ satisfying the orthogonality relation

$$\int \operatorname{div}(\underset{\sim}{v} - \Pi \underset{\sim}{v}) g\, dx = 0\,, \qquad \underset{\sim}{v} \in \underset{\sim}{H}^1\,, \quad g \in M^0\,, \tag{3.2}$$

and the approximation property

$$\|\underset{\sim}{v} - \Pi \underset{\sim}{v}\|_0 \le Ch \|\underset{\sim}{v}\|_1\,, \qquad \underset{\sim}{v} \in \underset{\sim}{H}^1\,, \tag{3.3}$$

where C depends only on Ω and θ (the angle bound for $\mathscr{T}$). We define $\Pi: \underline{\underline{H}}^1 \to \underline{\underline{S}} \subset \underline{\underline{V}}$ as the Cartesian product of this operator and infer directly from (3.2) and (3.3) that

$$\int \underset{\sim}{\operatorname{div}}(\underline{\underline{\tau}} - \Pi\underline{\underline{\tau}}) \cdot \underline{v}\, dx = 0\,, \qquad \underline{\underline{\tau}} \in \underline{\underline{H}}^1\,, \quad \underline{v} \in \underline{M}^0\,, \tag{3.4}$$

$$\|\underline{\underline{\tau}} - \Pi\underline{\underline{\tau}}\|_0 \le Ch\|\underline{\underline{\tau}}\|_1\,, \qquad \underline{\underline{\tau}} \in \underline{\underline{H}}^1\,. \tag{3.5}$$

Note, moreover, that $\underset{\sim}{\operatorname{div}} \underline{\underline{B}} = 0$, so that $\underset{\sim}{\operatorname{div}} \underline{\underline{V}} = \underset{\sim}{\operatorname{div}} \underline{\underline{S}}_0 = \underline{M}^0$. Consequently, letting $P: \underline{H}^0 \to \underline{M}^0$ denote the orthogonal projection, we have the relation

$$\int \underset{\sim}{\operatorname{div}} \underline{\underline{\tau}} \cdot (\underline{v} - P\underline{v})\, dx = 0\,, \qquad \underline{\underline{\tau}} \in \underline{\underline{V}}\,, \quad \underline{v} \in \underline{H}^0\,. \tag{3.6}$$

It then follows from (3.4) that

$$\underset{\sim}{\operatorname{div}} \Pi\underline{\underline{\tau}} = P \,\underset{\sim}{\operatorname{div}} \underline{\underline{\tau}}\,, \qquad \underline{\underline{\tau}} \in \underline{\underline{H}}^1\,. \tag{3.7}$$

From the approximation property (with $\|\cdot\|_{-1}$ indicating the norm in $H^1(\Omega)'$)

$$\|\underline{v} - P\underline{v}\|_{-s} \le C\|\underline{v}\|_r h^{r+s}\,, \qquad \underline{v} \in \underline{H}^r\,, \quad r, s \in \{0, 1\}\,, \tag{3.8}$$

we see also that

$$\|\underset{\sim}{\operatorname{div}}(\underline{\underline{\tau}} - \Pi\underline{\underline{\tau}})\|_{-s} \le C\|\underset{\sim}{\operatorname{div}} \underline{\underline{\tau}}\|_r h^{r+s}\,, \qquad \underline{\underline{\tau}} \in \underline{\underline{H}}^{r+1}\,, \quad r, s \in \{0, 1\}\,. \tag{3.9}$$

4. Error Analysis

Recall that Ω is a convex polygon, θ is an angle bound for $\mathscr{T}$, and μ_1 and μ_0 are positive upper and lower bounds for the Lamé coefficient μ. Henceforth the symbol C denotes a generic constant which may depend only on Ω, θ, μ_0, and μ_1.

For the analysis it is useful to notice that the approximate solution can be defined equivalently through a variant of (3.1). For $\underline{\underline{X}} \subset \underline{\underline{H}}^0$ let

$$\hat{\underline{\underline{X}}} = \{\underline{\underline{\tau}} \in \underline{\underline{X}} \mid \textstyle\int \operatorname{tr}(\underline{\underline{\tau}})\, dx = 0\}\,.$$

Then $\underline{\underline{V}}$ may be decomposed as the direct sum of $\hat{\underline{\underline{V}}}$ and $\boldsymbol{R}\underline{\underline{\delta}}$, and the choice $\underline{\underline{\tau}} = \underline{\underline{\delta}}$ in (3.1a) implies that $\underline{\underline{\sigma}}_h \in \hat{\underline{\underline{V}}}$. Hence, the approximate solution can be defined as the triple $(\underline{\underline{\sigma}}_h, \underline{u}_h, \gamma_h) \in \hat{\underline{\underline{V}}} \times \underline{M}^0 \times M_0^1$ such that

$$a(\underline{\underline{\sigma}}_h, \underline{\underline{\tau}}) + \int \underline{u}_h \cdot \underset{\sim}{\operatorname{div}} \underline{\underline{\tau}}\, dx + \int \gamma_h \operatorname{as}(\underline{\underline{\tau}})\, dx = 0\,, \qquad \underline{\underline{\tau}} \in \hat{\underline{\underline{V}}}\,, \tag{4.1}$$

holds along with (3.1b) and (3.1c). Note that an analogous reinterpretation of the continuous problem is valid and, in particular, $\underline{\underline{\sigma}} \in \hat{\underline{\underline{H}}}$.

We begin by proving a quasioptimal estimate for PEERS. Again we emphasize that the constant in this estimate, as in all the estimates below, depends on the positive upper and lower bounds μ_1 and μ_0 for the Lamé constant μ but is independent of $\lambda \in [0, \infty)$.

THEOREM 4.1. *There exists a unique element* $(\underline{\underline{\sigma}}_h, \underline{u}_h, \gamma_h) \in \text{PEERS}$ *satisfying* (3.1). *Moreover, there exists a constant C such that*

$$\|\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h\|_{\underset{\approx}{H}}+\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_0+\|\gamma-\gamma_h\|_0$$
$$\leq C\inf\{\|\underset{\approx}{\sigma}-\underset{\approx}{\tau}\|_{\underset{\approx}{H}}+\|\underset{\sim}{u}-\underset{\sim}{v}\|_0+\|\gamma-\eta\|_0 \mid (\underset{\approx}{\tau},\underset{\sim}{v},\eta)\in\mathrm{PEERS}\}\,.$$

COROLLARY 4.2. *Suppose* $\underset{\sim}{f}\in\underset{\sim}{H}^1$. *Then,*

$$\|\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h\|_{\underset{\approx}{H}}+\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_0+\|\gamma-\gamma_h\|_0\leq Ch\|\underset{\sim}{f}\|_1\,.$$

Proof of Corollary 4.2. We take $\underset{\approx}{\tau}=\Pi\underset{\approx}{\sigma}$, $\underset{\sim}{v}=P\underset{\sim}{u}$, and $\eta=R\gamma$, R denoting the L^2 projection into M_0^1, in Theorem 4.1. From (3.5) and (3.9) we have

$$\|\underset{\approx}{\sigma}-\underset{\approx}{\tau}\|_0\leq Ch\|\underset{\approx}{\sigma}\|_1$$

and

$$\|\underset{\sim}{\operatorname{div}}(\underset{\approx}{\sigma}-\underset{\approx}{\tau})\|_0\leq Ch\|\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}\|_1=Ch\|\underset{\sim}{f}\|_1\,.$$

Clearly,

$$\|\underset{\sim}{u}-\underset{\sim}{v}\|_0\leq Ch\|\underset{\sim}{u}\|_1$$

and

$$\|\gamma-\eta\|_0\leq Ch\|\gamma\|_1\leq Ch\|\underset{\sim}{u}\|_2\,.$$

Since $\|\underset{\approx}{\sigma}\|_1+\|\underset{\sim}{u}\|_2\leq C\|\underset{\sim}{f}\|_0$ by Theorem 2.1, the corollary follows.

The proof of Theorem 4.1 will be based on the abstract stability theory for mixed methods of [5] applied to the alternate characterization of the discrete solution using (4.1). It suffices to prove the following two lemmas.

LEMMA 4.3. *There exists a constant C such that* $Ca(\underset{\approx}{\tau},\underset{\approx}{\tau})\geq\|\underset{\approx}{\tau}\|_{\underset{\approx}{H}}^2$ for all $\underset{\approx}{\tau}\in Z_h=\{\underset{\approx}{\sigma}\in\underset{\approx}{\hat{V}}\mid\int\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}\cdot\underset{\sim}{v}\,dx+\int\operatorname{as}(\underset{\approx}{\tau})\eta\,dx=0,\ \underset{\sim}{v}\in\underset{\sim}{M}^0,\ \eta\in M_0^1\}$. *In fact, the inequality holds for all divergence-free* $\underset{\approx}{\tau}\in\underset{\approx}{\hat{H}}$.

LEMMA 4.4. *There exists a constant C such that, for all pairs* $(\underset{\sim}{v},\eta)\in\underset{\sim}{M}^0\times M_0^1$, *there exists a nonzero* $\underset{\approx}{\tau}\in\underset{\approx}{\hat{V}}$ *such that*

$$C\{\int\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\cdot\underset{\sim}{v}\,dx+\int\operatorname{as}(\underset{\approx}{\tau})\eta\,dx\}\geq\|\underset{\approx}{\tau}\|_{\underset{\approx}{H}}(\|\underset{\sim}{v}\|_0+\|\eta\|_0)\,.$$

Moreover, $\underset{\approx}{\tau}$ *can be chosen so that* $\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}=\underset{\sim}{v}$ *and*

$$\int(\operatorname{as}(\underset{\approx}{\tau})-\eta)\alpha\,dx=0\,,\qquad\alpha\in M_0^1\,.$$

Before proving these two lemmas, let us note that Lemma 4.3 establishes one of the two conditions of [5] when $\underset{\approx}{\hat{V}}$ is considered in place of $\underset{\approx}{V}$. The second conditions of [5] follows from Lemma 4.4, so that the error estimate of Theorem 4.1 is valid, except that the range of $\underset{\approx}{\tau}$ in the infimum must be restricted to $\underset{\approx}{\hat{V}}$. Since $\underset{\approx}{\sigma}\in\underset{\approx}{\hat{H}}$,

$$\inf_{\underset{\approx}{\tau}\in\underset{\approx}{V}}\|\underset{\approx}{\sigma}-\underset{\approx}{\tau}\|_{\underset{\approx}{H}}=\inf_{\underset{\approx}{\tau}\in\underset{\approx}{V}}\|\underset{\approx}{\sigma}-\underset{\approx}{\tau}\|_{\underset{\approx}{H}}\,,$$

and Theorem 4.1 thus follows from the two lemmas.

Proof of Lemma 4.3. For $\underset{\approx}{\tau} \in \underset{\approx}{V}$, $\operatorname{div} \underset{\approx}{\tau} \in \underset{\sim}{M}^0$; hence, the condition $\int \underset{\sim}{\operatorname{div}} \underset{\approx}{\tau} \cdot \underset{\sim}{v} dx = 0$ for $\underset{\sim}{v} \in \underset{\sim}{M}^0$ implies that

$$\underset{\sim}{\operatorname{div}} \underset{\approx}{\tau} = 0 . \tag{4.2}$$

Consequently, it suffices to demonstrate the inequality for divergence-free tensors in $\underset{\approx}{\hat{H}}$; i.e., $\underset{\approx}{\tau} \in \underset{\approx}{H}$ such that

$$\underset{\sim}{\operatorname{div}} \underset{\approx}{\tau} = 0 \qquad \text{and} \qquad \int \operatorname{tr}(\underset{\approx}{\tau}) dx = 0 . \tag{4.3}$$

Take $\underset{\sim}{v} \in \underset{\sim}{\mathring{H}}^1$ such that $\underset{\sim}{\operatorname{div}} v = \operatorname{tr}(\underset{\approx}{\tau})$ and $\|\underset{\approx}{v}\|_1 \le C(\Omega) \|\operatorname{tr}(\underset{\approx}{\tau})\|_0$; this is possible in light of (4.3) and the assumption that Ω is convex [4]. Now, setting $\underset{\approx}{\bar{\tau}} = \underset{\approx}{\tau} - \operatorname{tr}(\underset{\approx}{\tau}) \underset{\approx}{\delta}/2$, the deviatoric of $\underset{\approx}{\tau}$, we have

$$\begin{aligned} \|\operatorname{tr}(\underset{\approx}{\tau})\|_0^2 &= \int \operatorname{tr}(\underset{\approx}{\tau}) \underset{\approx}{\delta} : \underset{\approx}{\operatorname{grad}} \underset{\sim}{v} dx \\ &= -2 \int (\underset{\approx}{\bar{\tau}} : \underset{\approx}{\operatorname{grad}} \underset{\sim}{v} + \underset{\sim}{\operatorname{div}} \underset{\approx}{\tau} \cdot \underset{\sim}{v}) dx \\ &= -2 \int \underset{\approx}{\bar{\tau}} : \underset{\approx}{\operatorname{grad}} \underset{\sim}{v} dx \le 2 \|\underset{\approx}{\bar{\tau}}\|_0 \|v\|_1 , \end{aligned}$$

so that

$$\|\operatorname{tr}(\underset{\approx}{\tau})\|_0 \le C(\Omega) \|\underset{\approx}{\bar{\tau}}\|_0 . \tag{4.4}$$

Further,

$$\|\underset{\approx}{\tau}\|_{\underset{\approx}{H}}^2 = \|\underset{\approx}{\tau}\|_0^2 = \|\underset{\approx}{\bar{\tau}}\|_0^2 + \frac{1}{2} \|\operatorname{tr}(\underset{\approx}{\tau})\|_0^2 \le C(\Omega) \|\underset{\approx}{\bar{\tau}}\|_0^2 .$$

Finally, by the definition of the form a,

$$a(\underset{\approx}{\tau}, \underset{\approx}{\tau}) = \frac{1}{2\mu} \|\underset{\approx}{\bar{\tau}}\|_0^2 + \frac{1}{4(\mu+\lambda)} \|\operatorname{tr}(\underset{\approx}{\tau})\|_0^2 \ge \mu^{-1} C(\Omega)^{-1} \|\underset{\approx}{\tau}\|_{\underset{\approx}{H}}^2 ,$$

completing the proof of the lemma.

Proof of Lemma 4.4. Given $(\underset{\sim}{v}, \eta) \in \underset{\sim}{M}^0 \times M_0^1$, take $\underset{\approx}{\rho} \in \underset{\approx}{H}^1$ such that $\underset{\sim}{\operatorname{div}} \underset{\approx}{\rho} = \underset{\sim}{v}$ and $\|\underset{\approx}{\rho}\|_1 \le C \|\underset{\sim}{v}\|_0$. Let $\underset{\approx}{\tau}^1 = \Pi \underset{\approx}{\rho}$; then by (3.7) and (3.5),

$$\underset{\sim}{\operatorname{div}} \underset{\approx}{\tau}^1 = \underset{\sim}{v} , \qquad \|\underset{\approx}{\tau}^1\|_{\underset{\approx}{H}} \le C \|\underset{\sim}{v}\|_0 .$$

Set s equal to the mean value of $\eta - \operatorname{as}(\underset{\approx}{\tau}^1)$ on Ω; so, $|s| \le C(\|\underset{\approx}{\tau}^1\|_0 + \|\eta\|_0) \le C(\|\underset{\sim}{v}\|_0 + \|\eta\|_0)$, and $\beta = \eta - \operatorname{as}(\underset{\approx}{\tau}^1) - s$ has mean value zero. Thus, we can find $\underset{\sim}{q} \in \underset{\sim}{\mathring{H}}^1$ such that

$$\operatorname{div} \underset{\sim}{q} = \beta , \qquad \|\underset{\sim}{q}\|_1 \le C \|\beta\|_0 \le C(\|\underset{\sim}{v}\|_0 + \|\eta\|_0) . \tag{4.5}$$

By standard interpolation results [6] we can approximate $\underset{\sim}{q}$ by $\underset{\sim}{q}_h \in \underset{\sim}{M}_0^1 \cap \underset{\sim}{\mathring{H}}^1$ such that

$$\|\underset{\sim}{q}_h\|_1 \le C \|\underset{\sim}{q}\|_1 , \tag{4.6}$$

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{0,T}^2 \le C \|\underset{\sim}{q}\|_1^2 . \tag{4.7}$$

Set

$$\underset{\sim}{a}_T = \int_T (\underset{\sim}{q} - \underset{\sim}{q}_h)\,dx \in \boldsymbol{R}^2\,, \qquad T \in \mathscr{T}\,,$$

and note that

$$|\underset{\sim}{a}_T| \le C h_T \|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{0,T}\,. \tag{4.8}$$

Next, recall that b_T denotes the normalized bubble functions on T and set

$$\underset{\sim}{r} = \underset{\sim}{q}_h + \sum_{T \in \mathscr{T}} \underset{\sim}{a}_T b_T \in \mathring{\underset{\sim}{H}}{}^1\,;$$

so,

$$\int_T \underset{\sim}{r}\,dx = \int_T \underset{\sim}{q}_h\,dx + \underset{\sim}{a}_T = \int_T \underset{\sim}{q}\,dx\,, \qquad T \in \mathscr{T}\,. \tag{4.9}$$

Since $\|\underset{\sim}{\operatorname{grad}}\, b_T\|_{0,T} \le C h_T^{-2}$, it follows from (4.8) that

$$\|\underset{\approx}{\operatorname{grad}}\,(\underset{\sim}{a}_T b_T)\|_{0,T} \le C h_T^{-1} \|\underset{\sim}{q} - \underset{\sim}{q}_h\|_{0,T}\,,$$

and, by (4.7),

$$\left\| \underset{\approx}{\operatorname{grad}} \sum_T \underset{\sim}{a}_T b_T \right\|_0^2 = \sum_T \|\underset{\approx}{\operatorname{grad}}\,(\underset{\sim}{a}_T b_T)\|_{0,T}^2 \le C \|\underset{\sim}{q}\|_1^2\,. \tag{4.10}$$

Thus, the bound $\|\underset{\sim}{r}\|_1 \le C \|\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{r}\|_0 \le C(\|\underset{\sim}{v}\|_0 + \|\eta\|_0)$ results from (4.6), (4.10), and (4.5). Now, set

$$\underset{\approx}{\tau}^2 = \underset{\approx}{\tau}^1 + \begin{pmatrix} \underset{\sim}{\operatorname{rot}}\, r_1 \\ \underset{\sim}{\operatorname{rot}}\, r_2 \end{pmatrix} + \frac{s}{2} \underset{\approx}{\chi}\,.$$

One easily verifies that $\underset{\approx}{\tau}^2 \in \underset{\approx}{V}$ and that

$$\|\underset{\approx}{\tau}^2\|_0 \le C(\|\underset{\approx}{\tau}^1\|_0 + \|\underset{\sim}{r}\|_1 + |s|) \le C(\|\underset{\sim}{v}\|_0 + \|\eta\|_0)\,.$$

Moreover,

$$\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}^2 = \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}^1 = \underset{\sim}{v}$$

and, for $\alpha \in M_0^1$,

$$\begin{aligned} \int \operatorname{as}(\underset{\approx}{\tau}^2)\alpha\,dx &= \int (\operatorname{as}(\underset{\approx}{\tau}^1) + \operatorname{div} \underset{\sim}{r} + s)\alpha\,dx \\ &= \int (\eta - \beta)\alpha\,dx - \int \underset{\sim}{r} \cdot \underset{\sim}{\operatorname{grad}}\,\alpha\,dx\,. \end{aligned} \tag{4.11}$$

But, $\underset{\sim}{\operatorname{grad}}\,\alpha \in \underset{\sim}{M}^0$, so that (4.9) and (4.5) imply that

$$\int \underset{\sim}{r} \cdot \underset{\sim}{\operatorname{grad}}\,\alpha\,dx = \int \underset{\sim}{q} \cdot \underset{\sim}{\operatorname{grad}}\,\alpha\,dx = -\int \operatorname{div} \underset{\sim}{q}\,\alpha\,dx = -\int \beta\alpha\,dx\,.$$

Combining this equation with (4.11) gives

$$\int \operatorname{as}(\underset{\approx}{\tau}^2)\alpha\,dx = \int \eta\alpha\,dx\,, \qquad \alpha \in M_0^1\,.$$

Finally, let t be the mean value of $\operatorname{tr}(\underset{\approx}{\tau}^2)$ and set $\underset{\approx}{\tau}=\underset{\approx}{\tau}^2-(t/2)\underset{\approx}{\delta}\in\underset{\approx}{\hat{V}}$. Then, $\|\underset{\approx}{\tau}\|_0\le\|\underset{\approx}{\tau}^2\|_0\le C(\|\underset{\sim}{v}\|_0+\|\eta\|_0)$, $\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}=\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}^2=\underset{\sim}{v}$, and $\operatorname{as}(\underset{\approx}{\tau})=\operatorname{as}(\underset{\approx}{\tau}^2)$. We thus have

$$\int\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\cdot\underset{\sim}{v}\,dx+\int\operatorname{as}(\underset{\approx}{\tau})\eta\,dx=\|\underset{\sim}{v}\|_0^2+\|\eta\|_0^2\ge C^{-1}\|\underset{\approx}{\tau}\|_{\underset{\approx}{H}}(\|\underset{\sim}{v}\|_0+\|\eta\|_0)\,,$$

and the proof of Lemma 4.4 has been accomplished.

The H^0 error estimates for the stress and displacement do not require the extra regularity $\underset{\sim}{f}\in\underset{\sim}{H}^1$, as we now show. At the same time we demonstrate higher order estimates in the space $(H^1)'$ (the dual space of H^1, with the dual norm denoted $\|\cdot\|_{-1}$) and also certain supercloseness of $\underset{\sim}{u}_h$ to the $\underset{\sim}{H}^0$-projection $P\underset{\sim}{u}$.

THEOREM 4.5. *For* $\underset{\sim}{f}\in\underset{\sim}{H}^0$,

$$\|\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h\|_0+\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_0+\|\gamma-\gamma_h\|_0\le Ch\|\underset{\sim}{f}\|_0\,.$$

If $\underset{\sim}{f}\in\underset{\sim}{H}^1$, *then also*

$$\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{-1}+\|P\underset{\sim}{u}-\underset{\sim}{u}_h\|_0\le Ch^2\|\underset{\sim}{f}\|_1$$

and, for $\varepsilon>0$,

$$\|\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h\|_{-1}\le C_\varepsilon h^{2-\varepsilon}\|\underset{\sim}{f}\|_1\,.$$

Proof. Recall that we have defined the orthogonal projections $R\colon H^0\to M_0^1$ and $P\colon\underset{\sim}{H}^0\to\underset{\sim}{M}^0$, as well as the operator $\Pi\colon\underset{\approx}{H}^1\to\underset{\approx}{V}$. Define the error quantities

$$\begin{aligned}\underset{\approx}{d}&=\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h\in\underset{\approx}{V}\,,\\ \underset{\sim}{d}&=P\underset{\sim}{u}-\underset{\sim}{u}_h\in\underset{\sim}{M}^0\,,\\ d&=R\gamma-\gamma_h\in M_0^1\,.\end{aligned}$$

From (2.3) and (3.1) we have

$$\begin{aligned}(4.12\text{a})\qquad & a(\underset{\approx}{d},\underset{\approx}{\tau})+\int\underset{\sim}{d}\cdot\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\,dx+\int d\operatorname{as}(\underset{\approx}{\tau})dx\\ &\quad=a(\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma},\underset{\approx}{\tau})+\int(P\underset{\sim}{u}-\underset{\sim}{u})\cdot\underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\,dx+\int(R\gamma-\gamma)\operatorname{as}(\underset{\approx}{\tau})dx\\ &\quad=a(\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma},\underset{\approx}{\tau})+\int(R\gamma-\gamma)\operatorname{as}(\underset{\approx}{\tau})dx\,,\qquad \underset{\approx}{\tau}\in\underset{\approx}{V}\,,\\ (4.12\text{b})\qquad & \int\underset{\sim}{\operatorname{div}}\,\underset{\approx}{d}\cdot\underset{\sim}{v}\,dx=\int\underset{\sim}{\operatorname{div}}\,(\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma})\cdot\underset{\sim}{v}\,dx=0\,,\qquad \underset{\sim}{v}\in\underset{\sim}{M}^0\,,\\ (4.12\text{c})\qquad & \int\operatorname{as}(\underset{\approx}{d})\eta\,dx=\int\operatorname{as}(\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma})\eta\,dx\,,\qquad \eta\in M_0^1\,,\end{aligned}$$

where we have used (3.6) and (3.7) to simplify (4.12a) and (4.12b), respectively.

Note that (4.12b) and (3.7) imply that

$$(4.13)\qquad \underset{\sim}{\operatorname{div}}\,\underset{\approx}{d}=0\,,\qquad \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}_h=\underset{\sim}{\operatorname{div}}\,\Pi\underset{\approx}{\sigma}=P\underset{\sim}{f}\,.$$

Now let t be the mean value of $\operatorname{tr}(\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma})$ on Ω and set $\underset{\approx}{\bar{d}}=\underset{\approx}{d}-(t/2)\underset{\approx}{\delta}$. Clearly,

$$(4.14)\qquad \|\underset{\approx}{d}\|_0\le\|\underset{\approx}{\bar{d}}\|_0+C\|\Pi\underset{\approx}{\sigma}-\underset{\approx}{\sigma}\|_0\,.$$

Moreover, the choice $\tau=\delta$ in (4.12a) shows that t is equal to the mean value of $\operatorname{tr}(d)$ in Ω, so that $\tilde{d}\in\hat{V}$. Also,

$$a(\tilde{d},\tilde{d})=a(d,\tilde{d})\,,\qquad \operatorname{div}\tilde{d}=\operatorname{div}d=0\,,\qquad \operatorname{as}(\tilde{d})=\operatorname{as}(d)\,.$$

Therefore, applying Lemma 4.3 with $\tau=\tilde{d}\in\hat{V}$ and then using (4.12a) and (4.12c), we see that

$$\begin{aligned}\|\tilde{d}\|_0^2\le Ca(\tilde{d},\tilde{d})&=Ca(d,\tilde{d})\\ &=C[-\textstyle\int\operatorname{as}(\Pi\sigma-\sigma)d\,dx+a(\Pi\sigma-\sigma,\tilde{d})+\int(R\gamma-\gamma)\operatorname{as}(\tilde{d})dx]\\ &\le C[\|\Pi\sigma-\sigma\|_0\,\|d\|_0+(\|\Pi\sigma-\sigma\|_0+\|R\gamma-\gamma\|_0)\|\tilde{d}\|_0]\,.\end{aligned}$$

Combining this bound with (4.14), we can then deduce that

$$\|d\|_0^2\le C(\|\Pi\sigma-\sigma\|_0^2+\|R\gamma-\gamma\|_0^2+\|\Pi\sigma-\sigma\|_0\,\|d\|_0)\,. \tag{4.15}$$

To estimate d we apply Lemma 4.4 with $v=0$ and $\eta=d$ to get a nonzero, divergence-free $\tau\in V$ such that

$$C\int d\operatorname{as}(\tau)dx\ge\|d\|_0\,\|\tau\|_0\,.$$

Substituting this τ into (4.12a) gives the inequality

$$\|d\|_0\le C(\|d\|_0+\|\sigma-\Pi\sigma\|_0+\|\gamma-R\gamma\|_0)\,. \tag{4.16}$$

From (4.15), (4.16), and Theorem 2.1, we then deduce the estimates

$$\begin{aligned}&\|d\|_0\le C(\|\sigma-\Pi\sigma\|_0+\|\gamma-R\gamma\|_0)\le Ch\|f\|_0\,,\\ &\|d\|_0\le Ch\|f\|_0\,.\end{aligned}$$

Hence,

$$\|\sigma-\sigma_h\|_0+\|\gamma-\gamma_h\|_0\le\|d\|_0+\|\sigma-\Pi\sigma\|_0+\|d\|_0+\|\gamma-R\gamma\|_0\le Ch\|f\|_0\,. \tag{4.17}$$

This establishes the desired H^0 estimates for the stress and $\operatorname{rot}u$. We now estimate the H^0 and H^{-1} error in the displacement. It suffices to prove that

$$\|d\|_0\le Ch^{s+1}\|f\|_s\,,\qquad s=0\quad\text{or}\quad 1\,, \tag{4.18}$$

for (3.8) and (4.18) can be combined to show that

$$\|u-u_h\|_{-s}\le Ch^{s+1}\|f\|_s\,,\qquad s=0\quad\text{or}\quad 1\,,$$

as desired. Note that, for $s=1$, (4.18) gives the desired bound on $\|Pu-u_h\|_0$.

To prove (4.18) we use a duality argument. Define the pair $v\in H^2$ and $\tau\in\{\xi\in\hat{H}^1:\ \xi_{12}=\xi_{21}\}$ as the solution of the elasticity system

$$\begin{aligned}&\tau=2\mu\varepsilon(v)+\lambda\operatorname{tr}(\varepsilon(v))\delta &&\text{on}\quad\Omega\,,\\ &\operatorname{div}\tau=d &&\text{on}\quad\Omega\,,\\ &v=0 &&\text{on}\quad\partial\Omega\,,\end{aligned}$$

and set $\eta=(\operatorname{rot} \underset{\sim}{v})/2$. We know from Theorem 2.1 that $\|\underset{\approx}{\tau}\|_1+\|\underset{\sim}{v}\|_2+\|\eta\|_1 \leq C\|\underset{\sim}{d}\|_0$. Now,

$$\begin{aligned}\|\underset{\sim}{d}\|_0^2 &= \int \operatorname{div} \underset{\approx}{\tau} \cdot \underset{\sim}{d} dx = \int \operatorname{div} \Pi \underset{\approx}{\tau} \cdot \underset{\sim}{d} dx \\ &= a(\underset{\approx}{\sigma}_h - \underset{\approx}{\sigma}, \Pi \underset{\approx}{\tau}) + \int (\gamma_h - \gamma) \operatorname{as}(\Pi \underset{\approx}{\tau}) dx \\ &= a(\underset{\approx}{\sigma}_h - \underset{\approx}{\sigma}, \underset{\approx}{\tau}) + a(\underset{\approx}{\sigma}_h - \underset{\approx}{\sigma}, \Pi \underset{\approx}{\tau} - \underset{\approx}{\tau}) + \int (\gamma_h - \gamma) \operatorname{as}(\Pi \underset{\approx}{\tau} - \underset{\approx}{\tau}) dx \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where we have used (3.4), (4.12a), and the symmetry of $\underset{\approx}{\tau}$. Applying (3.5) and (4.17), we get

$$|I_2| + |I_3| \leq C(\|\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h\|_0 + \|\gamma - \gamma_h\|_0) h \|\underset{\approx}{\tau}\|_1 \leq C h^2 \|\underset{\sim}{f}\|_0 \|\underset{\sim}{d}\|_0 .$$

To bound I_1 we use the defining equations for $\underset{\approx}{\tau}$, $\underset{\sim}{v}$, and η and those for $\underset{\approx}{\sigma}$ and $\underset{\approx}{\sigma}_h$ to compute

$$\begin{aligned} I_1 &= \int (\underset{\approx}{\sigma}_h - \underset{\approx}{\sigma}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v}) dx = \int \operatorname{div}(\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h) \cdot \underset{\sim}{v} dx + \int \operatorname{as}(\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h) \eta dx \\ &= \int (\underset{\sim}{f} - P\underset{\sim}{f}) \cdot (\underset{\sim}{v} - P\underset{\sim}{v}) dx + \int \operatorname{as}(\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h)(\eta - R\eta) dx ,\end{aligned}$$

so that

$$|I_1| \leq C(h\|\underset{\sim}{f} - P\underset{\sim}{f}\|_0 + h^2 \|\underset{\sim}{f}\|_0) \|\underset{\sim}{d}\|_0 .$$

Combining, we have

$$\|\underset{\sim}{d}\|_0 \leq C(h^2 \|\underset{\sim}{f}\|_0 + h \|\underset{\sim}{f} - P\underset{\sim}{f}\|_0) ,$$

from which (4.18) follows.

To complete the proof of Theorem 4.5 we must prove the $\underset{\approx}{H}^{-1}$ estimate

$$\|\underset{\approx}{e}\|_{-1} = \|\underset{\approx}{\sigma} - \underset{\approx}{\sigma}_h\|_{-1} \leq C_\varepsilon h^{2-\varepsilon} \|\underset{\sim}{f}\|_1$$

for the stress. We shall use the decomposition

$$\underset{\approx}{e} = \underset{\approx}{e}^A + \underset{\approx}{e}^B + \underset{\approx}{e}^C := \left[\underset{\approx}{e} - \frac{1}{2} \operatorname{as}(\underset{\approx}{e}) \underset{\approx}{\chi} - \frac{1}{2} \operatorname{tr}(\underset{\approx}{e}) \underset{\approx}{\delta}\right] + \frac{1}{2} \operatorname{as}(\underset{\approx}{e}) \underset{\approx}{\chi} + \frac{1}{2} \operatorname{tr}(\underset{\approx}{e}) \underset{\approx}{\delta} .$$

Hence, it suffices to show that

(4.19a) $$\|\underset{\approx}{e}^A\|_{-1} \leq C h^2 \|\underset{\sim}{f}\|_1 ,$$

(4.19b) $$\|\operatorname{as}(\underset{\approx}{e})\|_{-1} \leq C h^2 \|\underset{\sim}{f}\|_1 ,$$

(4.19c) $$\|\operatorname{tr}(\underset{\approx}{e})\|_{-1} \leq C_\varepsilon h^{2-\varepsilon} \|\underset{\sim}{f}\|_1 .$$

Since $\underset{\approx}{e}^A$ is symmetric, there exists a symmetric tensor $\underset{\approx}{\psi} \in \underset{\approx}{H}^1$ such that

$$\|\underset{\approx}{e}^A\|_{-1} = \int \underset{\approx}{e}^A : \underset{\approx}{\psi} dx , \qquad \|\underset{\approx}{\psi}\|_1 = 1 .$$

Moreover, since $\operatorname{tr}(\underset{\approx}{e}^A) = 0$, it follows that $\operatorname{tr}(\underset{\approx}{\psi}) = 0$. (For, were this not so, then the tensor $\underset{\approx}{\psi}-\operatorname{tr}(\underset{\approx}{\psi})\underset{\approx}{\delta}/2$, being the $\underset{\approx}{H}^1$ orthogonal projection of $\underset{\approx}{\psi}$ onto the subspace of trace-free tensors, would have $\underset{\approx}{H}^1$ norm strictly smaller than unity. But this cannot be, as the $\underset{\approx}{H}^0$ inner product of this tensor with $\underset{\approx}{e}^A$ is the same as that for $\underset{\approx}{\psi}$ with $\underset{\approx}{e}^A$, namely $\|\underset{\approx}{e}^A\|_{-1}$.) Since $\underset{\approx}{\psi}$ is symmetric and trace-free, the definition of the bilinear form a and the equations (2.3a) and (3.1a) show that

$$\begin{aligned}\frac{1}{2\mu}\|\underset{\approx}{e}^A\|_{-1}&=\frac{1}{2\mu}\int\underset{\approx}{e}^A:\underset{\approx}{\psi}dx=\frac{1}{2\mu}\int\underset{\approx}{e}:\underset{\approx}{\psi}dx=a(\underset{\approx}{e},\underset{\approx}{\psi})\\&=a(\underset{\approx}{e},\underset{\approx}{\psi}-\Pi\underset{\approx}{\psi})-\int\operatorname{div}\Pi\underset{\approx}{\psi}\cdot(\underset{\sim}{u}-\underset{\sim}{u}_h)dx-\int\operatorname{as}(\Pi\underset{\approx}{\psi})(\gamma-\gamma_h)dx\\&=:I_1+I_2+I_3\,.\end{aligned}$$

Now, by (4.17) and (3.5),

$$\begin{aligned}|I_1|&\le C\|\underset{\approx}{e}\|_0\,\|\underset{\approx}{\psi}-\Pi\underset{\approx}{\psi}\|_0\le Ch^2\|\underset{\sim}{f}\|_0\,;\\|I_2|&=|\textstyle\int\underset{\sim}{\operatorname{div}}\,\Pi\underset{\approx}{\psi}\cdot\underset{\sim}{d}dx|\le\|\underset{\approx}{\psi}\|_1\,\|\underset{\sim}{d}\|_0\le Ch^2\|\underset{\sim}{f}\|_1\,,\end{aligned}$$

by (3.7) and (4.18); and

$$|I_3|=|\textstyle\int\operatorname{as}(\Pi\underset{\approx}{\psi}-\underset{\approx}{\psi})(\gamma-\gamma_h)dx|\le\|\underset{\approx}{\psi}-\Pi\underset{\approx}{\psi}\|_0\,\|\gamma-\gamma_h\|_0\le Ch^2\|\underset{\sim}{f}\|_0\,,$$

by (3.5), (4.17), and the symmetry of $\underset{\approx}{\psi}$. This establishes (4.19a).

The estimate (4.19b) follows easily from (2.3c) and (3.1c): for $\varphi\in H^1$, $\|\varphi\|_1=1$,

$$|\textstyle\int\operatorname{as}(\underset{\approx}{e})\varphi dx|=|\int\operatorname{as}(\underset{\approx}{e})(\varphi-R\varphi)dx|\le Ch\|\underset{\approx}{e}\|_0\,\|\varphi\|_1\le Ch^2\|\underset{\sim}{f}\|_0\,.$$

Finally, we estimate the H^{-1} norm of $\operatorname{tr}(\underset{\approx}{e})$. Choose $\varphi\in H^1$ such that $\|\varphi\|_1=1$ and $\int\operatorname{tr}(\underset{\approx}{e})\varphi dx=\|\operatorname{tr}(\underset{\approx}{e})\|_{-1}$. By taking $\underset{\approx}{\tau}=\underset{\approx}{\delta}$ in (2.3a) and (3.1a), we have $\int\operatorname{tr}(\underset{\approx}{e})dx=0$, and it follows that $\int\varphi dx=0$. Hence, there exists $\underset{\sim}{v}\in\mathring{\underset{\sim}{H}}^1\cap\underset{\sim}{H}^{2-\varepsilon}$ with

$$\operatorname{div}\underset{\sim}{v}=-2\varphi\,,$$

and $\|\underset{\sim}{v}\|_{2-\varepsilon}\le C_\varepsilon\|\varphi\|_1\leqq C_\varepsilon$. (This regularity estimate is proved in [4]; in general, there is no such $\underset{\sim}{v}$ in $\underset{\sim}{H}^2$ unless φ satisfies a boundary constraint.) Then,

$$\begin{aligned}\|\operatorname{tr}(\underset{\approx}{e})\|_{-1}&=-\frac{1}{2}\int\operatorname{tr}(\underset{\approx}{e})\underset{\approx}{\delta}:\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{v}dx\\&=\int(\underset{\approx}{e}^A+\underset{\approx}{e}^B):\underset{\approx}{\operatorname{grad}}\,\underset{\sim}{v}dx+\int\underset{\sim}{\operatorname{div}}\,\underset{\approx}{e}\cdot\underset{\sim}{v}dx\\&\le(\|\underset{\approx}{e}^A\|_{-1+\varepsilon}+\|e^B\|_{-1+\varepsilon})\|\underset{\sim}{v}\|_{2-\varepsilon}+\|\underset{\sim}{f}-P\underset{\sim}{f}\|_0\,\|\underset{\sim}{v}-P\underset{\sim}{v}\|_0\\&\le Ch^{2-\varepsilon}\|\underset{\sim}{f}\|_1\,,\end{aligned}$$

using interpolation between (4.17) and (4.19a) and (4.19b). This completes the proof of Theorem 4.5.

The higher order H^{-1} convergence proved above is often associated with superconvergence phenomena. For example, it follows easily that on the average $\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h$ and $\underset{\sim}{u}-\underset{\sim}{u}_h$ converge to zero with second order (modulo, perhaps, a factor of $h^{-\varepsilon}$ for $\underset{\approx}{\sigma}-\underset{\approx}{\sigma}_h$).

The estimate on $\|P\underset{\sim}{u}-\underset{\sim}{u}_h\|_0$ given in the theorem has various applications. In [7], Douglas and Milner use an analogous result to obtain interior and superconvergence estimates. Also, Arnold and Brezzi [2] use such an estimate in deriving a higher order correct approximation to the scalar variable in a Raviart-Thomas mixed method. Here we apply it to derive a simple optimal order estimate on the displacement error in L^∞, which is analogous to an earlier result of Douglas and Roberts [9]. For this result (only) we require a quasiuniform mesh, so that the inverse property

$$\|\underset{\sim}{v}\|_{\underset{\sim}{L}^\infty}\le C_0h^{-1}\|\underset{\sim}{v}\|_0\,,\qquad \underset{\sim}{v}\in\underset{\sim}{M}^0\,, \tag{4.20}$$

is valid. Here $\underset{\sim}{L}^\infty$ is the space of (essentially) bounded vectors on Ω, $\underset{\sim}{W}^1_\infty$ the subspace thereof of vectors with bounded gradient.

THEOREM 4.6. *If $\underset{\sim}{f}\in\underset{\sim}{H}^1$, $\underset{\sim}{u}\in\underset{\sim}{W}^1_\infty$, and the inverse property* (4.20) *holds, then*

$$\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{\underset{\sim}{L}^\infty}\le CC_0h(\|\underset{\sim}{f}\|_1+\|\underset{\sim}{u}\|_{\underset{\sim}{W}^1_\infty})\,.$$

Proof. From (4.20) and Theorem 4.5,

$$\|P\underset{\sim}{u}-\underset{\sim}{u}_h\|_{\underset{\sim}{L}^\infty}\le C_0h^{-1}\|P\underset{\sim}{u}-\underset{\sim}{u}_h\|_0\le CC_0h\|\underset{\sim}{f}\|_1\,,$$

while it is easy to see that

$$\|\underset{\sim}{u}-P\underset{\sim}{u}\|_{\underset{\sim}{L}^\infty}\le Ch\|\underset{\sim}{u}\|_{\underset{\sim}{W}^1_\infty}\,.$$

5. Implementation via Multipliers and Higher Order Correct Approximation of the Displacements

We now discuss the implementation of PEERS. Let $\mathscr{E}$ denote the set of edges of triangles in $\mathscr{T}$ and $\mathscr{V}$ the set of vertices. We set

$$\mathscr{E}_0=\{e\in\mathscr{E}\,|\,e\not\subset\partial\Omega\}\,.$$

For each $T\in\mathscr{T}$ let $\underset{\sim}{n}_T$ denote the outward unit normal to T (constant on each edge of T), and for each $e\in\mathscr{E}$ let m_e denote the midpoint of e and $\underset{\sim}{n}_e$ one of the unit vectors normal to e. As is well-known, an element $\underset{\sim}{v}\in\underset{\sim}{S}_0$ is uniquely determined by taking as degrees of freedom the values of $\underset{\sim}{v}\cdot\underset{\sim}{n}_e$ at m_e, $e\in\mathscr{E}$. Thus, we can define nodal basis functions $\underset{\sim}{\varphi}_e\in\underset{\sim}{S}_0$ by the equations

$$\underset{\sim}{\varphi}_e\cdot\underset{\sim}{n}_e(m_e)=1\,,$$

$$\underset{\sim}{\varphi}_e\cdot\underset{\sim}{n}_g(m_g)=0\,,\qquad g\in\mathscr{E}\backslash\{e\}\,.$$

A basis for $\underset{\approx}{V}$ is then given by the tensors

$$\begin{pmatrix}\varphi_e\\ \underset{\sim}{0}\end{pmatrix}, \begin{pmatrix}\underset{\sim}{0}\\ \varphi_e\end{pmatrix}, \qquad e\in\mathscr{E},$$

$$\begin{pmatrix}\underset{\sim}{\operatorname{rot}} b_T\\ \underset{\sim}{0}\end{pmatrix}, \begin{pmatrix}\underset{\sim}{0}\\ \underset{\sim}{\operatorname{rot}} b_T\end{pmatrix}, \qquad T\in\mathscr{T}.$$

The spaces $\underset{\sim}{M}^0$ and M_0^1 have familiar nodal bases. We can reduce (3.1) to a sparse linear system for the coefficients of $(\underset{\approx}{\sigma}_h, \underset{\sim}{u}_h, \gamma_h)$ with respect to these bases in the usual way. The resulting matrix is symmetric but indefinite.

However, PEERS can be implemented in another fashion which leads to a positive definite linear system, resembling that of a displacement method. The essential idea is to remove the constraint of interelement continuity of the normal components of elements of $\underset{\approx}{V}$ and reimpose this continuity via a Lagrange multiplier on each interelement edge. In this implementation each degree of freedom associated with the approximate stress field is internal to a single element, and so it can be eliminated inexpensively. The smaller linear system which results is positive definite. It can be reduced further by elimination of the displacement field, which also has only internal degrees of freedom.

The Lagrange multiplier introduced to enforce interelement continuity can be interpreted as another approximation to the displacement. Moreover, it turns out that this approximation is of second order accuracy in $\underset{\sim}{L}^2$, in contrast to $\underset{\sim}{u}_h$, which affords only first order accuracy.

An analogous implementation can be used with many mixed methods (essentially those for which the first variable—the stress field in the present case—is subject to interelement continuity constraints only across interelement edges and not at vertices). This concept is discussed in detail by Arnold and Brezzi [2], who also demonstrate an analogous higher order convergence associated with the new multipliers for two widely used mixed methods. Here we sketch briefly the application to PEERS. For more details as well as the techniques required for proof of the claims made here see [2].

Let $\underset{\sim}{V}(T)$ be the space of vector functions on T which are restrictions of functions in $\underset{\sim}{\mathscr{S}}+\boldsymbol{R}\,\underset{\sim}{\operatorname{rot}}\, b_T$, and set $\underset{\approx}{V}(T)=\{\underset{\approx}{\tau} \mid (\tau_{i1}, \tau_{i2})\in \underset{\sim}{V}(T),\ i=1,2\}$. Note that $\dim\, \underset{\approx}{V}(T)=8$. Set

$$\underset{\approx}{V}^* = \{\underset{\approx}{\tau}\in \underset{\approx}{H}^0 \mid \tau|_T\in \underset{\approx}{V}(T),\ T\in\mathscr{T}\}.$$

We enforce no interelement continuity on elements of $\underset{\approx}{V}^*$. Considering the elements of $\underset{\approx}{V}(T)$ to be extended to Ω by zero, we have $\underset{\approx}{V}(T)\subset \underset{\approx}{V}^*$ and a basis for $\underset{\approx}{V}^*$ can be taken as the union of the bases of the $\underset{\approx}{V}(T)$, $T\in\mathscr{T}$. Define also the space $\mathring{M}_N^1$ as the nonconforming piecewise linear approximation to $\mathring{H}^1$; i.e., the subset of M^1 consisting of functions continuous at m_e, $e\in\mathscr{E}_0$, and vanishing at m_e, $e\in\mathscr{E}\backslash\mathscr{E}_0$. A set of degrees of freedom for a function in $\mathring{M}_N^1$ is given by its values at m_e, $e\in\mathscr{E}_0$. Finally, for $T\in\mathcal{T}$ define a_T: $\underset{\approx}{H}^0(T)\times\underset{\approx}{H}^0(T)\to\boldsymbol{R}$ by restricting the integral in (2.4) to T.

Consider the following problem:

Find $(\underset{\approx}{\sigma}_h, \underset{\sim}{u}_h, \gamma_h, \underset{\sim}{\lambda}_h)\in\underset{\approx}{V}^*\times\underset{\sim}{M}^0\times M_0^1\times\underset{\sim}{\mathring{M}}_N^1$ such that

(5.1a) $$a_T(\underset{\approx}{\sigma}_h, \underset{\approx}{\tau})+\int_T \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\tau}\cdot\underset{\sim}{u}_h dx+\int_T \operatorname{as}(\underset{\approx}{\tau})\gamma_h dx-\int_{\partial T}(\underset{\approx}{\tau}\underset{\sim}{n}_T)\cdot\underset{\sim}{\lambda}_h ds=0\,,\qquad \underset{\approx}{\tau}\in\underset{\approx}{V}(T)\,,\quad T\in\mathcal{T}\,,$$

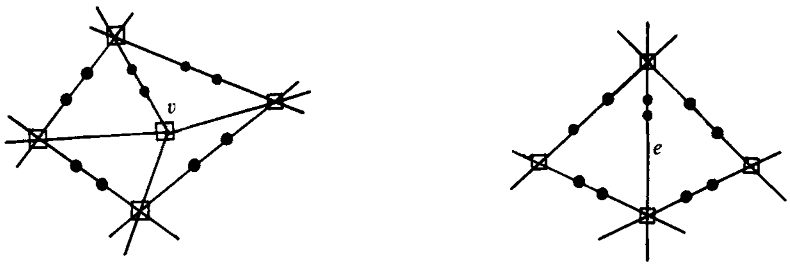
(5.1b) $$\int_T \underset{\sim}{\operatorname{div}}\,\underset{\approx}{\sigma}_h\cdot\underset{\sim}{v}dx=\int_T\underset{\sim}{f}\cdot\underset{\sim}{v}dx\,,\qquad \underset{\sim}{v}\in\mathcal{P}_0\,,\quad T\in\mathcal{T}\,,$$

(5.1c) $$\int \operatorname{as}(\underset{\approx}{\sigma}_h)\eta dx=0\,,\qquad \eta\in M_0^1\,,$$

(5.1d) $$-\sum_{T\in\mathcal{T}}\int_T(\underset{\approx}{\sigma}_h\underset{\sim}{n}_T)\cdot\underset{\sim}{\mu}dx=0\,,\qquad \underset{\sim}{\mu}\in\underset{\sim}{\mathring{M}}_N^1\,.$$

This problem has a unique solution and, as suggested by the notation, the first three components of the solution coincide with the solution to (3.1). (The fourth component, $\underset{\sim}{\lambda}_h$, may be reasonably viewed as another approximation to $\underset{\sim}{u}$, as can be seen by multiplying (2.2a) by $\underset{\approx}{\tau}\in\underset{\approx}{V}(T)$ and integrating over T.) In particular, the $\underset{\approx}{\sigma}_h$ determined by (5.1), a priori only to be in $\underset{\approx}{V}^*$, in fact belongs to $\underset{\approx}{V}$; i.e., it has continuous interelement normal components. To see this note that for each $\underset{\approx}{\tau}\in\underset{\approx}{V}(T)$, $\underset{\approx}{\tau}\underset{\sim}{n}_T$ is constant on each edge of T. (This is clearly true for $\underset{\approx}{\tau}\in\underset{\approx}{\mathcal{S}}$, while $(\underset{\sim}{\operatorname{rot}}\, b_T)\cdot\underset{\sim}{n}_T$ is the tangential derivative of b_T on ∂T and hence vanishes.) Now, fix an edge $e\in\mathcal{E}_0$ and take $\underset{\sim}{\mu}$ in (5.1d) to be zero at the midpoints of all other edges and successively to be $(1, 0)$ and $(0, 1)$ at m_e. The resulting equations show that the jump of $\underset{\approx}{\sigma}_h\underset{\sim}{n}_e$ across e vanishes. Consequently, (5.1) reduces to (3.1) when the test function $\underset{\approx}{\tau}$ in (5.1a) is restricted to $\underset{\approx}{V}$ and the relations of (5.1a) are summed over $T\in\mathcal{T}$.

Although the system in (5.1) is larger than that of (3.1), its structure allows for a more efficient solution process. For each $T\in\mathcal{T}$, (5.1a) reduces to a system of eight equations for the eight unknown coefficients of $\underset{\approx}{\sigma}_h|_T$. Inverting an 8×8 positive definite matrix gives $\underset{\approx}{\sigma}_h|_T$ in terms of $\underset{\sim}{u}_h|_T$, γ_h, and $\underset{\sim}{\lambda}_h$. Substituting this into (5.1b) gives a 2×2 system for the coefficients of $\underset{\sim}{u}_h|_T$ in terms of γ_h, $\underset{\sim}{\lambda}_h$, and the data $\underset{\sim}{f}$; inverting the corresponding 2×2 matrix and substituting the results into the previously derived expression for $\underset{\approx}{\sigma}_h|_T$ leads to a linear expression for $\underset{\approx}{\sigma}_h|_T$ in terms of γ_h, $\underset{\sim}{\lambda}_h$, $\underset{\sim}{f}$. Finally, this expression can be substituted into (5.1c) and (5.1d) to obtain a linear system for γ_h and $\underset{\sim}{\lambda}_h$. This final linear system can be viewed as a sparse linear system for the unknown coefficients of these functions with one linear equation associated with each vertex $v\in\mathcal{V}$ and two with each edge $e\in\mathcal{E}$. Its sparsity structure is indicated in Figure 1. Moreover, the matrix of the final system is positive definite, as can be seen as follows. If Σ denotes the vector of parameters representing $\underset{\approx}{\sigma}_h$ and Ψ the vector representing $\underset{\sim}{u}$, γ_h, and $\underset{\sim}{\lambda}_h$, then (5.1) takes the matricial form



The degrees of freedom entering the equations associated with the vertex v

The degrees of freedom entering the equations associated with the edge e

—□—: Value of γ_h at a vertex.
–•–•– : Value of $\underset{\sim}{\lambda}_h$ at the midpoint of an edge.

Fig. 1.

$$\begin{bmatrix} A & X \\ X^* & 0 \end{bmatrix} \begin{bmatrix} \Sigma \\ \Psi \end{bmatrix} = \Phi ,$$

where A is positive definite. Thus, the matrix $X^*A^{-1}X$ is positive definite, as it is both semi-definite and nonsingular. Then the block elimination of the $\underset{\sim}{u}_h$-parameters gives a positive definite matrix for the remaining parameters, which define γ_h and $\underset{\sim}{\lambda}_h$.

Having computed γ_h and $\underset{\sim}{\lambda}_h$, $\underset{\approx}{\sigma}_h$ and $\underset{\sim}{u}_h$ can be recovered element by element, using the element stiffness matrices already calculated. Hence the introduction of $\underset{\sim}{\lambda}_h$ can be regarded as a computational device to render more efficient the computation of the original variables $\underset{\approx}{\sigma}_h$, $\underset{\sim}{u}_h$, and γ_h. However, as shown for other mixed methods in [2], $\underset{\sim}{\lambda}_h$ actually provides asymptotically better approximation to $\underset{\sim}{u}$ than does $\underset{\sim}{u}_h$. The proof is a slight modification of Theorem 2.1 of [2]. The result is as follows.

THEOREM 5.1. *There exists a constant C such that*

$$\|\underset{\sim}{u} - \underset{\sim}{\lambda}_h\|_0 \leq Ch^2 \|\underset{\sim}{f}\|_1 .$$

6. The Incompressible Case

If we set $\lambda = +\infty$ in the weak formulation of the elasticity problem (2.3), we get a weak formulation of the equations of incompressible elasticity. The solution of this system is not uniquely determined, since a constant multiple of $\underset{\approx}{\delta}$ can be added to $\underset{\approx}{\sigma}$. A unique solution holds under the additional condition that

$$\int \mathrm{tr}\,(\underset{\approx}{\sigma})dx = 0 , \tag{6.1}$$

which is valid automatically for $\lambda < \infty$. See [3] for details.

Since our error estimates are independent of $\lambda \in [0, \infty)$, one way to approximate the incompressible problem with PEERS is to set λ to some very large but finite value and solve the resulting problem. The additional error so introduced will be proportional to $1/\lambda$.

Alternatively, one can attack the incompressible problem directly, setting $\lambda = +\infty$ in (3.7) and, in analogy with (6.1), imposing the additional condition

$$\int \operatorname{tr}(\underset{\approx}{\sigma}_h)dx = 0 .$$

It is not difficult to see that the resulting system has a unique solution and that all the error estimates of section 4 remain valid in this case. Such a program is carried out in more detail for another element in [3].

7. Plasticity Problems

As mentioned in the introduction, the treatment of plasticity problems was one of our chief motivations in developing PEERS. We sketch here how PEERS can be applied to a model plasticity problem. The essential ideas are well-known; however, we include this discussion to clarify their applicability to PEERS especially for those readers less familiar with plasticity.

We consider an incremental plasticity problem as formulated in [14] and [15]. Let $\underset{\approx}{K}$ denote a closed convex set in $\underset{\approx}{R}$ which is symmetric with respect to its (1, 2)- and (2, 1)-entries, and for $T \subset \Omega$ set $\mathscr{K}(T) = \{\underset{\approx}{\tau} \in \underset{\approx}{H}^0(T) \mid \underset{\approx}{\tau}(x) \in \underset{\approx}{K} \text{ a.e. in } T\}$. Then $\mathscr{K} = \mathscr{K}(\Omega)$ is the convex set of plastically admissible stress fields. Using a simple time discretization, we have to solve at each virtual time step a stationary problem of the following structure.

Find $\underset{\approx}{\sigma}^n \in \underset{\approx}{H}_s \cap \mathscr{K}$ and $\underset{\sim}{u}^n \in \underset{\sim}{H}^0$ such that

$$\text{(7.1a)} \qquad k^{-1} a(\underset{\approx}{\sigma}^n - \underset{\approx}{\sigma}^{n-1}, \underset{\approx}{\tau} - \underset{\approx}{\sigma}^n) + \int \underset{\sim}{\operatorname{div}}(\underset{\approx}{\tau} - \underset{\approx}{\sigma}^n) \cdot \underset{\sim}{u}^n dx \geq 0 , \qquad \underset{\approx}{\tau} \in \underset{\approx}{H}_s \cap \mathscr{K},$$

$$\text{(7.1b)} \qquad \int \underset{\sim}{\operatorname{div}}\, \underset{\approx}{\sigma}^n \cdot \underset{\sim}{v} dx = \int \underset{\sim}{f} \cdot \underset{\sim}{v} dx , \qquad \underset{\sim}{v} \in \underset{\sim}{H}^0 .$$

Here $k > 0$ denotes the virtual timestep and $\underset{\approx}{H}_s$ is the symmetric subspace of $\underset{\approx}{H}$, and $\underset{\sim}{f}$ and $\underset{\approx}{\sigma}^{n-1}$ are known.

We discretize this problem by PEERS using the formulation (5.1). The use of this formulation is essential, because it permits the condition of plastic admissibility of the stresses to be discretized in a local fashion, as will be seen below. Denoting by $\underset{\approx}{\sigma}_h$ and $\underset{\sim}{u}_h$ the approximations of $\underset{\approx}{\sigma}^n$ and $\underset{\sim}{u}^n$ (to be calculated) and by $\underset{\approx}{\bar{\sigma}}$ the approximation of $\underset{\approx}{\sigma}^{n-1}$ (known), we arrive at the following discrete problem:

Find $(\underset{\approx}{\sigma}_h, \underset{\sim}{u}_h, \gamma_h, \underset{\sim}{\lambda}_h) \in (\underset{\approx}{V}^* \cap \mathscr{K}(\Omega)) \times \underset{\sim}{M}^0 \times M_0^1 \times \underset{\sim}{\mathring{M}}_N^1$ such that

$$\text{(7.2a)} \qquad k^{-1} a_T(\underset{\approx}{\sigma}_h - \underset{\approx}{\bar{\sigma}}, \underset{\approx}{\tau} - \underset{\approx}{\sigma}_h) + \int_T \underset{\sim}{\operatorname{div}}(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h) \cdot \underset{\sim}{u}_h dx$$
$$+ \int_T \operatorname{as}(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h)\gamma_h dx - \int_{\partial T} (\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h)\underset{\sim}{n}_T \cdot \underset{\sim}{\lambda}_h dx \geq 0 ,$$
$$\underset{\approx}{\tau} \in \underset{\approx}{V}^*(T) \cap \mathscr{K}(T), \quad T \in \mathscr{T} ,$$

(7.2b) $$\int_T \operatorname{div} \underset{\approx}{\sigma}_h \cdot \underset{\sim}{v}\, dx = \int_T \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx, \qquad \underset{\sim}{v} \in \underset{\sim}{\mathscr{P}}_0, \quad T \in \mathscr{T},$$

(7.2c) $$\int \operatorname{as}(\underset{\approx}{\sigma}_h)\eta\, dx = 0, \qquad \eta \in M_0^1,$$

(7.2d) $$-\sum_T \int_{\partial T} (\underset{\approx}{\sigma}_h \underset{\sim}{n}_T) \cdot \underset{\sim}{\mu}\, dx = 0, \qquad \underset{\sim}{\mu} \in \underset{\sim}{\mathring{M}}{}_N^1.$$

Using self-explanatory notation, we write (7.2) more compactly as

(7.3a) $$k^{-1} a(\underset{\approx}{\sigma}_h - \underset{\approx}{\bar\sigma}, \underset{\approx}{\tau} - \underset{\approx}{\sigma}_h) + b(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h, \underset{\sim}{u}_h) + c(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h, \gamma_h) + d(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h, \underset{\sim}{\lambda}_h) \geq 0,$$
$$\underset{\approx}{\tau} \in \underset{\approx}{V}^* \cap \mathscr{K},$$

(7.3b) $$b(\underset{\approx}{\sigma}_h, \underset{\sim}{v}) = \int \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx, \qquad \underset{\sim}{v} \in \underset{\sim}{M}^0,$$

(7.3c) $$c(\underset{\approx}{\sigma}_h, \eta) = 0, \qquad \eta \in M_0^1,$$

(7.3d) $$d(\underset{\approx}{\sigma}_h, \underset{\sim}{\mu}) = 0, \qquad \underset{\sim}{\mu} \in \underset{\sim}{\mathring{M}}{}_N^1.$$

This system can be solved iteratively by using a form of Uzawa's algorithm, as we now explain. First, initial approximations $\underset{\sim}{u}_h^0$, γ_h^0, $\underset{\sim}{\lambda}_h^0$ are chosen (for instance, as the solution from the previous virtual step). Then for $r = 1, 2, \cdots$, $\underset{\approx}{\sigma}_h^r$, $\underset{\sim}{u}_h^r$, γ_h^r, and $\underset{\sim}{\lambda}_h^r$ are computed from $\underset{\sim}{u}_h^{r-1}$, γ_h^{r-1}, and $\underset{\sim}{\lambda}_h^{r-1}$ in the following manner: $\sigma_h^r \in V^* \cap \mathscr{K}$ is the solution of

(7.4a) $$a(\underset{\approx}{\sigma}_h^r - \underset{\approx}{\bar\sigma}, \underset{\approx}{\tau} - \underset{\approx}{\sigma}_h^r) + b(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h^r, \underset{\sim}{u}_h^{r-1}) + c(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h^r, \gamma_h^{r-1})$$
$$+ d(\underset{\approx}{\tau} - \underset{\approx}{\sigma}_h^r, \underset{\sim}{\lambda}_h^{r-1}) \geq 0, \qquad \underset{\approx}{\tau} \in \underset{\approx}{V}^* \cap \mathscr{K};$$

$\underset{\sim}{u}_h^r \in \underset{\sim}{M}^0$ is the solution of

(7.4b) $$(\underset{\sim}{u}_h^r - \underset{\sim}{u}_h^{r-1}, \underset{\sim}{v})_{\underset{\sim}{M}^0} = \rho[b(\underset{\approx}{\sigma}_h, \underset{\sim}{v}) - \int \underset{\sim}{f} \cdot \underset{\sim}{v}\, dx], \qquad \underset{\sim}{v} \in \underset{\sim}{M}^0;$$

$\gamma_h^r \in M_0^1$ is the solution of

(7.4c) $$(\gamma_h^r - \gamma_h^{r-1}, \eta)_{M_0^1} = \rho c(\underset{\approx}{\sigma}_h^r, \eta), \qquad \eta \in M_0^1;$$

and $\underset{\sim}{\lambda}_h^r \in \underset{\sim}{\mathring{M}}{}_N^1$ is the solution of

(7.4d) $$(\underset{\sim}{\lambda}_h^r - \underset{\sim}{\lambda}_h^{r-1}, \underset{\sim}{\mu})_{\underset{\sim}{\mathring{M}}{}_N^1} = \rho d(\underset{\approx}{\sigma}_h^r, \underset{\sim}{\mu}), \qquad \underset{\approx}{\mu} \in \underset{\sim}{\mathring{M}}{}_N^1.$$

Here $(\cdot, \cdot)_M$ denotes a scalar product in M ($M = \underset{\sim}{M}^0$, M_0^1, or $\underset{\sim}{\mathring{M}}{}_N^1$). Usually for convenience one selects the scalar product in $\boldsymbol{R}^s$, $s = \dim M$, multiplied by a scaling factor. The discretization parameter ρ is a fixed positive real number; it is allowed to depend on r. (The Arrow-Hurwitz generalization of the Uzawa algorithm can also be applied.)

The continuous problem (7.1) need not have a solution. However, it can be shown that, if the origin lies in the interior of $\mathscr{K}$ and $\|\underset{\sim}{f}\|_0$ is sufficiently small, then both (7.1) and its discretization (7.2) have a solution and the stress fields $\underset{\approx}{\sigma}^n$ and $\underset{\approx}{\sigma}_h$ are uniquely determined (see [14] and [16] for details). It is also easy to show that, if (7.2) has a solution, then the iterates determined by the Uzawa algorithm (7.4) converge to a solution as r tends to infinity when the parameter ρ is sufficiently small—how small depends on the choice of the scalar products in (7.4b-d). See [15] for a discussion in a similar situation and [10] for the general theory.

In (7.4) only the first step (7.4a) involves solving a nonlinear problem and presents computational difficulty. It consists in minimizing a quadratic functional over the set $\underset{\approx}{V}^*(T) \cap \mathscr{K}(T)$, a convex set in an eight dimensional linear space, for each element T, and can be resolved by various algorithms [10]. Note that these computations can be performed independently in each element (and in parallel if desired), while a discretization based directly on the formulation (3.1) would involve a minimization over a convex set of much larger dimension.

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